Journal of Sound and Vibration (1999) **223**(5), 759–774
Article No. jsvi.1998.2128, available online at http://www.idealibrary.com on IDEAL®

A BEST MATRIX APPROXIMATION METHOD FOR UPDATING THE ANALYTICAL MODEL

ZHANG XIANMIN
Department of Mechatronic Engineering, *Shantou University*, *Shantou*, *Guangdong* 515063, *People's Republic of China*

(*Received* 29 *August* 1997, *and in final form* 11 *December* 1998)

A best matrix approximation technique for updating the analytical model is developed using the known modal parameters. Firstly, the known modal matrix is decomposed by means of the singular-value decomposition technique. Secondly, the general updating equations for the analytical model are obtained on the basis of the singular-value decomposition results, the eigenequation, and the modal orthogonality relations. Thirdly, the best updating solution is defined according to the best approximation theory, and the existence and uniqueness of the best modification results relative to the analytical model are studied. Finally, the concrete form of the best modification and two updating algorithms are presented. Numerical examples demonstrate that the proposed method possesses high modificatory accuracy compared with some other methods, and it possesses the ability to modify larger error models.

1. INTRODUCTION

The finite element technique has been widely used to perform static and dynamic analysis in various fields for a long time. The accuracy of the analysis result depends heavily on the knowledge and experience of setting up an accurate model as well as the limitations of individual finite element code. In recent decades, dynamic characteristics of mechanical systems such as natural frequencies and mode shapes can be estimated accurately by modal testing [1]. Ideally, the modal parameters obtained from the analytical model and modal testing should be reasonably close to each, provided that the modal testing is performed carefully. However, most modal data obtained by the analytical model do not agree with those measured by modal testing. Therefore, modification of the original analytical model is necessary in order to obtain the most accurate and reliable model.

In the past, various methods have been proposed to minimize the difference between the analytical and the testing data. Baruch and Itzhack [2] assumed that the mass matrix is correct and introduced a constrained minimization procedure to modify the stiffness and flexibility matrices. Berman [3] introduced a formulation that modifies the mass matrix and assumed that the measured modes are exact. Subsequently, Berman and Nagy [4] combined the mass matrix

0022–460X/99/250759 + 16 $30.00/0

adjustment procedure of reference [3] with the stiffness matrix adjustment procedure of reference [2] to establish the so-called analytical model improvement (AMI) procedure. Ross [5] introduced a procedure for deriving both the mass and stiffness matrices from measured natural frequencies and a square modal matrix composed of measured mode vectors supplemented by arbitrary linearly independent vectors. A similar concept was developed by Zak [6] who supplements the measured mode data with information from analytically predicated modes. In addition, identification procedures based on matrix perturbation theory have been proposed by Chen and Garba [7], and Chen *et al.* [8]. Kabe [9] introduced a procedure that uses the mode data and structural connectivity information to optimally adjust deficient stiffness matrices. The adjustments performed are such that the percentage change to each stiffness coefficient is minimized. In reference [10], Caesar gave an overview on the methods used for the generation of system matrices fitted to test results by updating and direct identification of dynamic mathematical models. A more general mathematical formulation of the linear least squares problem was discussed. Wei and Zhang [11] introduced an analytical mass matrix modification procedure using the element correction method. This method preserves the original characteristics of the mass matrix. Wei [12] presented a dynamic model improvement method based on the Lagrange multiplier method. The dynamic equation and the orthogonality constraints are satisfied during the analytical derivation. Zhang *et al.* [13] developed a best matrix approximation technique to modify the analytical model, in which the interaction effects of mass stiffness matrices are taken into account, but the weight matrix is not included. Guo and Hemingway [14] introduced an orthogonality sensitivity method for analytical dynamic model correction using modal test data in the same reference. The method is valid for model correction in conjunction with the reduction method. Chen *et al.* [15] presented a two-stage method of updating the finite element model, in which the local physical parameters and the structural joint parameters are corrected using the dynamic test data.

In this paper, a new analytical model updating method is developed on the basis of the theory of the singular-value decomposition and the matrix approximation technique. In this method; the identified modal matrix is decomposed by means of the singular-value decomposition technique. The general updating equations for the analytical model are obtained according to the decomposition results, the eigenequation of the system, as well as the modal orthogonality relations. The existence and uniqueness of the best approximation relative to the analytical model are studied, and the concrete form of the best modification and two algorithms are presented. Examples demonstrate that the new model modification method possesses high modificatory accuracy compared with some other methods, and it possesses the ability to modify larger error models.

