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A SEMI-ANALYTICAL APPROACH TO THE NON-LINEAR DYNAMIC RESPONSE PROBLEM OF S–S AND C–C BEAMS AT LARGE VIBRATION AMPLITUDES PART I: GENERAL THEORY AND APPLICATION TO THE SINGLE MODE APPROACH TO FREE AND FORCED VIBRATION ANALYSIS

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In a previous series of papers [1–3], a general model based on Hamilton's principle and spectral analysis was developed for non-linear free vibrations occurring at large displacement amplitudes of fully clamped beams and rectangular homogeneous and composite plates. As an introduction to the present work, concerned with the forced non-linear response of C–C and S–S beams, the above model has been derived using spectral analysis, Lagrange's equations and the harmonic balance method. Then, the forced case has been examined and the analysis led to a set of non-linear partial differential equations which reduces to the classical modal analysis forced response matrix equation when the non-linear terms are neglected. On the other hand, if only one mode is assumed, this set reduces to the Duffing equation, very well known in one mode analyses of non-linear systems having cubic non-linearities. So, it appeared sensible to consider such a formulation as the multidimensional Duffing equation.

In order to solve the multidimensional Duffing equation in the case of harmonic excitation of beam like structures, a method is proposed, based on the harmonic balance method, and a set of non-linear algebraic equations is obtained whose numerical solution leads in each case to the basic function

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contribution coefficients to the displacement response function. These coefficients depend on the excitation frequency and the distribution of the applied forces. The frequency response curve obtained here exhibits qualitatively a classical non-linear behaviour, with multivalued regions in which the jump phenomenon could occur. Quantitatively, the analytical result obtained here, without assuming any limitation to the scale of the excitation, is identical to that obtained by the multiple scale method which assumes small values of the scaling parameter.

Attention was focused on the assumed one mode in order to improve the results obtained. In the case of free vibrations, the analytical solution obtained by elliptic functions has been expanded into power series of higher orders using the symbolic manipulation program "Maple". It has been shown that extreme care must be taken in the choice of the polynomial approximation which is valid only in a zone limited by a radius of convergence. The use of Padé approximants permitted considerable increase in the zone of validity of the solution obtained for very large vibration amplitudes.

1. INTRODUCTION

In modern engineering problems, large vibration amplitudes of beam-like or plate-like structures very often occur [4] inducing a dynamic behaviour which is different in many ways from that predicted by linear structural dynamics theories. Some of these differences, such as the jump phenomenon, the amplitude dependence of mode shapes, response harmonic distortion and the harmonic distortion amplitude dependence have been examined experimentally [1–8]. Also, a considerable amount of published theoretical research is available which shows some success in the analysis of many aspects of the non-linear dynamic behaviour described above. However, as pointed out in [4], no general and systematic approach to non-linear problems is available which allows all or at least most of the known non-linear effects to be described in a unified manner. In most of the theoretical studies, a one mode solution is often assumed. This assumption has been shown both theoretically and experimentally to be inaccurate for beams in references [5] and [6]. In reference [6], a theoretical model for large vibration amplitudes of thin elastic structures was developed, based on Hamilton's principle and spectral analysis, to obtain numerical results. The theory effectively reduced a non-linear free vibration problem to a set of non-linear algebraic equations depending on the classical rigidity and mass matrices, and a fourth order tensor due to the non-linearity. If the non-linearity tensor due to finite displacements was neglected, the classical eigenvalue problem known in the linear theory was obtained. By choosing the convenient basic function in each case, results have been obtained for various boundary conditions. The theory has been applied to simply supported and clamped–clamped beams in reference [1]. In an improved version of the model, the spatial distribution of the harmonic distortion was also included in the analytical and numerical formulation and some results were obtained, which are presented in reference [4]. However, although the above models succeeded well in analysing via this formulation the effects of large vibration amplitudes on the mode shapes of simply supported beams and clamped–clamped beams, they were restricted in a sense that only the free response problem was considered in the formulation.

