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LETTERS TO THE EDITOR

GENERALIZATION OF THE SENATOR–BAPAT METHOD TO SYSTEMS HAVING LIMIT CYCLES

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Many techniques exist for constructing analytical approximations to the solutions of oscillatory systems modelled by the equation

$$\ddot{x} + x = \varepsilon f(x, \dot{x}), \tag{1}$$

where ε is a small and positve parameter: the Lindstedt–Poincaré method [1], harmonic balancing [2], averaging techniques [3], and iteration procedures [4]. Recently, similar work has begun on systems that have large non-linearities, i.e., systems that do not have a linear limiting case. A particular example is the equation

$$\ddot{x} + x^3 = \mu f(x, \dot{x}). \tag{2}$$

For this case, even if μ is small, no standard perturbation procedure can be applied since $\mu = 0$ gives a non-linear differential equation. A first attempt to resolve this situation was provided by Mickens and Oyedeji [5]; they used a generalized form of the first approximation of Krylov and Bogoliubov [1, 3] to derive expressions for the time derivatives of the "averaged" amplitude and phase. This result was then extended by Yuste and Bejarano [6] to include the use of Jacobi elliptic functions [7]. The most recent results have been obtained by Senator and Bapat [8]. Their method, as presented in the paper [8], applies to equations of the form

$$\ddot{x} + g(x) = 0, \tag{3}$$

where $f(x)$ satisfies the condition

$$g(-x) = -g(x). \tag{4}$$

The purpose of this paper is to generalize the Senator–Bapat method to the case where limit cycles are possible. In particular, the following equation is considered:

$$\ddot{x} + x^3 = \mu(1 - x^2)\dot{x}, \tag{5}$$

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where μ is a small positive parameter. However, the method of this paper can also be applied to more general forms of equation (5),

$$\ddot{x} + g(x) = \mu \prod_{k=1}^{N} (a_k - x^2)\dot{x}, \tag{6}$$

where $\{a_k;\ k = 1, 2, \dots, N\}$ are positive parameters, $\mu > 0$, and the function $g(x)$ has the property given by equation (4).

Before proceeding, it should be indicated that equation (5) can be easily shown to have a unique and stable limit cycle, for $\mu > 0$, using standard results from the theory of differential equations. See section 2 of Appendix G in Mickens [1].

The basis of the generalized Senator–Bapat method is to rewrite equation (5) as

$$\ddot{x} + \phi x = \phi x - x^3 + \mu(1 - x^2)\dot{x}, \tag{7}$$

where ϕ is, for the moment, an unspecified positive constant. Next, a parameter ε is introduced, such that for $\varepsilon = 1$, the original equation (5) is obtained, i.e.,

$$\ddot{x} + \phi x = \varepsilon[\phi x - x^3 + \mu(1 - x^2)\dot{x}]. \tag{8}$$

At this point, the Lindstedt–Poincaré method is applied to equation (8). After calculating to the desired order in ε, the resulting expression for x is determined with ε put equal to one.

The following gives a summary of the calculations for equation (8). First, $x(t)$ is transformed to $x(\theta)$ where to order ε^2,

$$\theta = \omega t = [\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3)]t, \tag{9}$$

$$x(\theta) = x_0(\theta) + \varepsilon x_1(\theta) + \varepsilon^2 x_2(\theta) + O(\varepsilon^3), \tag{10}$$

and x is taken to be periodic with period 2π in the independent variable θ:

$$x(\theta + 2\pi) = x(\theta) \quad \text{or} \quad x_k(\theta + 2\pi) = x_k(\theta), \quad k = 0, 1, 2, \dots \tag{11}$$

Using the notation $(') = \mathrm{d}/\mathrm{d}\theta$ and the fact that

$$\mathrm{d}/\mathrm{d}t = \omega \mathrm{d}/\mathrm{d}\theta, \tag{12}$$

equation (8) becomes

$$\omega^2 z'' + \phi x = \varepsilon[\phi x - x^3 + \omega\mu(1 - x^2)x']. \tag{13}$$

Substituting equations (9) and (10) into equation (8), and setting the coefficients of the resulting expansion in ε to zero, the following relations are obtained:

$$\varepsilon^0 : \omega_0^2 x_0'' + \phi x_0 = 0, \tag{14}$$

$$\varepsilon : \omega_0^2 x_1'' + \phi x_1 = -2\omega_0\omega_1 x_0'' + \phi x_0 - x_0^3 + \omega_0\mu x_0' - \mu\omega_0 x_0^2 x_0', \tag{15}$$

$$\varepsilon^2 : \omega_0^2 x_2'' + \phi x_2 = -2\omega_0\omega_1 x_1'' - (\omega_1^2 + 2\omega_0\omega_2)x_0'' + \phi x_1 - 3x_0^2 x_1 + \omega_0\mu x_1' + \omega_1\mu x_0' - 2\omega_0 x_0 x_1 x_0' - \omega_0 x_0^2 x_1' - \omega_1 x_0^2 x_0'. \quad (16)$$