2. UPDATING THEORY

Let $R^{n\times m}$ represent the set of $n\times m$ real matrix, $SR^{n\times n}$ represent the set of $n\times n$ symmetric real matrix, and I_m represent the $m\times m$ identity matrix. $[K_0]\in SR^{n\times n}$, $[M_0]\in SR^{n\times n}$ represent the mass and stiffness matrices of the analytical model, $[K]\in SR^{n\times n}$, $[M]\in SR^{n\times n}$ represent the modified matrices corresponding to matrices $[M_0]$, $[K_0]$, and $[\Phi]\in R^{n\times m}$, $[\Lambda]\in SR^{m\times m}$ represent the tested eigenvector matrix and eigenvalue matrix. The term m is the number of the tested modes, and n is the dimension of the system. The system should satisfy the following eigenequation and orthogonality relations

$$[K][\Phi] = [M][\Phi][\Lambda], \tag{1}$$

$$[\Phi]^{\mathrm{T}}[M][\Phi] = I_m, \tag{2}$$

$$[\Phi]^{\mathrm{T}}[K][\Phi] = [\Lambda]. \tag{3}$$

As the eigenvectors are linear independent, so the matrix $[\Phi]$ is full column ranked, i.e., rank $[\Phi]=m$. From the singular-value decomposition technique [16], the eigenvector matrix $[\Phi]$ can be expressed as:

$$[\Phi] = [U]\begin{bmatrix} D \\ 0 \end{bmatrix}[V]^T, \tag{4}$$

where matrices $[U]$ and $[V]$ are $n\times n$ and $m\times m$ orthogonal matrices, $D=\mathrm{diag}[\alpha_1, \alpha_2, \ldots, \alpha_m]$, and α is the positive singular-value.

Substituting equation (4) into equation (1), and premultiplying matrix $]U]^{\mathrm{T}}$ yields:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} D \\ 0 \end{bmatrix}[V]^{\mathrm{T}} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} D \\ 0 \end{bmatrix}[V]^{\mathrm{T}}[\Lambda], \tag{5}$$

where

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = [U]^{\mathrm{T}}[K][U], \tag{6}$$

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = [U]^{\mathrm{T}}[M][U]. \tag{7}$$

From equation (5) one can obtain:

$$[K_{11}][D][V]^{\mathrm{T}} = [M_{11}][D][V]^{\mathrm{T}}[\Lambda], \tag{8}$$

$$[K_{21}][D][V]^{\mathrm{T}} = [M_{21}][D][V]^{\mathrm{T}}[\Lambda]. \tag{9}$$

Substituting equation (4) into equation (1), and considering equations (6) and (7) yields:

$$[M_{11}] = [D]^{-\mathrm{T}}[D]^{-1}. \tag{10}$$

From equations (8) and (9) one can obtain:

$$[K_{11}] = [D]^{-\mathrm{T}}[V]^{\mathrm{T}}[\Lambda][V]^{-\mathrm{T}}[D]^{-1}, \tag{11}$$

$$[K_{21}] = [M_{21}][D][V]^{\mathrm{T}}[\Lambda][V]^{-\mathrm{T}}[D]^{-1}, \tag{12}$$

By substituting equation (4) into equation (3), one can also obtain equation (10), which indicates that matrices $[M]$ and $[K]$ obtained from equations (1) and (2) will satisfy equation (3) automatically. From the above process, the following conclusion is obtained.

Conclusion 1. Given eigenvector matrix $[\Phi]$ and eigenvalue matrix $[\Lambda]$, using the singular-value decomposition relation (4), one can obtain the solution set of equations (1)–(3) $G([M], [K])$, in which $[M]$ and $[K]$ can be expressed as:

$$[K] = [U]\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}[U]^{\mathrm{T}}, \tag{13}$$

$$[M] = [U]\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}[U]^{\mathrm{T}}, \tag{14}$$

where the submatrices $[M_{11}]$, $[K_{11}]$, $[K_{21}]$ are determined by equations (10), (11) and (12).

Now, the best modification result which satisfies the following definition will be discussed.

Definition. Let $[M_0]$, $[K_0]$ be the mass matrix and the stiffness matrix of the analytical model, then matrices $([\hat{M}], [\hat{K}]) \in G$ are said to be the best approximation updated matrices of $[M_0]$ and $[K_0]$ if $([\hat{M}], [\hat{K}])$ satisfy the relation:

$$\begin{aligned} &\|N(\hat{K} - K_0)N\|_F + \|N(\hat{M} - M_0)N\|_F \\ &\quad = \inf_{([M],\,[K])\in G}(\|N(K - K_0)N\|_F + \|N(M - M_0)N\|_F), \end{aligned} \tag{15}$$

i.e., $([\hat{M}], [\hat{K}])$ are the best approximation matrices of $[M_0]$, $[K_0]$ in set G, where $\|\bullet\|_F$ indicates the Frobenius norm [16] and $[N]$ is the weight matrix. According to the problem, the weight matrix can be the identity matrix, diagonal matrix, or $[M_0]^{-1/2}$ and $[K_0]^{-1/2}$.

Theorem. Given the mass matrix and the stiffness matrix of the analytical model $[M_0]$ and $[K_0]$, then the best updated matrices $([\hat{M}], [\hat{K}])$ that satisfy the above definition in set G exist uniquely.

