Journal of Sound and Vibration (1999) **224**(1), 172–182
Article No. jsvi.1998.2143, available online at http://www.idealibrary.com on IDEAL®

NEW CLOSED-FORM SOLUTIONS FOR BUCKLING OF A VARIABLE STIFFNESS COLUMN BY MATHEMATICA®

I. ELISHAKOFF
Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431-0991, *U.S.A.*

AND

O. ROLLOT
Laboratoire de Recherches et Applications en Mécanique Avancée, Institut Français de Mécanique Avancée, Aubière, F-63175 *France*

(*Received* 2 *November* 1998, *and in final form* 17 *December* 1998)

1. INTRODUCTION

The buckling of uniform columns under various loading and boundary conditions is a well studied topic. As far the columns with variable cross section, several exact solutions are available, in terms of logarithmic and trigonometric [1, 2], Bessel [3, 4], and Lommel [5–7] functions. Exact solution in terms of series for buckling load for variable cross-section columns with variable axial forces was furnished by Eisenberger [8]. The closed-form solutions are extremely rare. Two cases will be described as follows. For the column [9] that is simply supported at both its ends and possesses the following stiffness $D(x)$

$$D(x) = 4x(L - x)D_0/L^2, \tag{1}$$

where L is the length, x is the axial co-ordinate, the governing differential equation reads

$$[D_0(4x(L - x))/L^2]\,\mathrm{d}^2w/\mathrm{d}x^2 + Pw = 0, \tag{2}$$

where $w(x)$ is a displacement. Substitution of the mode

$$w(x) = A[4x(L - x)/L^2] \tag{3}$$

where A is a constant, in equation (2) results in

$$A[D_0 4x(L - x)/L^2(-8/L^2) + P4x(L - x)/L^2] = 0. \tag{4}$$

Since for buckling $A \neq 0$, one obtains the buckling load

$$P_{cl} = 8D_0/L^2. \tag{5}$$

A second example belongs to Duncan [10]. Here the stiffness varies as

0022–460X/99/260172 + 11 $30.00/0

$$D(x) = [1 - \tfrac{3}{7}(x/L)^2]D_0, \tag{6}$$

so that the governing differential equation is

$$D_0[1 - \tfrac{3}{7}(x/L)^2]\frac{d^2w}{dx^2} + Pw = 0. \tag{7}$$

The buckling mode is guessed by Duncan [10] as

$$w(x) = A[7(x/L) - 10(x/L)^3 + 3(x/L)^5]. \tag{8}$$

By substitution of equation (8) into equation (7) the classical buckling load becomes

$$P_{cl} = \tfrac{16}{7}D_0/L^2. \tag{9}$$

The present writers are unaware of other closed-form solutions for the columns with variable stiffness. Obtaining such solutions is worthwhile, since closed-form solutions could serve as benchmark solutions for the purpose of contrasting various approximate solutions with them. They also can be utilized for educational purposes. In what follows, one generalizes the above two closed-form solutions. This note should be viewed as an auxiliary study towards the authors' general objective to obtain closed-form solutions for vibrating beams in the presence of axial load [13].

2. FORMULATION OF THE PROBLEM

The column buckling is governed by the differential equation

$$D(x)d^2w/dx^2 + Pw = 0, \tag{10}$$

where $D(x)$ is defined as

$$D = D_0 r(z), \tag{11}$$

where D_0 is a constant and z is a non-dimensional co-ordinate defined

$$z = x/L. \tag{12}$$

The governing differential equation (10) can be rewritten

$$r\, d^2w/dz^2 + k^2 w = 0, \tag{13}$$

where k^2 is a constant defined as

$$k^2 = PL^2/D_0. \tag{14}$$

One deduces the buckling load from equation (14):

$$P = k^2 D_0/L^2. \tag{15}$$

In this study, $r(z)$ is assumed to be a polynomial of the second degree. Three different variations for $r(z)$ are discussed that lead to new closed-form solutions for the buckling load:

$$r = \beta z - \gamma z^2, \quad r = 1 + \beta z - \gamma z^2, \quad r = 1 - \gamma z^2. \tag{16a–c}$$

In this paper the displacement w is assumed to be a polynomial function that satisfies the differential equation and all boundary conditions. One finds new closed-form solutions for some particular choices of β and γ.