The initial conditions are taken to be

$$x(0) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + O(\varepsilon^3), \quad x'(0) = 0 + \varepsilon \cdot 0 + \varepsilon^2 \cdot 0 + O(\varepsilon^3), \quad (17a, b)$$

where (A_0, A_1, A_2) are, for the present, unknown constants. (See Mickens [1], p. 60, for the details as to why this particular set of initial conditions is required.) Thus, the initial conditions, respectively, for equations (14), (15), and (16) are

$$x_0(0) = A_0, \quad x_0'(0) = 0, \quad (18a)$$

$$x_1(0) = A_1, \quad x_1'(0) = 0, \quad (18b)$$

$$x_2(0) = A_2, \quad x_2'(0) = 0. \quad (18c)$$

The solution to equation (14), subject to the initial conditions of equation (18a) and the periodicity requirement of equation (11), is

$$x_0(\theta) = A_0 \cos\theta, \quad (19)$$

with

$$\omega_0^2 = \phi. \quad (20)$$

The central issue is what is ϕ^2. The Senator–Bapat paper [8] gives several suggestions for how it should be selected. The author's view is that ϕ should equal the square of the angular frequency, ω_{HB}^2, obtained from the application of the lowest order harmonic balance method to equation (6) with $\mu = 0$ and with the initial conditions $x(0) = A_0$, $\dot{x}(0) = 0$. Under these requirements, equation (5) becomes

$$\ddot{x} + x^3 = 0, \quad (21)$$

and

$$\omega_{HB}^2 = (\tfrac{3}{4})A_0^2 = \phi. \quad (22)$$

(See Mickens [1], section 4.3.1.)

Substituting equations (19) and (22) into (15), and simplifying the resulting expression gives

$$x_1'' + x_1 = \left(\frac{4\omega_1}{\sqrt{3}}\right)\cos\theta + \left(\frac{2\mu}{\sqrt{3}}\right)\left(\frac{A_0^2}{4} - 1\right)\sin\theta - \left(\frac{A_0}{3}\right)\cos 3\theta + \left(\frac{\mu A_0^2}{2\sqrt{3}}\right)\sin 3\theta. \quad (23)$$

The elimination of secular terms in the solution for $x_1(\theta)$ requires

$$\omega_1 = 0, \quad A_0 = 2. \quad (24)$$

Solving the resultant differential equation for $x_1(\theta)$, including both the particular and homogeneous solutions [1], and enforcing the initial conditions of equation (18b), gives

$$x_1(\theta) = \left(A_1 - \frac{1}{12}\right)\cos\theta + \left(\frac{\sqrt{3}\mu}{4}\right)\sin\theta + \left(\frac{1}{12}\right)\cos 3\theta - \left(\frac{\mu}{4\sqrt{3}}\right)\sin 3\theta. \quad (25)$$

Note that at this stage of the calculation A_0, ω_0 and ω_1 have been determined; they are

$$A_0 = 2, \quad \omega_0 = \sqrt{\phi} = \sqrt{3}, \quad \omega_1 = 0. \quad (26)$$

It should be clear that at the order ε^n calculation the values of A_{n-1} and ω_n can be determined. This is a general result which holds true for perturbation methods applied to systems having limit cycles [1].

Carrying out the similar calculation for $x_2(\theta)$ gives

$$x_2'' + x_2 = [4\omega_2/\sqrt{3} + \tfrac{1}{12} - 2A_1 - \mu/6 + \mu^2/4]\cos\theta$$
$$+ (1/6\sqrt{3})[6A_1(3-\mu) + 2\mu - 1]\sin\theta + \text{(higher order harmonics)}. \quad (27)$$

The absence of secular terms in the solution for $x_2(\theta)$ gives

$$A_1 = \left(\frac{1}{6}\right)\left(\frac{1-2\mu}{3-\mu}\right), \quad \omega_2 = \left(\frac{1}{16\sqrt{3}}\right)\left[\frac{1-\mu-11\mu^2+3\mu^3}{3-\mu}\right]. \quad (28, 29)$$

Thus to order ε for $x(\theta)$ and order ε^2 for $\omega(\varepsilon)$, the following expressions are obtained:

$$x(\theta) = 2\cos\theta + \left(\frac{\varepsilon}{12}\right)\left\{\left[2\left(\frac{1-2\mu}{3-\mu}\right) - 1\right]\cos\theta \right.$$
$$\left. + (3\sqrt{3})\sin\theta + \cos 3\theta - (\sqrt{3}\mu)\sin 3\theta\right\} + O(\varepsilon^2), \quad (30)$$

$$\omega(\varepsilon) = \sqrt{3} + (\varepsilon^2/17\sqrt{3})[(1-\mu-11\mu^2+3\mu^2)/(3-\mu)] + \omega(\varepsilon^3). \quad (31)$$

The solution to equation (5) according to the Senator–Bapat method [8] is now recovered by setting $\varepsilon = 1$ in equations (30) and (31). Observe that both $x(\theta)$ and $\omega(1)$ are functions of μ.

It should be noted that *a priori* the above approximation to the solution, $x(\theta) = x(\omega t)$, is expected to be correct only for small values of the parameter μ. However, it can be directly seen that both $x(\theta)$ and $\omega(1)$ vary little as μ changes value in the interval (0, 1). Denoting the coefficient of cos θ by $a_0(\mu)$, it follows from equations (30) and (31) that

$$a_0(0) = 2(0{\cdot}98611), \quad a_0(1) = 2(0{\cdot}91667), \quad (32a)$$

$$\omega(1)|_{\mu=0} = (1{\cdot}00694)\sqrt{3}, \quad \omega(1)|_{\mu=1} = (0{\cdot}91667)\sqrt{3}. \quad (32b)$$

These results can be compared to what is obtained from the first order harmonic balance method applied to equation (5),

$$x(t) = 2\cos(\sqrt{3}t); \tag{33}$$

see Mickens [1], section 4.3.4, and the similar result from an averaging technique [5]. The above calculations show that the coefficient of the dominant lowest harmonic changes only by about 10% in having μ go from 0 to 1. A change of equal magnitude occurs also for the angular frequency.

In summary, it has been shown that the perturbation technique of Senator and Bapat [8] can be easily generalized to the case where not only is the non-linearity not small, but also limit cycles exist. The possibility of further generalizing the Senator–Bapat technique is now being investigated for inclusion in a higher order averaging method.

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