Proof. Since set G is a solution set of equations (1–3), then set G is a linear subspace of $SR^{n\times n} \times SR^{n\times n}$. Because set G is a convex set, and G is a finite dimension set, set G is compact. According to the projection theorem (see Appendix) of the inner product spaces [17], it can be concluded that there exists a unique best updated $([\hat{M}, [\hat{K}])$ for $[M_0]$ and $[K_0]$ in set G.

The above theorem indicates that the best update satisfying the definition for the analytical model exists uniquely.

From equations (6), (7) and (15), one can obtain:

$$
\begin{aligned}
f &= \| N(K - K_0)N^2 \|_F^2 + \| N(M - M_0)N \|_F^2 \\
&= \left\| N\left(U \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} U^{\mathrm{T}} - [K_0] \right) N \right\|_F^2 \\
&+ \left\| N\left(U \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} U^{\mathrm{T}} - [M_0] \right) N \right\|_F^2 .
\end{aligned} \tag{16}
$$

Expressing the orthogonal matrix $[U]$ in the blocked form:

$$
[U] = [U_1 : U_2] \tag{17}
$$

Substituting equation (17) into equation (16) yields:

$$
\begin{aligned}
f = \mathrm{tr}(&[N][U_1][K_{11}][U_1]^{\mathrm{T}}[N]^2[U_1][K_{11}][U_1]^{\mathrm{T}}[N] \\
&+ 4[N][U_1][K_{11}][U_1]^{\mathrm{T}}[N]^2[U_2][M_{21}][D_1][U_1]^{\mathrm{T}}[N] \\
&+ 2[N][U_1][K_{11}][U_1]^{\mathrm{T}}[N]^2[U_2][K_{22}][U_2]^{\mathrm{T}}[N] \\
&- 2[N][U_1][K_{11}][U_1]^{\mathrm{T}}[N]^2[K_0][N] \\
&+ 4[N][U_1][D_1]^{\mathrm{T}}[M_{21}]^{\mathrm{T}}[U_2]^{\mathrm{T}}[N]^2[U_2][M_{21}][D_1][U_1]^{\mathrm{T}}[N] \\
&+ 4[N][U_2][M_{21}][D_1][U_1]^{\mathrm{T}}[N]^2[U_2][K_{22}][U_2]^{\mathrm{T}}[N] \\
&- 4[N][U_2][M_{21}][D_1][U_1]^{\mathrm{T}}[N]^2[K_0][N] \\
&- 2[N][U_2][K_{22}][U_2]^{\mathrm{T}}[N]^2[K_0][N] \\
&+ [N][U_2][K_{22}][U_2]^{\mathrm{T}}[N]^2[U_2][K_{22}][U_2]^{\mathrm{T}}[N] \\
&+ [N][K_0][N]^2[K_0][N]) \\
&+ \mathrm{tr}([N][U_1][M_{11}][U_1]^{\mathrm{T}}[N]^2[U_1][M_{11}][U_1]^{\mathrm{T}}[N] \\
&+ 4[N][U_1][M_{21}]^{\mathrm{T}}[U_2]^{\mathrm{T}}[N]^2[U_2][M_{21}][U_1]^{\mathrm{T}}[N] \\
&+ [N][U_1][M_{21}]^{\mathrm{T}}[U_2]^{\mathrm{T}}[N]^2[U_2][M_{21}][U_1]^{\mathrm{T}}[N] \\
&+ [N][M_0][N]^2[M_0][N] \\
&+ 4[N][U_1][M_{11}]^{\mathrm{T}}[U_1]^{\mathrm{T}}[N]^2[U_2][M_{21}][U_1]^{\mathrm{T}}[N] \\
&+ 2[N][U_1][M_{11}]^{\mathrm{T}}[U_1]^{\mathrm{T}}[N]^2[U_2][M_{22}][U_2]^{\mathrm{T}}[N]
\end{aligned}
$$

$$- 2[N][U_1][M_{11}][U_1]^{\mathrm{T}}[N]^2[M_0][N]$$
$$+ 4[N][U_2][M_{21}]^{\mathrm{T}}[U_1]^{\mathrm{T}}[N]^2[U_2][M_{22}][U_2]^{\mathrm{T}}[N]$$
$$- 4[N][U_2][M_{21}]^{\mathrm{T}}[U_1]^{\mathrm{T}}[N]^2[M_0][N]$$
$$- 2[N][U_2][M_{22}][U_2]^{\mathrm{T}}[N]^2[M_0][N]), \tag{18}$$

where

$$[D_1] = [D][V]^{\mathrm{T}}[A][V]^{-\mathrm{T}}[D]^{-1} \tag{19}$$

Differentiate equation (18) with respect to matrices $[K_{22}]$, $[M_{22}]$, $[M_{21}]$ and let them equal zero. One can obtain:

