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DYNAMIC RESPONSE OF A ROTATING BEAM SUBJECTED TO A RANDOM MOVING LOAD

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The problem of transverse vibrations of homogeneous isotropic rotating beams due to the passage of different types of loads is of considerable practical interest. Using analytical and numerical methods, this paper investigates the stochastic dynamic response of a rotating simply supported beam subjected to a random force with constant mean value moving with a constant speed along the beam. The beam is modelled by Euler–Bernoulli, Rayleigh, and Timoshenko beam models. The problem is formulated by means of partial differential equations. Closed form solutions for the mean and variance of the response for the three models are obtained. The results show the effect of load speed, beam rotating speed, and geometrical size of the beam on the random response of the beam represented by some random dynamic coefficients.

1. INTRODUCTION

The general problem of transverse vibrations of beams resulting from the passage of moving loads is of considerable practical interest in the dynamics of structures. It has a wide range of applications in the civil, mechanical and aircraft industries. This problem has been studied in the context of machinery operations and the behavior of bridges, runways, rails, roadways, pipelines, etc. Several investigations have been performed to study different aspects of such a problem.

Fryba [1, 2] studied the response of a simply supported beam, using an Euler–Bernoulli model, subjected to a moving single and continuous random forces. He studied the effects of the constant speed and damping on the response of the beam. Zibdeh [3] dealt with the random vibration of a simply supported elastic beam subjected to a random point load moving with time-varying velocity. The beam is also subjected to axial deterministic forces. Closed form solutions for the mean and variance of the response are obtained. Zibdeh and Rackwitz [4] investigated higher order moments of a simply supported beam subjected to a stream of random moving loads of Poissonian type. The stream of loads was assumed to move with a time varying velocity. The results were verified by simulation. They have also used analytical and numerical methods [5] to

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investigate the statistical response moments of beams with general boundary conditions subjected to a stream of random moving loading systems of Poissonian pulse type. Chang [6] presented a method to perform the deterministic and random vibration analysis of a Rayleigh Timoshenko beam on an elastic foundation, he used modal analysis to compute the dynamic responses of the structure, such as the displacement and bending moment and some statistical responses such as mean square values of the dynamic displacement and bending moment. Iwankiewicz and Sniady [7] studied the problem of the dynamic response of a beam to the passage of a train of concentrated forces with random amplitudes. They obtained explicit expressions for the expected value and the variance of the beam deflection. Katz *et al*. [8] have used the Euler–Bernoulli, Rayleigh and Timoshenko beam theories to model a rotating shaft. The shaft, which is simply supported, rotates at a constant rotational speed and is subjected to a deterministic load which is of constant magnitude and moving with a constant axial velocity. Closed form solutions of the response for the three models are obtained. Argento and Scott [9] studied the response of rotating and non-rotating Timoshenko beams subjected to an accelerating fixed direction distributed surface force. The solution for pinned supports is set up using multi-integral transforms, and the inversion is expressed in terms of convolution integrals. These are numerically integrated for a uniformly distributed load having an exponentially varying velocity function. Argento *et al*. [10] studied the response of a rotating Rayleigh beam with different boundary conditions subjected to an axially accelerating distributed surface line load. The effects of load speed, beam rotational speed and geometry are studied.

This paper presents an extension to the aforementioned work. The influence of a random load moving with a constant velocity on the random vibration characteristics of a simply-supported rotating beam is studied. The approach is based on Euler–Bernoulli, Rayleigh and Timoshenko beam theories in addition to stochastic methods. Closed form solutions for the mean and variance of the deflection are obtained and results are presented by means of some random dynamic coefficient. The effect of load speed, rotational speed of the beam and the Rayleigh beam coefficient on the dynamic coefficient are also studied. Comparisons with known solutions of random loads moving with uniform velocity are made.

2. FORMULATION

The beam considered is assumed to be homogenous and isotropic with a uniform cross-section (A) rotating about its longitudinal axis with a constant angular velocity (Ω) and simply supported at its two ends. A load is assumed to move along the beam with a constant velocity which is usually called the axial velocity of the moving load. The equations of motion for the different beam models are presented below:

(1) Euler–Bernoulli beam model:

$$EI\frac{\partial^4 v(x,\ t)}{\partial x^4}+\rho A\frac{\partial^2 v(x,\ t)}{\partial t^2}=P(x,\ t), \tag{1}$$

where E is the modulus of elasticity, I is the moment of inertia of the cross-section, ρ is the density of the material, A is the cross-sectional area, $v(x,\ t)$ is the transverse deflection of the beam at point x and time t and $P(x,\ t)$ is the applied random force moving with a constant speed, written as

$$P(x,\ t)=\delta(x-ct)P(t). \tag{2}$$

Here $\delta(.)$ denotes the familiar delta function, c is the speed of the force, and $P(t)$ is the concentrated force which is randomly variable in time defined as

$$P(t)=P_o+P^o(t), \tag{3}$$

where P_o is the constant mean value representing the deterministic part of the force and $P^o(t)$ is the random part of the force to be defined later.

The boundary and initial conditions are

$$v(0,\ t)=0, \qquad \frac{\partial^2 v(0,\ t)}{\partial x^2}=0, \qquad v(l,\ t)=0, \qquad \frac{\partial^2 v(l,\ t)}{\partial x^2}=0, \tag{4}$$

$$v(x,\ 0)=0, \qquad \frac{\partial v(x,\ 0)}{\partial t}=0. \tag{5}$$

(2) Rayleigh beam model:

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}-\rho I\left[\frac{\partial^2 v}{\partial x^2\,\partial t^2}-2\mathrm{i}\Omega\frac{\partial^3 v}{\partial x^2\,\partial t}\right]=P(x,\ t), \tag{6}$$

where the boundary and initial conditions are as given in equations (4) and (5), respectively.

(3) Timoshenko beam model:

$$\rho A\frac{\partial^2 v}{\partial t^2}-KAG\left[\frac{\partial^2 v}{\partial x^2}-\frac{\partial\psi}{\partial x}\right]=P(x,\ t), \tag{7}$$

$$EI\frac{\partial^2\psi}{\partial x^2}+KAG\left[\frac{\partial v}{\partial x}-\psi\right]-\rho I\left[\frac{\partial^2\psi}{\partial t^2}-2\mathrm{i}\Omega\frac{\partial\psi}{\partial t}\right]=0, \tag{8}$$

where K denotes the Timoshenko shear correction factor, G is the shear modulus and ψ is the slope of the deflection curve due to bending deformation alone. The bounday and initial conditions are given as

$$v(0,\ t)=0, \qquad \frac{\partial\psi(0,\ t)}{\partial x}=0, \qquad v(l,\ t)=0, \qquad \frac{\partial\psi(l,\ t)}{\partial x}=0, \tag{9}$$

$$v(x,\, 0) = 0, \qquad \frac{\partial v(x,\, 0)}{\partial t} = 0, \qquad \psi(x,\, 0), \qquad \frac{\partial \psi(x,\, 0)}{\partial t} = 0, \tag{10}$$

Equations (7) and (8) represent the required governing equations. They neglect damping effects but include rotary inertia and gyroscopic moment which appear as $\ddot{\psi}$ and $\dot{\psi}$ terms, respectively. The effect of gyroscopic term is to induce a displacement component perpendicular to the direction of the load. In general the equations that govern the behavior of the beam in the three models have the following general form

$$L[v(x,\, t)] = P(x,\, t), \tag{11}$$

where $L[.]$ is the operator defined in equations (1), (6) or (7) and (8).