In most of the papers concerned with the non-linear dynamic response of beams, the single mode assumption was made as a tool for investigation of the effect of the geometric non-linearities on resonant phenomena. This is due to the simplification it introduces in the theory on one hand and on the other hand because the error it introduces in the estimation of the non-linear frequency remains small. Application of Galerkin's method to the governing equation leads to the single Duffing equation in time. The latter equation can be treated by using the elliptical functions as clearly presented in references [9–11]. The methods which are often used due to their simplicity are the harmonic balance method and the perturbation method [12–19]. Some authors use the multiple scale method [20, 21] or the direct numerical integration method [22]. In order to obtain an ''exact'' solution of the forced Duffing equation, some possibilities have been given [23, 24]. In other works, the time variable of the problem is assumed to be a harmonic function. The resulting non-linear differential equation in the spatial variables can be solved by the Ritz method [25], by combining the finite element method and linearising functions [26–29] or by the FEM and continuation method [30, 31]. Without the above mentioned simplifying assumptions, various methodological approaches can be used [32, 33]. Some experimental models can be found in references [34, 35]. Following the same procedure presented in references [1, 4], a study of non-linear forced vibrations of beams has been recently done [36].

The purpose of the present work was to extend the formulation presented in reference [4] to the non-linear dynamic forced response and to give some immediate and interesting applications of this semi-analytical approach. The formulation has been established using Lagrange's equations and the harmonic balance method. The mathematical formulation of the multimode approach has been presented for free and forced vibration of beams. A set of non-linear algebraic equations is obtained whose numerical solution leads in each case to the basic function contribution coefficients to the displacement response function. When the coupling effect of the normal co-ordinates can reasonably be neglected, as in the case of S–S beams, the one assumed mode represents the resonant response with good accuracy. Various solutions based on the single mode assumption obtained from this model for the forced case are examined. For the free case, the exact solution given by elliptic functions is presented. Attention is focussed on the amplitude frequency dependence, i.e., backbone curves, obtained using various modelling techniques and solutions. The adequacy of the results obtained has been carefully examined in the present work. It has been shown that, the first approximation solution obtained by the harmonic balance method is identical to the exact solution of the Duffing equation obtained in terms of elliptic functions. It has been established that increasing the number of terms in the approximate solution by using perturbation techniques does not improve the results. The solution remains valid only in the zone of convergence of the power series expansion which is limited by the radius of convergence which is generally small. A mathematical technique, based on Padé approximants is proposed in order to increase the range of amplitudes in which power series expansion can be used. Formulations of the frequency–amplitude relation are presented. These approximants permitted considerable improvement of the zone of validity. The results given by this technique match very well with those obtained by various authors in the case of free and forced vibration of beams.

2. GENERAL MULTIDIMENSIONAL THEORY

2.1. FREE VIBRATION MODEL BASED ON LAGRANGE'S EQUATION

Before giving the new formulation of the non-linear forced response problem, corresponding to large vibration amplitudes of thin beams, it is shown first that the numerical model obtained in reference [1] by applying Hamilton's principle to the free response problem can also be derived using Lagrange's equations and the harmonic balance method. This preliminary result will make formulation easier and clearer for the non-linear forced response case.

It is well known that Lagrange's equations, for a conservative system, having n degrees of freedom corresponding to n parameters $q_r(t)$, $r = 1$ to n can be written, if no forcing term is considered, as [37]

$$(\mathrm{d}/\mathrm{d}t)(\partial T/\partial \dot{q}_r) + \partial T/\partial q_r - \partial V/\partial q_r = 0, \qquad r = 1 \text{ to } n. \tag{1}$$

In the above equation, T is the kinetic energy, V is the total strain energy (which may include or not the non-linear terms depending on the problem considered) and $\dot{q}_i$ the derivatives of q_i with respect to time.

The non-linear strain–displacement relationships of a uniform beam undergoing large deflections are

$$\epsilon_x = \partial u/\partial x + \tfrac{1}{2}(\partial W/\partial x)^2, \qquad K_x = \partial^2 W/\partial x^2 \tag{2}$$

where u and W are the axial and transverse displacements respectively. ϵ_x is the axial strain and K_x is the curvature. Using Hooke's law, one can write the following relationships for the axial resultant force N_x and the bending moment M.

$$N_x = EA\epsilon_x, \qquad M = EIK_x, \tag{3}$$

where E, A and I are Young's modulus, the area, and the second moment of area of the cross-section of the beam. The elastic strain energy V of the beam is

$$V = \frac{1}{2}\int_0^L (N_x\epsilon_x + MK_x)\,\mathrm{d}x. \tag{4}$$

The inertia in the in-plane direction u is expected to be small compared to the inertia in the transverse direction W and hence it is neglected. The in-plane equation of motion is then given by [28, 31]

$$\frac{\partial}{\partial x}\left[EA\left(\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial^2 W}{\partial x^2}\right)\right]=0 \quad \text{or} \quad N_x = EA\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right)=\text{constant}. \tag{5.1, 2}$$