$$[K_{22} = [A]^{-1}([U_2]^{\mathrm{T}}[N]^2[K_0][N]^2[U_2] - [C][K_{11}][C]^{\mathrm{T}}$$
$$- 2[C][D_1]^{\mathrm{T}}[M_{21}]^{\mathrm{T}}[A])[A]^{-1}, \tag{20}$$

$$[M_{22}] = [A]^{-1}([U_2]^{\mathrm{T}}[N]^2[M_0][N]^2[U_2] - [C][K_{11}][C]^{\mathrm{T}}$$
$$- 2[C][M_{21}]^{\mathrm{T}}[A])[A]^{-1}, \tag{21}$$

$$[M_{21}] = \tfrac{1}{2}[A]^{-1}([U_2]^{\mathrm{T}}[N]^2[M_0][N]^2[U_1] - [C][K_{11}][B][D_1]^{\mathrm{T}}$$
$$+ [U_2]^{\mathrm{T}}[N]^2[K_0][N]^2[U_1][D_1]^{\mathrm{T}} - [C][M_{11}][B]$$
$$- [U_2]^{\mathrm{T}}[N]^2[K_0][N]^2[U_2][A]^{-1}[C][D_1]^{\mathrm{T}}$$
$$- [U_2]^{\mathrm{T}}[N]^2[M_0][N]^2[U_2][A]^{-1}[C] + [C][M_{11}][C]^{\mathrm{T}}[A]^{-1}[C]$$
$$+ [C][K_{11}][C]^{\mathrm{T}}[A]^{-1}[C][D_1]^{\mathrm{T}})([B] + [D_1][B][D_1]^{\mathrm{T}}$$
$$- [D_1][C]^{\mathrm{T}}[A]^{-1}[C][D_1]^{\mathrm{T}} - [C]^{\mathrm{T}}[A]^{-1}[C])^{-1}, \tag{22}$$

where

$$[A] = [U_2]^{\mathrm{T}}[N]^2[U_2],$$
$$[B] = [U_1]^{\mathrm{T}}[N]^2[U_1],$$
$$[C] = [U_2]^{\mathrm{T}}[N]^2[U_1].$$

When the weight matrix is the identity matrix, equations (20), (21) and (22) can be simplified as:

$$[K_{22}] = [K_{022}] = [U_2]^{\mathrm{T}}[K_0][U_2], \tag{23}$$

$$[M_{22}] = [M_{022}] = [U_2]^{\mathrm{T}}[M_0][U_2], \tag{24}$$

$$[M_{21}] = \tfrac{1}{2}([U_2]^{\mathrm{T}}[M_0][U_1] + [U_2]^{\mathrm{T}}[K_0][U_1][D_1]^{\mathrm{T}})(I_m + [D_1][D_1]^{\mathrm{T}})^{-1}. \tag{25}$$

From the results, the following conclusion is obtained.

Conclusion 2. Given the mass matrix and the stiffness matrix of the analytical model $[M_0]$ and $[K_0]$, the best updated matrices ($[\hat{M}]$, $[\hat{K}]$) that satisfy the definition in set G exist uniquely. The updated results can be expressed as:

$$[\hat{M}] = [U]\begin{bmatrix} [D]^{\mathrm{T}}[D]^{-1} & \hat{M}_{21}^{\mathrm{T}} \\ \hat{M}_{21}^{\mathrm{T}} & \hat{M}_{22} \end{bmatrix}[U]^{\mathrm{T}}, \tag{26}$$

$$[\hat{K}] = [U]\begin{bmatrix} [D]^{-\mathrm{T}}[V]^{\mathrm{T}}[\Lambda][V]^{-\mathrm{T}}[D]^{-1} & D_1^{\mathrm{T}}\hat{M}_{21}^{\mathrm{T}} \\ \hat{M}_{21}^{\mathrm{T}}D_1 & \hat{K}_{22} \end{bmatrix}[U]^{\mathrm{T}}, \tag{27}$$

where $[\hat{K}_{22}]$, $[\hat{M}_{22}]$, $[\hat{M}_{21}]$ are calculated from equations (20), (21) and (22).

According to conclusions 1 and 2, the basic updating algorithm can be summarized as follows.

Algorithm 1

(1) Obtain the initial mass matrix and stiffness matrix of the analytical model using the finite element method etc.

(2) Obtain the modal parameters $[\Phi]$ and $[\Lambda]$ from mode tests.

(3) Decompose matrix $[\Phi]$ using the singular-value decomposition technique, and calculate the terms: $[U]^{\mathrm{T}}[K_0][U]$ and $[U]^{\mathrm{T}}[M_0][U]$.

(4) Calculate $[\hat{K}_{22}]$, $[\hat{M}_{22}]$, and $[\hat{M}_{21}]$ from equations (20), (21) and (22);

(5) Calculate $[\hat{M}]$ and $[\hat{K}]$ from equations (26) and (27).