The solution of equation (11) is assumed as

$$v(x,\, t) = E[v(x,\, t)] + v^{\mathrm{o}}(x,\, t). \tag{12}$$

Substituting equations (3) and (12) into equation (11) gives

$$L\{E[v(x,\, t)] + v^{\mathrm{o}}(x,\, t)\} = P_{\mathrm{o}} + P^{\mathrm{o}}(x,\, t), \tag{13}$$

which yields

$$L\{E[v(x,\, t)]\} = P_{\mathrm{o}}, \quad L\{v^{\mathrm{o}}(x,\, t)\} = P^{\mathrm{o}}(x,\, t). \tag{14, 15}$$

Deterministic solutions of the above three models can be obtained by considering equation (14) and such solutions may be found in reference [8]. The random part of the response is obtained by considering equation (15). In other words, the statistical characteristics of the first order (mean value) may be obtained from equation (14) while the statistical characteristics of the second order are described by, equation (15). For example, the mean value or the deterministic part of the deflection for the Euler–Bernoulli model is obtained by writing the deflection as

$$v(x,\, t) = \sum_{n=1}^{\infty} v_n(x) q_n(t), \tag{16}$$

where $q_n(t)$ are the generalized deflections or the modal responses, and $v_n(x)$ are the comparison functions. The comparison functions that satisfy the boundary conditions can be chosen as

$$v_n(x) = \sin(n\pi x/l). \tag{17}$$

Following normal procedure, the differential equation of the nth mode of the generalized deflection is written as

$$\ddot{q}_n(t) + \omega_n^2 q_n(t) = Q_n(t), \tag{18}$$

where

$$\omega_n = \sqrt{\frac{EI}{\rho A}}\left(\frac{n\pi}{l}\right)^2 \tag{19}$$

and

$$Q_n(t) = \frac{2}{l\rho A}\int_0^l P^o \delta(x - ct) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x. \tag{20}$$

Carrying out the integration in equation (20) one obtains

$$Q_n(t) = \frac{2}{l\rho A} P_o \sin \theta_n t. \tag{21}$$

This is known as the generalized force in which

$$\theta_n = \frac{n\pi c}{l}. \tag{22}$$

The solution of equation (18) can be written as:

$$q_n(t) = \int_0^t h_n(t-\tau)Q_n(\tau)\mathrm{d}\tau = \int_0^t h_n(t)Q_n(t-\tau)\,\mathrm{d}\tau, \tag{23}$$

where $h_n(t)$ is the impulse response function defined as

$$h_n(t) = \frac{1}{\omega_n}\sin(\omega_n t). \tag{24}$$

This equation represents the response of the system in equation (18) to an impulse $\delta(t)$ with zero initial conditions. To obtain the stochastic response, the covariance of the force is defined as [1],

$$C_{PP}(t_1, t_2) = E[P^o(t_1)P^o(t_2)], \tag{25}$$

where $E[.]$ denotes expectations. It follows that the covariance of the moving force becomes

$$C_{pp}(x_1, x_2, t_1, t_2) = [\delta(x_1 - ct_1)\delta(x_2 - ct_2)C_{PP}(t_1, t_2)]. \tag{26}$$

Using equation (20), the covariance of the generalized moving force is obtained as

$$C_{Q_nQ_m}(t_1, t_2) = \frac{4}{(\rho A l)^2} v_n(ct_1)v_m(ct_2)C_{PP}(t_1, t_2). \tag{27}$$

It follows from equation (23) that the covariance of the generalized deflection becomes

$$C_{q_nq_m} = \frac{4}{(\rho A l)^2}\int_0^t\int_0^t h_n(t_1-\tau_1)h_m(t_2-\tau_2)v_n(c\tau_1)v_m(c\tau_2)C_{PP}(\tau_1,\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2. \tag{28}$$

In light of equation (16), the covariance of the deflection can be written as

$$C_{vv}(x_1, x_2, t_1, t_2) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} v_n(x_1)v_m(x_2)C_{q_nq_m}(t_1, t_2). \tag{29}$$

The variance of the deflection becomes

$$\sigma_v^2(x, t) = C_{vv}(x, x, t, t) = \sum_{n=1}^{\infty} v_n^2(x)C_{q_nq_m}(t, t). \tag{30}$$

Upon substituting equation (28) into equation (30), the variance of the deflection is written as

$$\sigma_v^2(x, t) = \frac{4}{(\rho A l)^2} \times \frac{v_n^2(x)}{\omega_n^2}\int_0^t\int_0^t \sin\omega_n(t-\tau_1)\sin\omega_n(t-\tau_2)\sin(\theta_n\tau_1)$$
$$\times \sin(\theta_n\tau_2)C_{pp}(\tau_1, \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{31}$$

Equation (31) represents the general form of the variance of the Euler–Bernoulli beam subjected to a random moving force with constant velocity.

The numerical results will be arranged in the form of a ratio of the standard deviation, of the deflection at mid-span point of the beam to the value of v_{o}, the static deflection of a simply supported beam at its midspan produced by a concentrated force P acting at the midspan, defined as [8]

$$v_{\mathrm{o}} = \frac{Pl^3}{48EI} \tag{32}$$

This ratio can be expressed as a product of the coefficient of variation V_p of the force $P(t)$ and the function $V_{vp}(t)$, which is written as

$$V_v(x, t) = \sigma_v(l/2, t)/v_{\mathrm{o}} = V_pV_{vp}(t). \tag{33}$$

The coefficient of variation $V_v(x, t)$ is similar to the dynamic coefficient in the deterministic approach. In what follows the function $V_{vp}(t)$, dependent on time (t) is calculated at the mid-span of the beam for several types of covariances of the force $P(t)$ to obtain the coefficient of variation of the deflection of the beam, equation (34). The force $P(t)$ is assumed to be a stationary random function of (t). Two basic types of covariances are considered [1].

(a) White noise

$$C_P(\tau) = S_P\delta(\tau), \quad S_P(\omega) = S_P. \tag{34}$$

(b) Constant covariance

$$C_P(\tau) = \sigma_P^2, \quad S_P(\omega) = 2\Pi\sigma_P^2\delta(w). \tag{35}$$