3. UNCOVERED CLOSED-FORM SOLUTIONS

3.1. *Case* 1: $r = \beta z - \gamma z^2$

The variation of $D(x)$ is given by

$$D = D_0(\beta z - \gamma z^2) \tag{17}$$

and the displacement is a polynomial of degree two

$$w = az + bz^2. \tag{18}$$

The boundary conditions for a simply supported beam are:

$$w(0) = 0, \quad D_0 w''(0) = 0, \quad w(1) = 0, \quad D_0 w''(1) = 0. \tag{19a–d}$$

Equations (19a) and (19b) are always satisfied, equations (19c) and (19d) lead to

$$b = -a \tag{20}$$

and

$$\beta = \gamma. \tag{21}$$

Taking into account the boundary conditions in equation (19), one defines

$$r = z\gamma - z^2\gamma \tag{22}$$

and

$$w = az - az^2. \tag{23}$$

w has to satisfy the differential equation (13) for any z. This problem is solvable with the aid of Mathematica® command *SolveAlways* [11, 12]. Whereas for solving the problem the use of symbolic algebra is not absolutely necessary, it is an extremely convenient tool. *SolveAlways* yields parameter values for which the given equation or system of equations which depend on a set of parameters is valid for all variable values. The result of *SolveAlways* is given in the form of a list of all possible sets of values. *SolveAlways* works primarily with linear and polynomial equations.

For this case two sets are obtained. The first one leads to a trivial solution with $a = 0$. The second set leads to

$$k^2 = 2\gamma. \tag{24}$$

Finally, using equation (15) one deduces the buckling load,

$$P = 2\gamma D_0/L^2. \tag{25}$$

The buckling mode reads

$$w = a(z - z^2), \tag{26}$$

that corresponds to the following definition of the stiffness

$$D = D_0\gamma(z - z^2). \tag{27}$$

One can now relate to the first example described in the introduction for $\gamma = 4$ (see equation (1)). One uncovers the same buckling load,

$$P = 8D_0/L^2. \tag{28}$$

For $a = 4$, the same buckling mode

$$w = 4(z - z^2) \tag{29}$$

is found as in equation (3). Equation (25) allows one to optimize the column in the presence of the buckling constraint

$$P_{cl} \geqslant \hat{P}. \tag{30}$$

This yields the admissible region of variation of the parameter γ,

$$\gamma \geqslant \hat{P}L^2/2D_0, \tag{31}$$

so that the buckling load of the column will satisfy the inequality (30).

If one assumes the displacement is a polynomial of higher degree, one can find higher buckling loads. The method proposed below is the base of the algorithm of the Mathematica® function *SolveAlways*. One assumes that the displacement is the form

$$w = w_0 \sum_{j=1}^{N} a_j z^j. \tag{32}$$

This displacement equals zero at $z = 0$ so one satisfies the first boundary condition. The second derivative of w reads

$$w'' = w_0 \sum_{j=2}^{N} a_j j(j-1) z^{j-2}. \tag{33}$$

One defines the stiffness as

$$D = D_0\gamma(z - z^2). \tag{34}$$

This definition is very interesting because the stiffness equals zero at both ends. So the bending moment Dw'' equals zero at the ends identically, irrespective of the definition of the displacement. The differential equation becomes

$$\gamma(z - z^2)w'' + k^2 w = 0 \tag{35}$$

and taking into account equations (32) and (33), one obtains

$$\gamma(z - z^2)\sum_{j=2}^{N} a_j j(j-1)z^{j-2} + k^2 \sum_{j=1}^{N} a_j z^j = 0. \tag{36}$$