From algorithm 1 it is known that the modification results $[\hat{M}]$ and $[\hat{K}]$ are full matrices. In order to make matrices $[\hat{M}]$ and $[\hat{K}]$ have the same band form as that of $[M_0]$ and $[K_0]$, an iterative algorithm is developed based on algorithm 1.

Algorithm 2

(1) Calculate $[\hat{M}]_p$ and $[\hat{K}]_p$ using algorithm 1 according to the given terms $[\Phi]$, $[\Lambda]$, $[M_0]$ and $[K_0]$, where p is the iterative number,

(2) Ignore the elements of the matrices $[\hat{M}]_p$ and $[\hat{K}]_p$ that do not belong to the band of the matrices $[M_0]$ and $[K_0]$. The new matrices are named $[M]_p$ and $[K]_p$.

(3) Calculate $[\Phi]^{\mathrm{T}}[M]_p[\Phi] = [m]_p$ and $[\Phi]^{\mathrm{T}}[K]_p[\Phi] = [k]_p$.

(4) Determine if the following condition is satisfied:

$$|m_{ij}| \leqslant \varepsilon_1,\ |k_{ij}| \leqslant \varepsilon_2,\ i, j = 1, 2, \ldots, m,\ i \neq j, \tag{28}$$

where ε_1 and ε_2 are two prescribed tolerances If the condition (28) has been satisfied, then the iteration is finished. Otherwise let $[M_0] = [M]_p$, $[K_0] = [K]_p$, and go back to step (1).

3. NUMERICAL EXAMPLES

Several numerical examples are presented to demonstrate the feasibility of the model updating method.

Example 1. A lumped mass matrix and a finite element stiffness matrix were developed as an analytical model. The two matrices are as follows:

$$[M_0] = \mathrm{diag}[2{\cdot}1,\ 1{\cdot}9,\ 1{\cdot}1,\ 0{\cdot}9],$$

$$[K_0] = \begin{bmatrix} 5{\cdot}1 & -4{\cdot}0 & 1{\cdot}0 & 0{\cdot}0 \\ -4{\cdot}0 & 6{\cdot}1 & -4{\cdot}0 & 1{\cdot}0 \\ 1{\cdot}0 & -4{\cdot}0 & 6{\cdot}1 & -4{\cdot}0 \\ 0{\cdot}0 & 1{\cdot}0 & -4{\cdot}0 & 5{\cdot}1 \end{bmatrix}.$$

The first two modal parameters are shown in Table 2. Table 1 gives the modification results of the Berman method [4] and algorithm 1 of the new method.

The relative errors of eigenvalues and eigenvectors are as follows [13]:

$$\text{relative error of eigenvalue} = \frac{|\omega_t - \omega|}{\omega_t} \times 100\%, \tag{29}$$

$$\text{relative error of eigenvector} = \left(\frac{\| \Phi_t - \Phi \|}{\| \Phi_t \|}\right)^{1/2} \times 100\%, \tag{30}$$

where ω_t and $\{\Phi_t\}$ are the experimental results, and ω and $\{\Phi\}$ are the analytical results.

Table 2 indicates that the modificatory accuracy of the presented method is significantly improved when compared with the Berman method. Substituting

TABLE 1

Modification results of the mass matrix and stiffness matrix

Model updating method	Results of mass matrix				Results of stiffness matrix			
Berman AMI method	2·0912	−0·0028	0·0044	0·0044	5·1679	−4·0488	0·9097	0·0668
	0·0028	1·8993	0·0016	0·0016	−4·0488	6·0373	−4·0276	1·0191
	0·0044	0·0016	1·0981	−0·0020	0·9097	−4·0276	6·1274	−4·0531
	0·0044	0·0016	−0·0020	0·8979	0·0668	1·0191	−4·0531	4·9712
The new method	2·0640	−0·0008	0·0358	−0·0011	5·0736	−4·0434	0·9842	0·0330
	−0·0008	1·9031	−0·0003	−0·0131	−4·0434	6·0383	−4·0275	1·0095
	0·0358	−0·0003	1·0613	0·0019	0·9842	−4·0275	6·0735	−4·0437
	−0·0011	−0·0131	0·0019	0·9587	0·0330	1·0095	−4·0437	5·0145

TABLE 2

Comparison of the modal parameters before and after modification

Modal parameters	Analytical model	Accurate value	Berman AMI method	The new method	Relative errors (%) Analytical model	AMI method	The new method
Eigenvalues 1	0·16193	0·09654	0·09650	0·09654	67·74	0·041	0·0
2	1·45368	1·39147	1·41657	1·39147	4·47	1·80	0·0
	−0·31916	−0·31263	−0·31217	−0·31263			
	−0·49771	−0·49548	−0·49536	−0·49548			
1	−0·47165	−0·47912	−0·47963	−0·47912	1·71	0·00013	0·0
	−0·28034	−0·28979	−0·29037	−0·28979			
Eigenvectors							
	0·44951	0·44527	0·44484	0·44527			
	0·10478	0·12444	0·12402	0·12444			
2	−0·50115	−0·48944	−0·48988	−0·48944	3·51	0·10	0·0
	−0·55632	−0·57702	−0·57752	−0·57702			