For the case of white noise, the variance is obtained by substituting equation (34) into equation (31). Carrying out the integration and rearranging in accordance with equation (33), the function $V_{vp}(t)$ is written as

$$V_{vp}(t) = \sum_{n=1}^{\infty} \frac{96}{\pi^4} v_n(l/2) \times \left\{ \frac{\theta_n^2}{8n^6(\omega_n^2 - \theta_n^2)} \left[\sin(2\omega_n)t - \left(\frac{\omega_n}{\theta_n}\right)^3 \sin(2\theta_n)t + \sin\frac{2\omega_n}{\theta_n^2}(\omega_n^2 - \theta_n^2)t \right] \right\}^{1/2}, \tag{36}$$

where the coefficient of variation V_p in equation (33), which is analogous to the coefficient of variation of the force is written for the white noise case as [2]

$$V_p = \sqrt{S_p \omega_l}/P. \tag{37}$$

Following a similar procedure, the function $V_{vp}(t)$ for the constant covariance case is written as

$$V_{vp}(t) = \sum_{n=1}^{\infty} \frac{96}{n^4 \pi^4} \times v_n(l/2) \times \frac{\omega_n^2}{\omega_n^2 - \theta_n^2} \left[\sin\theta_n t - \frac{\theta_n}{\omega_n} \sin\omega_n t \right] \tag{38}$$

where the coefficient of variation of the force V_p is [2]

$$V_p = \sigma_p / P. \tag{39}$$

The random response of the Rayleigh beam model is obtained by substituting equation (16) into the random part of equation (6) which yields upon rearrangement

$$\ddot{q}_n(t) + 2\omega_{dn}\dot{q}_n(t) + \omega_n^2 q_n(t) = Q_n(t), \tag{40}$$

where

$$\omega_n^2 = \frac{EI\left(\frac{n\pi}{l}\right)^4}{\left[\rho A + \rho I\left(\frac{n\pi}{l}\right)^2\right]}, \qquad \omega_{dn} = -\frac{\mathrm{i}\Omega\rho l\left(\frac{n\pi}{l}\right)^2}{\left[\rho A + \rho I\left(\frac{n\pi}{l}\right)^2\right]}, \tag{41, 42}$$

$$Q_n(t) = \frac{1}{M_n}\int_0^t P^{\mathrm{o}}\delta(x - ct)\sin\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x, \qquad M_n = \frac{\left[\rho A l + \rho I l\left(\frac{n\pi}{l}\right)^2\right]}{2}. \tag{43, 44}$$

Using the convolution integral the solution of equation (40) can be found as

$$q_n(t) = \int_{-\infty}^{\infty} h_n(\tau)Q_n(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} h_n(t-\tau)Q_n(\tau)\,\mathrm{d}\tau, \tag{45}$$

where

$$h_n(t) = \left\{ \begin{matrix} \frac{1}{\omega_n'}\ \mathrm{e}^{-\omega_{dn}t} \sin(\omega_n' t..t \geqslant 0 \\ 0......................t < 0 \end{matrix} \right\}, \tag{46}$$

and

$$\omega_n'^2 = \omega_n^2 - \omega_{dn}^2. \tag{47}$$

Following exactly the same procedure as in the case of the Euler beam, the variance of the deflection can be written as

$$\sigma_v^2(x,\ t) = \sum_{n=1}^{\infty} \frac{v_n^2(x)}{M_n^2 \omega_n'^2} \int_0^t \int_0^t \mathrm{e}^{-\omega_{dn}(t-\tau_1)} \sin \omega_n'(t-\tau_1)\, \mathrm{e}^{-\omega_{dn}(t-\tau_2)} \sin \omega_n'(t-\tau_2)$$
$$\times \sin(\theta_n \tau_1) \sin(\theta_n \tau_2) C_{PP}(\tau_1,\ \tau_2)\ \mathrm{d}\tau_1 \tau_2, \tag{48}$$

where θ_n is as given in equation (22).

The expression $V_{vp}(t)$ for the case of white noise is obtained by substituting equation (34) into equation (48). Carrying out the integration and using trigonometric identities to obtain

$$V_{vp}(t) = \sum_{n=1}^{\infty} \frac{24}{(n\pi)^2} \frac{v_n(l/2)}{M_n \omega_n'} \frac{\rho A}{l^3} \left\{ \frac{1}{2\omega_{dn}} (1 - \mathrm{e}^{-2\omega_{dn}t}) + \frac{\omega_{dn}^2}{\omega_{dn}^2 + \omega_n'^2} \frac{1}{2\omega_{dn}} \right.$$
$$\times \left\{ 1 - \mathrm{e}^{-\omega_{dn}t} \times \left[\cos 2\omega_n' t - \frac{\omega_n'}{\omega_{dn}} \sin 2\omega_n' t \right] \right\} + \frac{\omega_{dn}^2}{\omega_{dn}^2 + (\theta_n - \omega_n')^2} \frac{(\theta_n - \omega_n')}{2\omega_{dn}^2}$$
$$\times \left\{ -\sin 2\theta_n t + \mathrm{e}^{-2\omega_{dn}t} \sin 2\omega_n' t + \frac{\omega_{dn}}{(\theta_n - \omega_n')} [\cos 2\theta_n t - \mathrm{e}^{-\omega_{dn}t} \cos 2\omega_n' t] \right.$$
$$+ \frac{\omega_{dn}^2}{\omega_{dn}^2 + (\omega_n' + \theta_n)^2} \frac{(\omega_n' + \theta_n)}{2\omega_{dn}^2} \left\{ \sin 2\theta_n t + \mathrm{e}^{-2\omega_{dn}t} \sin 2\omega_n' t \right.$$
$$\left. - \frac{\omega_{dn}}{(\omega_n' + \theta_n)} [\cos 2\theta_n t - \mathrm{e}^{-2\omega_{dn}t} \cos 2\omega_n' t] \right\}^{1/2}. \tag{49}$$

In a similar fashion the expression $V_{vp}(t)$ for the constant covariance case is obtained by substituting equation (35) into equation (48) to yield

$$V_{vp}(t) = \sum_{n=1}^{\infty} \frac{Pv_n(l/2)}{2M_n\omega'_n v_o} \left[\frac{(\omega'_n + \theta_n)^2}{(\omega'_n + \theta_n)^2 + \omega_{dn}^2} \left\{ \frac{1}{(\omega'_n + \theta_n)} [\sin\theta_n t + \mathrm{e}^{-\omega_{dn} t} \right.\right.$$

$$\times \sin\omega'_n t] + \frac{\omega_{dn}}{(\omega'_n + \theta_n)^2} [\cos\theta_n t - \mathrm{e}^{-\omega_{dn} t} \cos\omega'_n t]$$

$$- \frac{(\omega'_n - \theta_n)^2}{(\omega'_n - \theta_n)^2 + \omega_{dn}^2} \left\{ \frac{1}{(\omega'_n - \theta_n)} [\mathrm{e}^{-\omega_{dn} t} \sin\omega'_n t \right.$$

$$\left.\left. - \sin\theta_n t + \frac{\omega_{dn}}{(\omega'_n - \theta_n)^2} [\cos\theta_n t - \mathrm{e}^{-\omega_{dn} t} \cos\omega'_n t] \right\} \right] \tag{50}$$

The random response of a Timoshenko beam is solved by considering the random part of equations (7) and (8). Upon using the finite integral transformation approach and then using a Laplace transformation [8], one obtains the following form of the generalized deflection

$$q^{\mathrm{o}}(n, s) = L[P^{\mathrm{o}}(t)] \frac{1}{\rho A \Delta s} \frac{\theta_n}{s^2 + \theta_n^2} [s^2 - 2\mathrm{i}\Omega s + b^2], \tag{51}$$

where

$$b^2 = C_1^2 \left(\frac{n\pi}{l}\right)^2 + \left(\frac{C_2}{r_{\mathrm{o}}}\right) \tag{52}$$

in which r_{o} is the radius of gyration of the cross-section of the beam, and

$$C_1^2 = E/\rho \quad \text{and} \quad C_2^2 - KAG/\rho A, \tag{53}$$

$$\Delta s = (s - \mathrm{i}\omega_{n1})(s - \mathrm{i}\omega_{n2})(s - \mathrm{i}\omega_{n3})(s - \mathrm{i}\omega_{n4}). \tag{54}$$