One expands the relation

$$\sum_{j=2}^{N} \gamma a_j j(j-1)z^{j-1} - \sum_{j=2}^{N} \gamma a_j j(j-1)z^j + \sum_{j=1}^{N} k^2 a_j z = 0. \tag{37}$$

Now the equation is modified to have only z^j terms:

$$\sum_{j=1}^{N-1} \gamma a_{j+1} j(j+1)z^j - \sum_{j=2}^{N} \gamma a_j j(j-1)z^j + \sum_{j=1}^{N} k^2 a_j z^j = 0, \tag{38}$$

so that one writes the sum from $j = 2$ to $j = N - 1$:

$$2\gamma a_2 z + \sum_{j=2}^{N-1} \gamma a_{j+1} j(j+1)z^j - \sum_{j=2}^{N-1} \gamma a_j j(j-1)z^j - \gamma a_N N(N-1)z^N$$

$$+ k^2 a_1 z + \sum_{j=1}^{N} k^2 a_j z^j + k^2 a_N z^N = 0. \tag{39}$$

Regrouping the terms of the same degree yields

$$(k^2 a_1 + 2\gamma a_2)z + \sum_{j=2}^{N-1} [\gamma a_{j+1} j(j+1) - \gamma a_j j(j-1) + k^2 a_j] z^j$$

$$+ (k^2 - \gamma N(N-1))a_N z^N = 0. \tag{40}$$

This equation must equal zero for any z. For z^N,

$$k^2 = \gamma N(N-1). \tag{41}$$

One deduces

$$P = \gamma N(N-1)D_0/L^2. \tag{42}$$

The first buckling load is for $N = 2$,

$$P_{cl} = 2\gamma D_0/L^2. \tag{43}$$

As a result, a polynomial of degree N leads to the mth buckling load with

$$N = m + 1. \tag{44}$$

The other terms of the polynomial lead to the global definition of the coefficients. For z in the first power, one obtains

$$a_2 = -m(m+1)a_1/2, \tag{45}$$

whereas for z in power j,

$$a_{j+1} = \{[j(j-1) - m(m+1)]/j(j+1)\}a_j. \tag{46}$$

One notices that for $j = 1$, equation (46) reduces equation (45). Hence equation (46) is valid for any a_j. One chooses $a_1 = 1$. One can verify via Mathematica® that the sum of the coefficients a_j equals zero, and thus the boundary condition of the displacement at the end $z = 1$, is satisfied.

Now, one can consider that one has a column clamped beam at $z = 0$ and it has arbitrary boundary conditions at $z = 1$. If one assumes w is a polynomial, as in equation (32), then the coefficient a_1 must vanish to have a slope equal to zero at $z = 0$. One has to take into account equations (46) and (45). These relations impose that all a_j must equal zero. Finally one arrives at a trivial solution. One concludes that the present method is inefficient for the column that has a clamped end at $z = 0$.

3.2. *Case* 2: $r = 1 + \beta z - \gamma z^2$

In this case, the stiffness is given by

$$D = D_0(1 + \beta z - \gamma z^2) \tag{47}$$

and the displacement is a polynomial of degree five,

$$w = az + bz^2 + cz^3 + dz^4 + ez^5. \tag{48}$$

After simplifications, the boundary conditions (19) lead to the conditions

$$b = 0, \quad a + c + d + e = 0, \quad (3c + 6d + 10e)(1 + \beta - \gamma) = 0. \tag{49a–c}$$

Mathematica®, leads to three different sets of solutions which respect the differential equation. The second and the third one yield the same buckling mode. Hence, one has to consider two sets which correspond to different buckling loads and modes. The first set is defined as

$$k^2 = 12\beta^2, \quad \gamma = \beta^2, \quad d = \beta^3 a, \quad c = -2\beta^2 a, \quad e = 0. \tag{50}$$