TABLE 3
Comparison of the updating results for larger model errors

Modal parameters		Analytical model	Accurate value	Berman AMI method	The new method	Relative errors (%)		
						Analytical model	AMI method	The new method
Eigenvalues	1	0·18024	0·09654	0·11142	0·09654	86·70	15·41	0·0
	2	1·36962	1·39147	1·79927	1·39147	1·57	29·31	0·0
		−0·27955	−0·31263	−0·33569	−0·31263			
		−0·46642	−0·49548	−0·50080	−0·49548			
	1	−0·49306	−0·47912	−0·45254	−0·47912	6·33	5·60	0·0
		−0·31215	−0·28979	−0·26169	−0·28979			
Eigenvectors								
		0·47281	0·44527	0·42505	0·44527			
		0·13762	0·12444	0·10261	0·12444			
	2	−0·42572	−0·48944	−0·51129	−0·48944	18·88	5·59	0·0
		−0·44640	−0·57702	−0·60086	−0·57702			

TABLE 4

Comparison of the modal parameters before and after modification

Modal parameters		Analytical model	Tested value	Algorithm 1 of the new method	Relative errors (%) Analytical model	Relative errors (%) Algorithm 1 of the new method
Eigenvalues (Hz)	1	9·19493	8·99104	8·99104	2·27	0·0
	2	25·20976	27·85248	27·85248	9·49	0·0
Eigenvectors	1	0·63745	0·62550	0·62550	3·12	0·0
		0·63408	0·62079	0·62079		
		0·61390	0·60206	0·60206		
		0·58722	0·57881	0·57881		
		0·52944	0·51779	0·51779		
		0·46604	0·44898	0·44898		
		0·39770	0·37679	0·37679		
		0·31251	0·29137	0·29137		
		0·21076	0·19498	0·19498		
		0·10566	0·09761	0·09761		
	2	−0·59230	−0·58377	−0·58377	16·13	0·0
		−0·56871	−0·54160	−0·54160		
		−0·43188	−0·38207	−0·38207		
		−0·26065	−0·19495	−0·19495		
		0·01439	0·11421	0·11421		
		0·28827	0·40688	0·40688		
		0·53920	0·67015	0·67015		
		0·61833	0·70753	0·70753		
		0·45122	0·48939	0·48939		
		0·23019	0·24767	0·24767		

the results from Table 1 into equation (18), one can obtain the Frobenius norm of the Berman method f_{BERMAN}, and the Frobenius norm of the new method f_{SVD} as follows:

$$f_{BERMAN} = 0{\cdot}2665033, \quad f_{SVD} = 0{\cdot}2521787. \tag{31}$$

From equation (31) it is known that $f_{BERMAN} > f_{SVD}$, which indicates that the modification result of the Berman method is not the best modification result for the analytical model.

In order to test the ability of modifying a larger error model for the new method of this paper, the errors of matrices $[M_0]$ and $[K_0]$ are amplified as follows:

TABLE 5

Modal parameters before and after modification of algorithm 2

Modal parameters		Analytical model	Tested value	Algorithm 2 of the new method	Relative errors (%) Analytical	Algorithm 2
Eigenvalues (Hz)	1	9·19493	8·99104	9·09650	2·27	1·17
	2	25·20976	27·85248	27·50197	9·49	1·26
Eigenvectors	1	0·63745	0·62550	0·62550	3·12	0·49
		0·63408	0·62079	0·62079		
		0·61390	0·60206	0·60206		
		0·58722	0·57881	0·57881		
		0·52944	0·51779	0·51779		
		0·46604	0·44898	0·44898		
		0·39770	0·37679	0·37679		
		0·31251	0·29137	0·29137		
		0·21076	0·19498	0·19498		
		0·10566	0·09761	0·09761		
	2	−0·59230	−0·58377	−0·58377	16·13	1·26
		−0·56871	−0·54160	−0·54160		
		−0·43188	−0·38207	−0·38207		
		−0·26065	−0·19495	−0·19495		
		0·01439	0·11421	0·11421		
		0·28827	0·40688	0·40688		
		0·53920	0·67015	0·67015		
		0·61833	0·70753	0·70753		
		0·45122	0·48939	0·48939		
		0·23019	0·24767	0·24767		

$$[M_0] = \mathrm{diag}[2{\cdot}4,\ 1{\cdot}7,\ 1{\cdot}5,\ 0{\cdot}8],$$

$$[K_0] = \begin{bmatrix} 5{\cdot}5 & -4{\cdot}2 & 1{\cdot}1 & 0{\cdot}0 \\ -4{\cdot}2 & 6{\cdot}7 & -4{\cdot}3 & 1{\cdot}0 \\ 1{\cdot}0 & -4{\cdot}3 & 6{\cdot}5 & -4{\cdot}4 \\ 0{\cdot}0 & 1{\cdot}0 & -4{\cdot}4 & 5{\cdot}6 \end{bmatrix}.$$