ω_{n1}, ω_{n2}, ω_{n3} and ω_{n4} are defined in Appendix C. Using the convolution theorem to obtain

$$q^{\mathrm{o}}(t) = \left(\frac{\theta_n}{\rho A}\right) \int_0^t P^{\mathrm{o}}(\tau)[A_1 \mathrm{e}^{\mathrm{i}\omega_{n1}(t-\tau)} + A_2 \mathrm{e}^{\mathrm{i}\omega_{n3}(t-\tau)} + A_3 \mathrm{e}^{\mathrm{i}\omega_{n3}(t-\tau)}$$

$$+ A_4 \mathrm{e}^{\mathrm{i}\omega_{n4}(t-\tau)} + A_5 \mathrm{e}^{\mathrm{i}\theta_n(t-\tau)} + A_6 \mathrm{e}^{-\mathrm{i}\theta_n(t-\tau)}] \, \mathrm{d}\tau. \tag{55}$$

So the covariance of the generalized deflection can be written as:

$$C_{q_nq_j}(t_1,\ t_2) = \left(\frac{\theta_n}{\rho A}\right)^2 \int_0^t \int_0^t [A_1 e^{i\omega_{n1}(t_1-\tau_1)} + A_2 e^{i\omega_{n2}(t_1-\tau_1)} + A_3 e^{i\omega_{n3}(t_1-\tau_1)}$$
$$+ A_4 e^{i\omega_{n4}(t_1-\tau_1)} + A_5 e^{i\theta_n(t_1-\tau_1)} + A_6 e^{-i\theta_n(t_1-\tau_1)}]$$
$$\times [A_1 e^{i\omega_{n1}(t_2-\tau_2)} + A_2 e^{i\omega_{n2}(t_2-\tau_2)} + A_3 e^{i\omega_{n3}(t_3-\tau_3)}$$
$$+ A_4 e^{i\omega_{n4}(t_2-\tau_2)} + A_5 e^{i\theta_n(t_2-\tau_2)} + A_6 e^{-i\theta_n(t_2-\tau_2)}] C_{PP}(\tau_1,\ t_2)\, d\tau_1\, d\tau_2. \quad (56)$$

It follows that the variance of the deflection can be calculated from

$$\sigma_v^2(x,\ t) = \left(\frac{2\theta_n v_n(x)}{\rho A}\right)^2 \int_0^t \int_0^t [A_1 e^{i\omega_{n1}(t_1-\tau_1)} + A_2 e^{i\omega_{n2}(t_1-\tau_1)} + A_3 e^{i\omega_{n3}(t_1-\tau_1)}$$
$$+ A_4 e^{i\omega_{n4}(t_1-\tau_1)} + A_5 e^{i\theta_n(t_1-\tau_1)} + A_6 e^{-i\theta_n(t_1-\tau_1)}]$$
$$\times [A_1 e^{i\omega_{n1}(t_2-\tau_2)} + A_2 e^{i\omega_{n2}(t_2-\tau_2)} + A_3 e^{i\omega_{n3}(t_3-\tau_3)}$$
$$+ A_4 e^{i\omega_{n4}(t_2-\tau_2)} + A_5 e^{i\theta_n(t_2-\tau_2)} + A_6 e^{-i\theta_n(t_2-\tau_2)}] C_{PP}(\tau_1,\ t_2)\, d\tau_1\, d\tau_2. \quad (57)$$

This equation represents a general form of the variance of a Timoshenko beam subjected to a random moving force with constant velocity where A_1–A_6 can be found in Appendix A. Substituting equation (34) into equation (57), it follows that the expression $V_{vp}(t)$ for the case of white noise is written as

$$V_{vp}(t) = \sum_{n=1}^{\infty} \left(\frac{-96i\theta_n v_n(l/2)}{l^4(\omega_1)^{1/2}}\right) \left[\frac{EI}{\rho A}\right] \left\{ \frac{B_1}{\omega_1}(e^{i\omega_1 t} - 1) + \frac{B_2}{\omega_2}(e^{i\omega_2 t} - 1) + \frac{B_3}{\omega_3}(e^{i\omega_3 t} - 1)\right.$$
$$+ \frac{B_4}{\omega_4}(e^{i\omega_4 t} - 1) + \frac{B_5}{\omega_5}(e^{i\omega_5 t} - 1) + \frac{B_6}{\omega_6}(e^{i\omega_6 t} - 1) + \frac{B_7}{\omega_7}(e^{i\omega_7 t} - 1)$$
$$+ \frac{B_8}{\omega_8}(e^{i\omega_8 t} - 1) + \frac{B_9}{\omega_9}(e^{i\omega_9 t} - 1) + \frac{B_{10}}{\omega_{10}}(e^{i\omega_{10} t} - 1) + \frac{B_{11}}{\omega_{11}}(e^{i\omega_{11} t} - 1)$$
$$+ \frac{B_{12}}{\omega_{12}}(e^{i\omega_{12} t} - 1) + \frac{B_{13}}{\omega_{13}}(e^{i\omega_{13} t} - 1) + \frac{B_{14}}{\omega_{14}}(e^{i\omega_{14} t} - 1) + \frac{B_{15}}{\omega_{15}}(e^{i\omega_{15} t} - 1)$$
$$+ \frac{B_{16}}{\omega_{16}}(e^{i\omega_{16} t} - 1) + \frac{B_{17}}{\omega_{17}}(e^{i\omega_{17} t} - 1) + \frac{B_{18}}{\omega_{18}}(e^{i\omega_{18} t} - 1) + \frac{B_{19}}{\omega_{19}}(e^{i\omega_{19} t} - 1)$$
$$\left. + \frac{B_{20}}{\omega_{20}}(e^{i\omega_{20} t} - 1) + \frac{B_{21}}{\omega_{21}}(e^{i\omega_{21} t} - 1) \right\}^{1/2} \quad (58)$$

where the coefficients B_1–B_{21} are defined in Appendix B.