This set of solutions implies the following definition of the stiffness:

$$D = D_0(1 + \beta z - \beta^2 z^2), \tag{51}$$

with attendant buckling load

$$P = 12\beta^2 D_0/L^2 \tag{52}$$

and the buckling mode

$$w = a(z - 2\beta^2 z^3 + \beta^3 z^4). \tag{53}$$

These coefficients have to satisfy the boundary conditions (49). From there one finds three values of β that satisfy the above conditions:

$$\beta = 1, \quad \beta = (1 - \sqrt{5})/2 \quad \text{and} \quad \beta = (1 + \sqrt{5})/2. \tag{54}$$

For the particular case $\beta = 1$, the stiffness is defined by

$$D = D_0(1 + z - z^2), \tag{55}$$

so that the buckling load equals

$$P = 12D_0/L^2. \tag{56}$$

The buckling mode is

$$w = a(z - 2z^3 + z^4). \tag{57}$$

One notices that this function is a Duncan polynomial [10]. For the particular case $\beta = (1 - \sqrt{5})/2$, the stiffness is defined by

$$D = D_0[1 + (1 - \sqrt{5})z/2 - (1 - \sqrt{5})^2 z^2/4]. \tag{58}$$

The buckling load equals

$$P = 3(\sqrt{5} - 1)^2 D_0/L^2. \tag{59}$$

The buckling mode is

$$w = a\{z - [(1 - \sqrt{5})^2/2]z^3 + [(1 - \sqrt{5})^3/8]z^4\}. \tag{60}$$

For the particular case $\beta = (1 + \sqrt{5})/2$, the stiffness is defined by

$$D = D_0\{1 + [(1 + \sqrt{5})/2]z - [(1 + \sqrt{5})^2/4]z^2\}, \tag{61}$$

so that the buckling load equals

$$P = 3(\sqrt{5} + 1)^2 D_0/L^2 \tag{62}$$

and the buckling mode is

$$w = a\{z - [(1 + \sqrt{5})^2/2]z^3 + [(1 + \sqrt{5})^3/8]z^4\}. \tag{63}$$

The second set is defined as

$$k^2 = 20\gamma, \quad \beta = -\sqrt{3\gamma}, \quad d = -[5\gamma^{3/2}/\sqrt{3}]a, \quad c = -(10\gamma/3)a, \quad e = (2\gamma^2/3)a. \tag{64}$$

This set of solutions implies the definition of the stiffness

$$D = D_0(1 - \sqrt{3\gamma}z - \gamma z^2), \tag{65}$$

with the buckling load

$$P = 20\gamma D_0/L^2 \tag{66}$$

and the buckling mode

$$w = d[z - (10\gamma/3)z^3 - (5\gamma^{3/2}/\sqrt{3})z^4 + (2\gamma^2/3)z^5]. \quad (67)$$

These coefficients have to satisfy the boundary conditions (39). From there one finds a single value of γ that satisfies these equations: $\gamma = (5 - \sqrt{21})/2$. Then stiffness becomes

$$D = D_0\{1 - z\sqrt{[3(5 - \sqrt{21}/2] - [(5 - \sqrt{21})/2]z^2}\}. \quad (68)$$

The buckling load equals

$$P = 10(5 - \sqrt{21})D_0/L^2 \quad (69)$$

and the buckling mode becomes

$$w = a\{z - [5(5 - \sqrt{21})/3]z^3 - [5(5 - \sqrt{21})^{3/2}/2\sqrt{6}]z^4 + [(5 - \sqrt{21})^2/6]z^5\}. \quad (70)$$

3.3. *Case* 3: $r = 1 - \gamma z^2$

In this last part, one has

$$D = D_0(1 - \gamma z^2). \quad (71)$$

The displacement is a polynomial of degree 5 which is utilized to search for the buckling mode:

$$w = az + bz^2 + cz^3 + dz^4 + ez^5. \quad (72)$$

After simplifications, the boundary conditions (19) lead to the conditions

$$b = 0, \quad a + c + d + e = 0, \quad (3c + 6d + 10e)(1 - \gamma) = 0. \quad (73a\text{–}c)$$