The two matrices are modified on the basis of the first two modal eigenvalues and eigenvectors listed in Table 2. The comparisons are shown in Table 3, which indicate that algorithm 1 of the new method reproduces the eigenvalues and eigenvectors accurately once again, and the Berman method possesses larger modification errors.

Example 2. The first two eigenvalues and eigenvectors of the system are tested and listed in Table 4. The mass matrix of the analytical model is as follows:

$$[M_0] = \mathrm{diag}[0{\cdot}1,\ 0{\cdot}5,\ 0{\cdot}2,\ 1{\cdot}0,\ 0{\cdot}2,\ 0{\cdot}2,\ 0{\cdot}8,\ 1{\cdot}0,\ 0{\cdot}3,\ 0{\cdot}1]$$

The stiffness matrix is:

$$[K_0] = \begin{bmatrix} 63\cdot0 & -63\cdot0 & & & & & & & & \\ -63\cdot0 & 126\cdot0 & -63\cdot0 & & & & & & & \\ & -63\cdot0 & 126\cdot0 & -63\cdot0 & & & & & & \\ & & -63\cdot0 & 126\cdot0 & -63\cdot0 & & & & & \\ & & & -63\cdot0 & 126\cdot0 & -63\cdot0 & & & & \\ & & & & -63\cdot0 & 126\cdot0 & -63\cdot0 & & & \\ & & & & & -63\cdot0 & 126\cdot0 & -63\cdot0 & & \\ & & & & & & -63\cdot0 & 126\cdot0 & -63\cdot0 & \\ & & & & & & & -63\cdot0 & 126\cdot0 & -63\cdot0 \\ & & & & & & & & -63\cdot0 & 126\cdot0 \end{bmatrix} \times 10^3$$

The analytical model is updated using algorithm 1 based on the first two tested modal eigenvalues and eigenvectors. The comparisons before and after modification are shown in Table 4.

The analytical model is modified using algorithm 2 at the same time. Its updating accuracy is shown in Table 5 which indicates that algorithm 2 can also improve the analytical model effectively. The updated matrices for the analytical model of algorithm 2 after 10 iterations are as follows:

$$[M]_{p=10} = \text{diag}[0\cdot16146,\ 0\cdot46041,\ 0\cdot14622,\ 1\cdot25895,\ 0\cdot27001,\ 0\cdot15175,\ 0\cdot71139,$$
$$0\cdot76281,\ 0\cdot12584,\ 0\cdot08504],$$

$$[K]_{p=10} = \begin{bmatrix} 63{\cdot}0848 & -62{\cdot}9571 & & & & & & & & \\ -62{\cdot}9571 & 126{\cdot}0379 & -62{\cdot}9813 & & & & & & & \\ & -62{\cdot}9813 & 126{\cdot}0048 & -63{\cdot}0117 & & & & & & \\ & & -63{\cdot}0117 & 125{\cdot}9790 & -63{\cdot}0350 & & & & & \\ & & & -63{\cdot}0350 & 125{\cdot}9704 & -63{\cdot}0240 & & & & \\ & & & & -63{\cdot}0240 & 125{\cdot}9991 & -62{\cdot}9793 & & & \\ & & & & & -62{\cdot}9793 & 126{\cdot}0564 & -62{\cdot}9340 & & \\ & & & & & & -62{\cdot}9340 & 126{\cdot}0756 & 62{\cdot}9475 & \\ & & & & & & & -62{\cdot}9475 & 126{\cdot}0368 & -26{\cdot}9815 \\ & & & & & & & & -62{\cdot}9815 & 126{\cdot}0095 \end{bmatrix} \times 10^3.$$

4. CONCLUSION

A new analytical model updating method is developed on the basis of the theory of the singular-value decomposition and the matrix approximation technique. In this method, the identified modal matrix is decomposed by means of the singular-value decomposition technique. The general updating equations for the analytical model are obtained according to the decomposition results, the eigenequation of the system, as well as the modal orthogonality relations. The existence and uniqueness of the best approximation relative to the analytical model are studied, and two model updating algorithms are presented. Examples demonstrate that the new model updating method is effective, and possesses the ability to modify larger error models.

ACKNOWLEDGMENTS

This research was supported by the National Natural Science Foundation of China under grant 59605001 and the Natural Science Foundation of Guangdong Province under grant 970381. Their support is gratefully acknowledged.