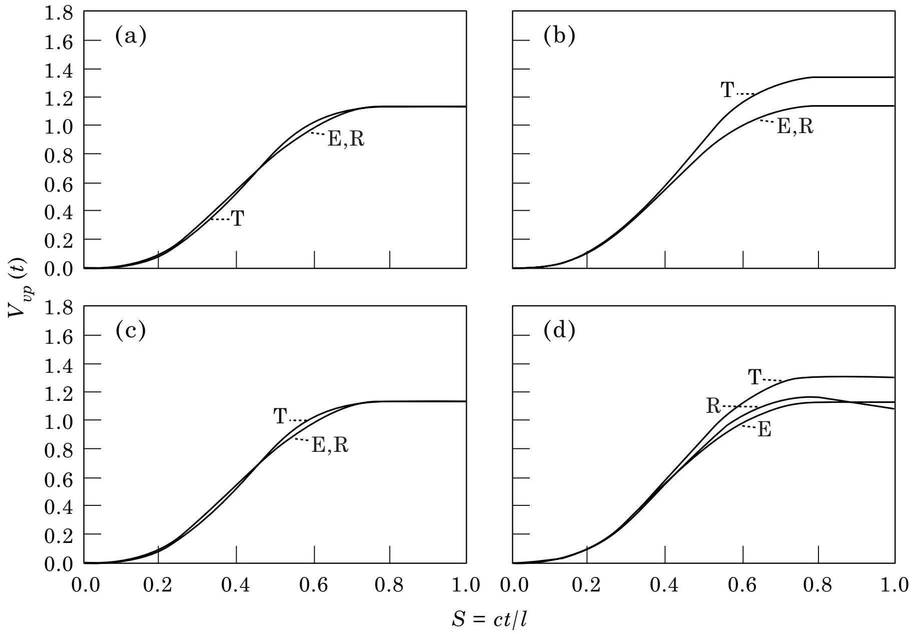

Figure 1. Random dynamic effect versus time, $\alpha = 0{\cdot}5$, white noise; T: Timoshenko, R: Rayleigh, E: Euler–Bernoulli model. (a) $\bar{\Omega} = 0{\cdot}0$, $\beta = 0{\cdot}075$; (b) $\bar{\Omega} = 0{\cdot}0$, $\beta = 0{\cdot}2$; (c) $\bar{\Omega} = 2{\cdot}5$, $\beta = 0{\cdot}075$; (d) $\bar{\Omega} = 2{\cdot}5$, $\beta = 0{\cdot}2$.

For the case of constant covariance, the expression $V_{vp}(t)$ is written as

$$V_{vp}(t) = \sum_{n=1}^{\infty}\left(\frac{-96\mathrm{i}\theta_n}{l^4}\right)\left(\frac{EI}{\rho A}\right)v_n(l/2)\left\{\frac{A_1}{\omega_{n1}}(\mathrm{e}^{\mathrm{i}\omega_{n_1}t} - 1) + \frac{A_2}{\omega_{n2}}(\mathrm{e}^{\mathrm{i}\omega_{n_2}t} - 1)\right.$$
$$\left. + \frac{A_3}{\omega_{n3}}(\mathrm{e}^{\mathrm{i}\omega_{n3}t} - 1) + \frac{A_4}{\omega_{n4}}(\mathrm{e}^{\mathrm{i}\theta_{n4}t} - 1) + \frac{A_5}{\theta_n}(\mathrm{e}^{\mathrm{i}\theta_n t} - 1) + \frac{A_6}{\theta_n}(1 - \mathrm{e}^{-\mathrm{i}\theta_n t})\right\} \quad (59)$$

3. NUMERICAL EXAMPLES AND DISCUSSION

In order to study the nature of the random response of a rotating shaft subje-t to a random load moving with onstant velocity, the following parameters are defined: $\alpha = c/c_{cr}$ is the load speed parameter, where c_{cr} is the critical speed of the load and corresponds physically to the case when the frequency of the load (θ) is equal to the fundamental frequency of a simply supported Euler–Bernoulli beam ω_{1EB}, $\bar{\Omega} = \Omega/\omega_1$ is the non-dimensional rotational speed of the beam, x/l is the non-dimensional position along the shaft, $\beta = \pi r_o/l$ is the Rayleigh beam coefficient, which for a circular cross-section relates the diameter of the beam to its length and $S = ct/l$ is the dimensionless time. In the numerical study the load speed parameter has been considered for two different values namely: (1) $\alpha = 0{\cdot}5$,

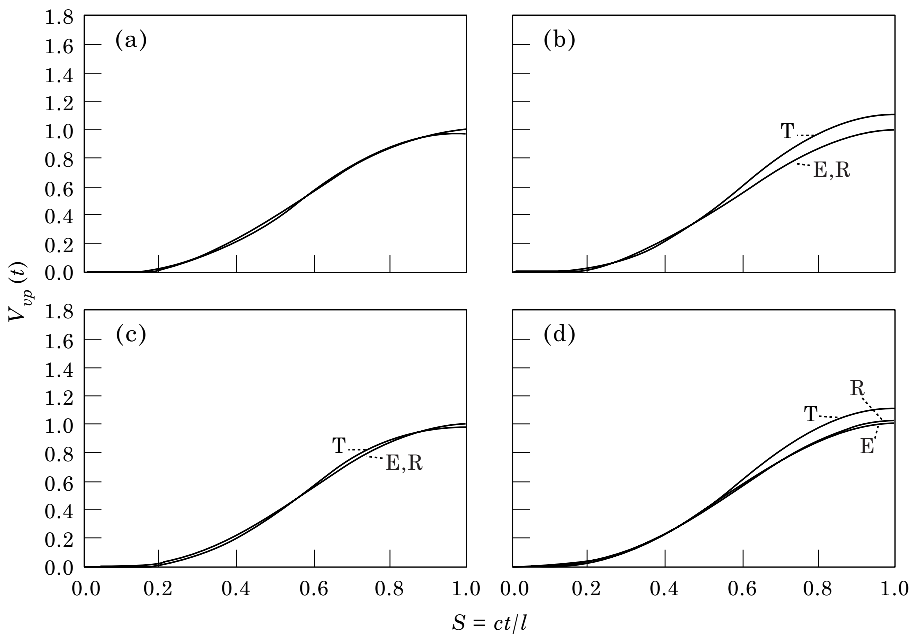

Figure 2. Random dynamic effect versus time, $\alpha = 1{\cdot}0$, white noise; T: Timoshenko, R: Rayleigh, E: Euler–Bernoulli model. (a) $\bar{\Omega} = 0{\cdot}0$, $\beta = 0{\cdot}075$; (b) $\bar{\Omega} = 0{\cdot}0$, $\beta = 0{\cdot}2$; (c) $\bar{\Omega} = 2{\cdot}5$, $\beta = 0{\cdot}075$; (d) $\bar{\Omega} = 2{\cdot}5$, $\beta = 0{\cdot}2$.

the load velocity is equal to half the critical speed c_{cr}; (2) $\alpha = 1{\cdot}0$, the load speed and critical speed are equal.

Two values for the rotational speed parameter are studied, $\bar{\Omega} = 0{\cdot}0$ and 2·5. The Rayleigh beam coefficient is studied for two values namely $\beta = 0{\cdot}075$ and 0·2. To clarify the analysis, the time variation of the functions $V_{vp}(t)$, corresponding to the coefficient of variation of the deflection at the mid-span of the beam for the three models, namely Euler–Bernoulli, Rayleigh and Timoshenko beam models are plotted on the same figure for comparison between the three models. Different combinations of the two types of covariances, namely, white noise and constant covariance are studied for the three beam models and the effect of the parameter β is considered. Also the effect of rotational speed of the beam as well as the relation between the speed parameter α and the maximum values of the function $V_{vp}(t)$ are studied.

Figures 1 and 2 show the relation between the function $V_{vp}(t)$ and the dimensionless time S for the case of white noise. Figure 1(a) shows the function $V_{vp}(t)$ for $\alpha = 0{\cdot}5$, and the non-rotating beam $\bar{\Omega} = 0{\cdot}0$ where the three models are identical in their behavior, especially for the Euler–Bernoulli and Rayleigh models. Increasing the Rayleigh beam coefficient β, Figure 1(b) shows an increase in the value of the $V_{vp}(t)$ for the Timoshenko model while the two other models remain unaffected. Figures 1(c) and (d) show the same relation but for $\bar{\Omega} = 2{\cdot}5$ where there is a large effect on the Rayleigh beam especially for $\beta = 0{\cdot}2$.