The solution of the governing differential equation (13) by Mathematica® leads to two different sets of solution which correspond to different buckling load and mode. The first set is given by

$$k^2 = 6\gamma, \quad c = -\gamma a, \quad d = e = 0. \quad (74)$$

This set of solutions implies the following definition of the buckling load:

$$P = 6\gamma D_0/L^2, \quad (75)$$

and the buckling mode is a polynomial of degree 3,

$$w = a(z - \gamma z^3). \quad (76)$$

These coefficients have to satisfy the boundary conditions (39). Thus one finds one value of γ that satisfies these equations: $\gamma = 1$. Therefore, the stiffness is defined by

$$D = D_0(1 - z^2). \quad (77)$$

The buckling is given by

$$P = 6D_0/L^2. \tag{78}$$

The buckling mode reads

$$w = a(z - z^3). \tag{79}$$

The second set is given by

$$k^2 = 20\gamma, \quad c = -(10\gamma/3)a, \quad e = (7\gamma^2/3)a, \quad d = 0. \tag{80}$$

This set of solutions implies the following definition of the buckling load:

$$P = 20\gamma D_0/L^2, \tag{81}$$

whereas the buckling mode is

$$w = a[z - (10\gamma/3)z^3 + (7\gamma^2/3)z^5]. \tag{82}$$

These coefficients have to satisfy the boundary conditions (73). This allows us to find two values of γ that satisfy these equations: $\gamma = \frac{3}{7}$ or $\gamma = 1$. For the particular case $\gamma = 1$, the stiffness is defined by

$$D = D_0(1 - z^2), \tag{83}$$

so the buckling load equals

$$P = 20D_0/L^2 \tag{84}$$

and the buckling mode is

$$w = a[z - (10/3)z^3 - (7/3)z^5]. \tag{85}$$

It is noticeable that this stiffness is the same as that for the first set of the third case (equations (78)–(80)), but a greater buckling load has been found implying that one determines the higher buckling loads by increasing the degree of freedom of the displacement. For the particular case $\gamma = 3/7$, the stiffness is defined by equation (6), so that the expression for buckling load in equation (82) reduces to that in equation (9) and the buckling mode is

$$w = a[z - (10/7)z^3 + (3/7)z^5]. \tag{86}$$

By choosing $a = 7\ A$, the displacement reduces to equation (8). This result and the buckling load match with the Duncan example presented in reference [10].

4. CONCLUSION AND SUMMARY

In this study example (1) is first generalized to a family of beams with a variable moment of inertia. Using the proposed approach, one also determines new closed-form solutions of columns with variable stiffness, including generalization of Duncan's solution [10]. Then a design criterion is discussed, so that the buckling load exceeds any prescribed value. Note that if one uses polynomials of higher degree, more degrees of freedom are allowed for the