REFERENCES

1. P. J. EWINS 1984 *Modal Testing: Theory and Practice*. Letchworth, Hertfordshire, England: Research Studies Press Ltd.
2. M. BARUCH and I. Y. BAR-ITZHACK 1978 *AIAA Journal* **17**, 346–351. Optimal weighted orthogonalization of measured modes.
3. A. BERMAN 1979 *AIAA Journal* **17**, 1147–1148. Mass matrix correction using an incomplete set of measured modes.
4. A. BERMAN and E. J. NAGY 1983 *AIAA Journal* **21**, 1168–1173. Improvement of a large analytical model using test data.
5. R. G. ROSS, JR. 1971 *SEA Paper* 710787, 2627–2635. Synthesis of stiffness and mass matrices from experimental vibration modes.
6. M. ZAK 1983 *ASCE Journal of Engineering Mechanics* **109**, 1437–1444. Discrete model improvement by eigenvector updating.
7. C. CHEN and G. A. GARBA 1979 *AIAA Paper* 79-0831. A matrix perturbation for analytical model improvement.
8. J. C. CHEN, C. P. KUO and G. A. GARBA 1983 *AIAA Paper* 83-0812. Direct structural parameter identification by test results.
9. A. M. KABE 1985 *AIAA Journal* **23**, 1431–1436. Stiffness matrix adjustment using mode data.
10. B. CAESAR 1986 *Proceedings of the International Modal Analysis Conference, Los Angeles, CA*, 394–401. Update and identification of dynamic mathematical models.
11. F. S. WEI and D. W. ZHANG 1989 *AIAA Journal* **27**, 119–12l. Mass matrix modification using element correction method.
12. F. S. WEI 1990 *AIAA Journal* **28**, 1686–1688 Mass and stiffness interaction effects in analytical modl modification.
13. ZHANG XIANMIN, LIU HONGZHAO and CAO WEIQING 1994 *Journal of Vibration Engineering* **7**, 70–79. A best matrix approximation technique to update analytical model through identified model parameters (in Chinese).
14. S. GUO and N. G. HEMINGWAY 1995 *Journal of Sound and Vibration* **187**, 771–780. An orthogonality sensitivity method for analytical dynamic model correction using modal test data.

15. CHEN XIN *et al*. 1996 *Proceedings of ASME* 1996 *Design Engineering Technical Conference, Irving, CA*. Updating finite element modal of combined structures on the basis of dynamic test results.
16. P. LANCASTER and M. TISMENETSKY 1985 *The Theory of Matrices*. New York: Academic Press; second edition.
17. R. D. MILNE 1980 *Applied Functional Analysis*. Pitman Publishing Ltd.

APPENDIX:

Theorem (the projection theorem) [17]. Let W be a closed subspace of a Hilbert space H and let $X \in H$ be a vector not in W. Then there exists a unique vector $Y_0 \in W$ such that

$$d(X, W) = \inf_{Y \in W} \|X - Y\| = \|X - Y_0\| .$$

Proof. Choose a sequence of vectors $Y^{(k)} \in W$ such that

$$\lim_{k \to \infty} \|X - Y^{(k)}\| = d.$$

By the parallelogram law

$$\|Y^{(k)} - Y^{(j)}\|^2 = 2\|Y^{(k)} - X\|^2 + 2\|Y^{(j)} - X\|^2 - \|(Y^{(k)} - X) + (Y^{(j)} - X)\|^2$$

$$= 2\|Y^{(k)} - X\|^2 + 2\|Y^{(j)} - X\|^2 - 4\left\|\frac{Y^{(k)} + Y^{(j)}}{2} - X\right\|.$$

Since W is a linear subspace, $\frac{1}{2}(Y^{(k)} + Y^{(j)}) \in W$ for all k, j and from the meaning of d, it follows that

$$\left\|\frac{Y^{(k)} + Y^{(j)}}{2} - X\right\| \geqslant d,$$

giving

$$\|Y^{(k)} - Y^{(j)}\|^2 \leqslant 2\|Y^{(k)} - X\|^2 + 2\|Y^{(j)} - X\|^2 - 4d^2.$$

Since

$$\|Y^{(k)} - X\|^2 \to d^2$$

as $k \to \infty$, one may conclude that $\{Y^{(k)}\}$ is a Cauchy sequence in W. But W is a closed subspace of the complete space H. Hence, the sequence has a limit $Y_0 \in W$. By the continuity of the norm, $d = \|Y_0 - X\|$.

Suppose there is another vector $Z_0 \in W$ such that $d = \|Z_0 - X\|$. Then

$$\|Y_0 - Z_0\|^2 = 2\|Y_0 - X\|^2 + 2\|Z_0 - X\|^2 - \|Y_0 + Z_0 - 2X\|^2$$

$$= 4d^2 - 4\left\|\frac{Y_0 + Z_0}{2} - X\right\| \leqslant 0,$$

which implies that Y_0 is unique.