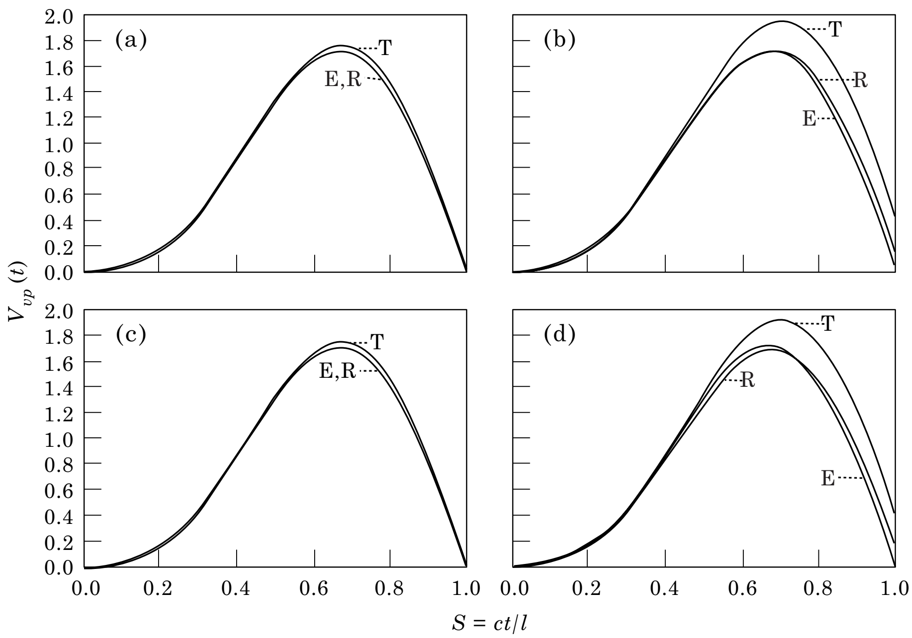

Figure 3. Random dynamic effect versus time, $\alpha = 0{\cdot}5$, constant covariance: T: Timoshenko, R: Rayleigh, E: Euler–Bernoulli model. (a) $\bar{\Omega} = 0{\cdot}0$, $\beta = 0{\cdot}075$; (b) $\bar{\Omega} = 0{\cdot}0$, $\beta = 0{\cdot}2$; (c) $\bar{\Omega} = 2{\cdot}5$, $\beta = 0{\cdot}075$; (d) $\bar{\Omega} = 2{\cdot}5$, $\beta = 0{\cdot}2$.

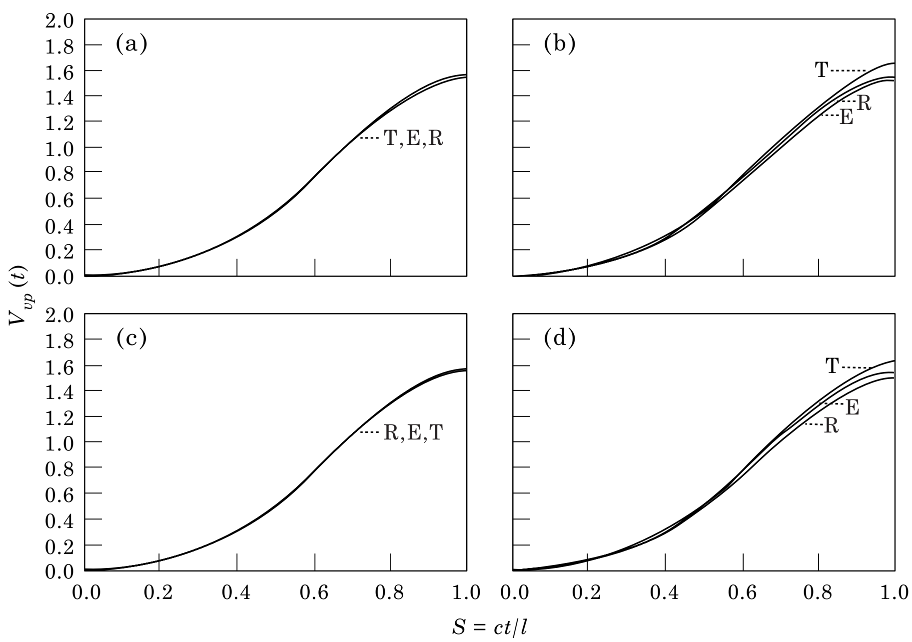

Figure 4. Random dynamic effect versus time, $\alpha = 1{\cdot}0$, constant covariance; T: Timoshenko, R: Rayleigh, E: Euler–Bernoulli model. (a) $\bar{\Omega} = 0{\cdot}0$, $\beta = 0{\cdot}075$; (b) $\bar{\Omega} = 0{\cdot}0$, $\beta = 0{\cdot}2$; (c) $\bar{\Omega} = 2{\cdot}5$, $\beta = 0{\cdot}075$; (d) $\bar{\Omega} = 2{\cdot}5$, $\beta = 0{\cdot}2$.

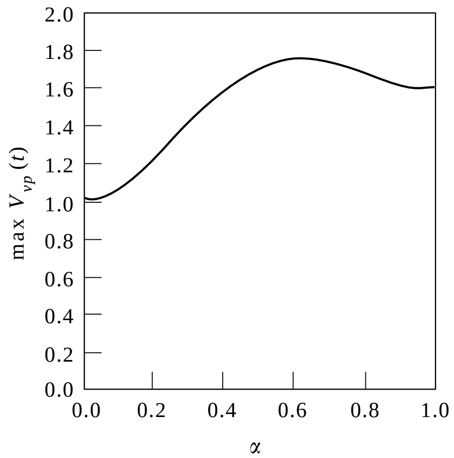

Figure 5. Maxima of $V_{vp}(t)$ as a function of the speed parameter α, Euler–Bernoulli model; constant covariance.

Figure 2 shows the case of Figure 1 but for $\alpha = 1{\cdot}0$. It is noticed that the dynamic effect is decreased with increasing α and the variations among different beam models are less pronounced at large values of α. It is also observed that the dynamic effect is more sensitive to $\bar{\Omega}$ at smaller values of α. Similar behavior is observed in Figures 3 and 4 in which the relation between the function $V_{vp}(t)$ and the dimensionless time for the case of constant covarince is shown. The behavior of the maximum value of the function $V_{vp}(t)$, max $V_{vp}(t)$, is shown in Figures 5–10. It is shown as a function of the speed parameter α for different Rayleigh beam coefficients. Figure 5 shows the max $V_{vp}(t)$ for the Euler–Bernoulli model and constant covariance case. It shows that the maximum variance of the beam first increases for values of the dimensionless speed parameter up to about $\alpha = 0{\cdot}5$ to $0{\cdot}7$ and then slightly decreases. This figure also shows as expected that the rotational speed and the Rayleigh beam coefficient