TABLE 1
Summary of uncovered solutions

Stiffness variation	Buckling load	Buckling mode
$D = D_0\gamma(z - z^2)$	$P_m = m(m+1)\gamma D_0/L^2$	$w = w_0 \sum_{j=1}^{m+1} a_j z^j, \quad a_{j+1} = \frac{j(j-1) - m(m+1)}{j(j+1)} a_j, \quad a_1 = 1$
$D = D_0(1 + z - z^2)$	$P_{cl} = 12D_0/L^2$	$w = w_0(z - 2z^3 + z^4)$
$D = D_0(1 + (1 - \sqrt{5})z/2 - (1 - \sqrt{5})^2 z^2/4)$	$P = 3(\sqrt{5} - 1)^2 D_0/L^2$	$w = w_0\left[z - \frac{(1-\sqrt{5})^2}{2} z^3 + \frac{(1-\sqrt{5})^3}{8} z^4\right]$
$D = D_0\left(1 + \frac{1+\sqrt{5}}{2} z - \frac{(1+\sqrt{5})^2}{4} z^2\right)$	$P = 3(\sqrt{5} + 1)^2 D_0/L^2$	$w = w_0\left[z - \frac{(1+\sqrt{5})^2}{2} z^3 + \frac{(1+\sqrt{5})^3}{8} z^4\right]$
$D = D_0\left[1 - \sqrt{\frac{3(5-\sqrt{21})}{2}} z - \frac{5-\sqrt{21}}{2} z^2\right]$	$P = 10(5 - \sqrt{21}) D_0/L^2$	$w = w_0\left[z - \frac{5(5-\sqrt{21})}{3} z^3 - \frac{5(5-\sqrt{21})^{3/2}}{2\sqrt{6}} z^4 + \frac{(5-\sqrt{21})^2}{6} z^5\right]$
$D = D_0(1 - z^2)$	$P_{cl} = 6D_0/L^2$	$w = w_0(z - z^3)$

displacement. In such circumstances, the method leads to higher buckling loads. It appears remarkable that the *closed-form* solutions obtained are simpler than *exact* solutions for many problems involving uniform columns. The summary of uncovered new solutions is given in Table 1.

ACKNOWLEDGMENT

This study was performed when Olivier Rollot was on a special internship program at the Florida Atlantic University during the Fall semester of the 1998/1999 academic year. The help of Dr. Çetin Çetinkaya of the Wolfram Research, Inc. is appreciated. Discussions with Leandro Espinosa and Massimiliano Zingales of Florida Atalantic University are gratefully acknowledged.

REFERENCES

1. A. N. Dinnik 1955 *Prodol'nyi Izgib. Kruchenie (Bending in Presence of Axial Forces Torsion)*. Moscow: Academy of USSR Publishing; 96–130 (in Russian).
2. H. Bleich 1952 *Buckling Strength of Metal Structures*. New York: McGraw-Hill; 187–191.
3. A. N. Dinnik 1932 *Transactions of the ASME, Applied Mechanics*. Design of column of varying cross-sections.
4. A. N. Dinnik 1927 *Vestnik Inzhenerov* **8**, 1929 **1-2**, 1930 **7**, 1931 **9**. About the bending in the presence of axial load, of columns with varying cross-section (in Russian).
5. I. Elishakoff and F. Pelligrini 1987 *Journal of Sound and Vibration* **114**, 144–148. Exact and effective approximate solutions of some divergent type non-conservative problems.
6. I. Elishakoff and F. Pellegrini 1987 *Journal of Sound and Vibration* **115**, 182–186. Application of Bessel and Lommel functions and undetermined multiplier Galerkin method version for instability of non-uniform column.
7. I. Elishakoff and F. Pelligrini 1988 *Computer Methods in Applied Mechanics and Engineering* **66**, 107–119. Exact solution for buckling of some divergence type non-conservative systems in terms of Bessel and Lommel functions.
8. M. Eisenberger 1991 *International Journal of Solids and Structures* **27**, 135–143. Buckling loads for variable cross-section members with variable axial forces.
9. I. Elishakoff 1998 *A First Course in Buckling of Structures*. Boca Raton: Florida Atlantic University; Lecture Notes, Department of Mechanical Engineering.
10. W. J. Duncan 1937 *Aeronautical Research Committee, Reports and Memoranda* **1798**. Galerkin's method in mechanics and differential equations.
11. S. Wolfram 1996 *The Mathematica® Book*. Cambridge University Press; third edition. See p. 1204.
12. M. L. Abell and J. P. Braselton 1992 *The Mathematica® Handbook*. New York: Academic Press. See p. 435.
13. S. Candan and I. Elishakoff 1999 Closed-form solutions for vibration of deterministically and stochastically inhomogeneous beams (in preparation).