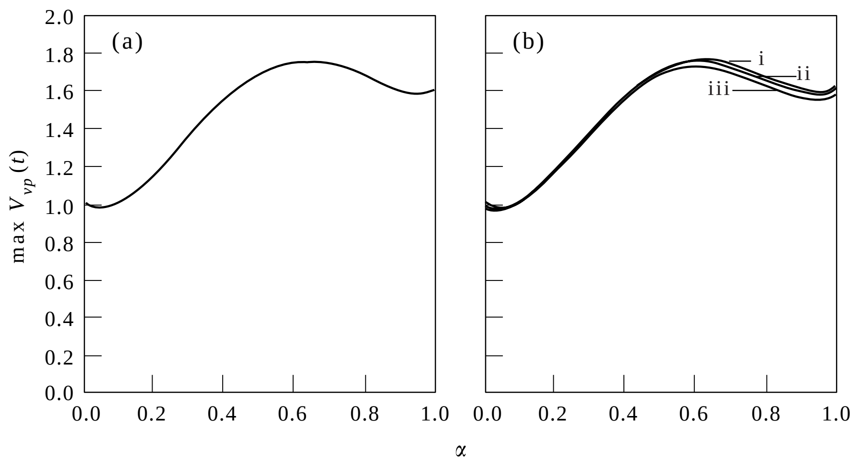

Figure 6. Maxima of $V_{vp}(t)$ as a function of the speed parameter α, Rayleigh model; constant covariance. (a) $\bar{\Omega} = 0{\cdot}0$; (b) $\bar{\Omega} = 2{\cdot}5$. (i) $\beta = 0{\cdot}2$; (ii) $\beta = 0{\cdot}15$; (iii) $\beta = 0{\cdot}075$.

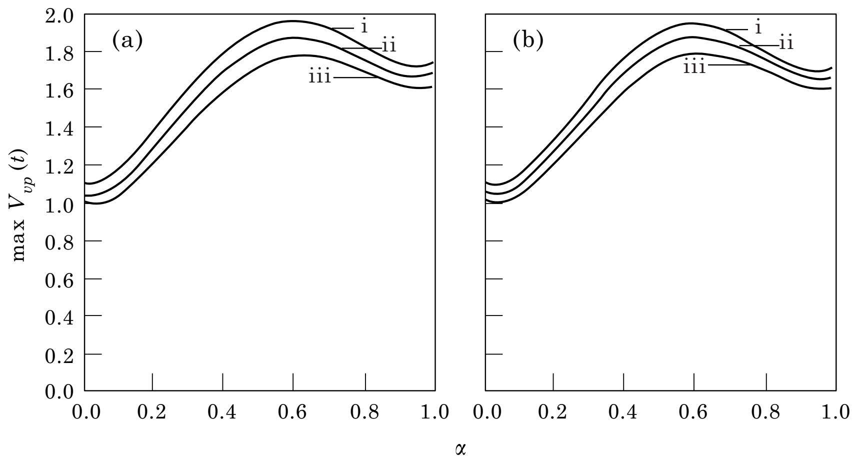

Figure 7. Maxima of $V_{vp}(t)$ as a function of the speed parameter α, Timoshenko model; constant covariance. (a) $\bar{\Omega}=0{\cdot}0$; (b) $\bar{\Omega}=2{\cdot}5$. (i) $\beta=0{\cdot}2$; (ii) $\beta=0{\cdot}15$; (iii) $\beta=0{\cdot}075$.

have no effect on the maximum value of variance. The max $V_{vp}(t)$ for the Rayleigh beam model $\bar{\Omega}=0{\cdot}0$, is shown in Figure 6(a). It is clear that the effect of β on the maximum of $V_{vp}(t)$ is negligible and it is the same as the Euler–Bernoulli case shown in Figure 5. Figure 6(b) shows the effect of β on the max $V_{vp}(t)$ for the Rayleigh beam model and $\bar{\Omega}=2{\cdot}5$. It is clear that the max $V_{vp}(t)$ increases when β increases. The effect of β on the max $V_{vp}(t)$ is more pronounced for the Timoshenko beam model than the other two models, as shown in Figures 7(a) and (b); however, the max $V_{vp}(t)$ is not sensitive to changes in $\bar{\Omega}$ in the case of the Timoshenko beam model. The maximum value of the variance for the white noise case is shown in Figures 8–10 for the three beam models and for different values of β. Different general behavior is observed in this case than in the case of constant covariance. The case of the Euler–

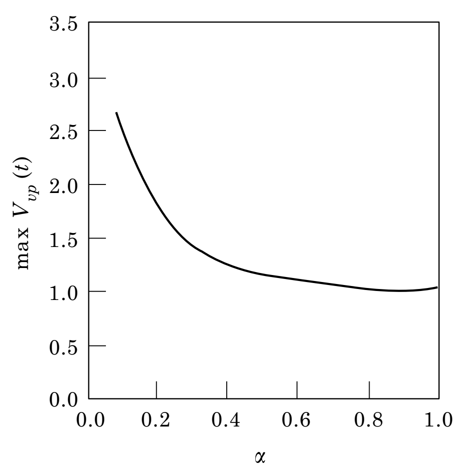

Figure 8. Maxima of $V_{vp}(t)$ as a function of the speed parameter α, Euler–Bernoulli model; white noise.

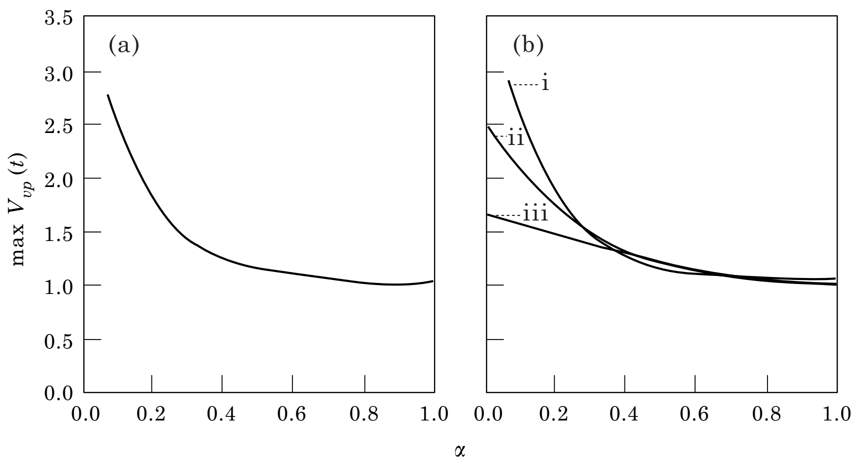

Figure 9. Maxima of $V_{vp}(t)$ as a function of the speed parameter α, Rayleigh model; white noise. (a) $\bar{\Omega}=0{\cdot}0$; (b) $\bar{\Omega}=2{\cdot}5$. (i) $\beta=0{\cdot}2$; (ii) $\beta=0{\cdot}15$; (iii) $\beta=0{\cdot}075$.

Bernoulli model is shown in Figure 8. It is clear that the variance decreases as α increases; β and $\bar{\Omega}$ have no effect on this model. Figure 9(a) shows the max $V_{vp}(t)$ for the Rayleigh model in the case of $\bar{\Omega}=0{\cdot}0$. Similarly, Figure 9(b) shows the case of Figure 9(a) but for $\bar{\Omega}=2{\cdot}5$. It is noticed that the max $V_{vp}(t)$ is more sensitive to β at smaller values of α, up to $\alpha=0{\cdot}26$. This sensitivity diminishes as α increases. Figures 10(a) and (b) show the maximum value of the response for the white noise case and for the Timoshenko beam model. Figure 10(a) shows the case of $\bar{\Omega}=0{\cdot}0$. The effect of β is clear as also shown in the constant covariance case, Figure 7(a). Figure 10(b) shows the case of $\bar{\Omega}=2{\cdot}5$. Again, this model is not sensitive to changes in $\bar{\Omega}$ as in the case of the Rayleigh beam model.

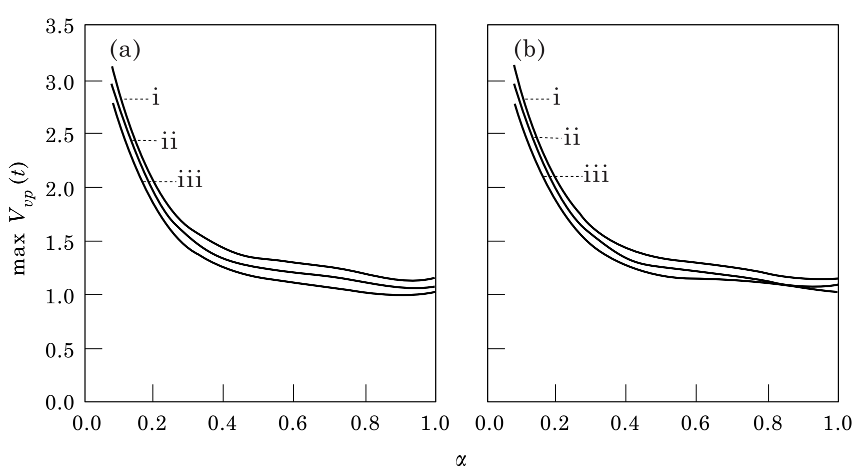

Figure 10. Maxima of $V_{vp}(t)$ as a function of the speed parameter α, Rayleigh model; white noise. (a) $\bar{\Omega}=0{\cdot}0$; (b) $\bar{\Omega}=2{\cdot}5$. (i) $\beta=0{\cdot}2$; (ii) $\beta=0{\cdot}15$; (iii) $\beta=0{\cdot}075$.

4. CONCLUSIONS

In conclusion, a rotating simply supported beam subjected to a random load moving with constant velocity is modelled using three different beam theories, namely Euler–Bernoulli, Rayleigh, and Timoshenko beam models. Analytical closed forms of the variance are obtained. The variance of the deflection at the mid-span point of the beam is calculated for different cases. The results indicate that the Timoshenko model has the largest dynamic effect among the three models considered. The dynamic effect is more sensitive to the chosen model at large β and as the force gets towards the end of the beam. The results show that the dynamic effect is larger at smaller values of α.

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APPENDIX A

The complex coefficients in equation (57):

$$A_1 = -\mathrm{i}(\omega_{n1}^2 - 2\Omega\omega_{n1} - b^2)/[(\omega_{n1}-\omega_{n2})(\omega_{n1}-\omega_{n3})(\omega_{n1}-\omega_{n4})(-\omega_{n1}^2+\theta_n^2)],$$

$$A_2 = -\mathrm{i}(\omega_{n2}^2 - 2\Omega\omega_{n2} - b^2)/[(\omega_{n2}-\omega_{n1})(\omega_{n2}-\omega_{n3})(\omega_{n2}-\omega_{n4})(-\omega_{n2}^2+\theta_n^2)],$$

$$A_3 = -\mathrm{i}(\omega_{n3}^2 - 2\Omega\omega_{n3} - b^2)/[(\omega_{n3}-\omega_{n1})(\omega_{n3}-\omega_{n2})(\omega_{n3}-\omega_{n4})(-\omega_{n3}^2+\theta_n^2)],$$

$$A_4 = -\mathrm{i}(\omega_{n4}^2 - 2\Omega\omega_{n4} - b^2)/[(\omega_{n4}-\omega_{n1})(\omega_{n4}-\omega_{n2})(\omega_{n4}-\omega_{n3})(-\omega_{n1}^2+\theta_n^2)],$$

$$A_5 = -\mathrm{i}(-\theta_n^2 + 2\Omega\theta_n + b^2)/[(2\theta_n(\theta_n-\omega_{n1})(\theta_n-\omega_{n2})(\theta_n-\omega_{n3})(\theta_n-\omega_{n4})],$$

$$A_6 = -\mathrm{i}(\theta_n^2 + 2\Omega\theta_n - b^2)/[2\theta_n(\theta_n+\omega_{n1})(\theta_n+\omega_{n2})(\theta_n+\omega_{n3})(\theta_n+\omega_{n4})].$$

APPENDIX B

The complex coefficients in equation (58)

$$B_1 = A_1^2, \qquad B_{11} = 2A_4 A_5,$$
$$B_2 = A_2^2, \qquad B_{12} = 2A_4 A_6,$$
$$B_3 = A_3^2, \qquad B_{13} = 2A_1 A_4,$$
$$B_4 = 2A_2 A_3, \qquad B_{14} = 2A_1 A_5,$$
$$B_5 = 2A_1 A_2, \qquad B_{15} = 2A_1 A_6,$$
$$B_6 = 2A_3 A_1, \qquad B_{16} = 2A_2 A_4,$$
$$B_7 = A_4^2, \qquad B_{17} = 2A_2 A_5,$$
$$B_8 = A_5^2, \qquad B_{18} = 2A_2 A_6,$$
$$B_9 = A_6^2, \qquad B_{19} = 2A_3 A_4,$$
$$B_{10} = 2A_5 A_6, \qquad B_{20} = 2A_3 A_5,$$
$$B_{21} = 2A_3 A_6.$$

APPENDIX C

The frequency terms that appear in equations (54–59)

$$\omega_1 = 2\omega_{n1}, \qquad \omega_{11} = \omega_{n4} + \theta_n,$$
$$\omega_2 = 2\omega_{n2}, \qquad \omega_{12} = \omega_{n4} - \theta_n,$$
$$\omega_3 = 2\omega_{n3}, \qquad \omega_{13} = \omega_{n1} + \omega_{n4},$$
$$\omega_4 = \omega_{n2} + \omega_{n3}, \qquad \omega_{14} = \omega_{n1} + \theta_n,$$
$$\omega_5 = \omega_{n1} + \omega_{n2}, \qquad \omega_{15} = \omega_{n1} - \theta_n,$$
$$\omega_6 = \omega_{n1} + \omega_{n3}, \qquad \omega_{16} = \omega_{n2} + \omega_{n4},$$
$$\omega_7 = \omega_{n4}, \qquad \omega_{17} = \omega_{n2} + \theta_n,$$
$$\omega_8 = 2\theta_n, \qquad w_{18} = w_{n2} - \theta_n,$$
$$\omega_9 = -2\theta_n, \qquad \omega_{19} = \omega_{n3} + \omega_{n4},$$
$$\omega_{10} = 0{\cdot}0, \qquad \omega_{20} = \omega_{n3} + \theta_n,$$
$$\omega_{21} = \omega_{n3} - \theta_n.$$