N_x represents the non-linear stretching force. Integrating (5.2) between the limits 0 and L and applying the boundary conditions $u(0) = u(L) = 0$ gives

$$N_x = \frac{EA}{2L}\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 \mathrm{d}x. \tag{6}$$

For immovable ends, if one neglects the in-plane inertia, the non-linear resultant axial force N_x can be written in terms of the transverse displacement W alone. The total strain energy of the beam is given by

$$V = \frac{EA}{8L}\left[\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 \mathrm{d}x\right]^2 + \frac{EI}{2}\int_0^L \left(\frac{\partial^2 W}{\partial x^2}\right)^2 \mathrm{d}x. \tag{7}$$

The kinetic energy is given by

$$T = \tfrac{1}{2}\rho A \int_0^L \left(\frac{\partial W}{\partial t}\right)^2 \mathrm{d}x. \tag{8}$$

Using a generalised parameterisation and the usual summation convention defined in [1], one can put

$$W(x, t) = q_i(t) w_i(x), \tag{9}$$

where $w_i(x)$ are the basic functions. Using these equations, the discretisation of the kinetic energy T and the total strain energy V leads to:

$$T = \tfrac{1}{2}\dot{q}_i\dot{q}_j m_{ij}, \qquad V = \tfrac{1}{2}q_i q_j k_{ij} + \tfrac{1}{2}q_i q_j q_k q_l b_{ij} k_l, \tag{10.1, 2}$$

where the terms m_{ij}, k_{ij}, and b_{ijkl} are as given in references [1, 4], by:

$$m_{ij} = \rho A \int_0^L w_i(x) w_j(x)\,\mathrm{d}x, \qquad k_{ij} = EI\int_0^L \frac{\mathrm{d}^2 w_i(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2 w_j(x)}{\mathrm{d}x^2}\,\mathrm{d}x, \tag{11.1, 2}$$

$$b_{ijkl} = \frac{EA}{4L}\int_0^L \frac{\mathrm{d}w_i(x)}{\mathrm{d}x}\frac{\mathrm{d}w_j(x)}{\mathrm{d}x}\,\mathrm{d}x \int_0^L \frac{\mathrm{d}w_k(x)}{\mathrm{d}x}\frac{\mathrm{d}w_l(x)}{\mathrm{d}x}\,\mathrm{d}x \tag{11.3}$$

Substituting equations (10) in (1) leads to the following set of non-linear partial differential equations:

$$\ddot{q}_i m_{ir} + q_i k_{ir} + 2q_i q_j q_k b_{ijkr} = 0, \qquad r = 1, \ldots, n, \tag{12}$$

which can be written in matrix form as

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} + 2[\mathbf{B}(\mathbf{q})]\{\mathbf{q}\} = \{\mathbf{0}\}, \tag{13}$$

where $[\mathbf{M}]$, $[\mathbf{K}]$, $[\mathbf{B}]$ and $\{\mathbf{q}\}$ and are the mass tensor, the linear rigidity tensor, the non-linear rigidity tensor and the column vector of generalised parameters $\{\mathbf{q}\}^t = [q_1, q_2, \ldots, q_n]$ respectively. If the non-linear term $[\mathbf{B}(\mathbf{q})]\{\mathbf{q}\}$ is neglected in equation (13), the classical linear modal analysis free vibration equation is obtained:

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} = \{\mathbf{0}\}. \tag{14}$$

Now, by considering the non-linear equation (13) and assuming harmonic motion, one can put

$$q_i(t) = a_i \cos(\omega t). \tag{15}$$

Substituting equation (15) into equation (13) leads to

$$-\omega^2[\mathbf{M}]\{\mathbf{A}\}\cos(\omega t) + [\mathbf{K}]\{\mathbf{A}\}\cos(\omega t) + 2[\mathbf{B}(\mathbf{A})]\{\mathbf{A}\}\cos^3(\omega t) = \{\mathbf{0}\}, \tag{16}$$

where $\{\mathbf{A}\}$ is the column vector $\{\mathbf{A}\}^t = [a_1, a_2, \ldots, a_n]$. Expanding the trigonometric function $\cos^3(\omega t)$ as

$$\cos^3(\omega t) = \tfrac{3}{4}\cos(\omega t) - \tfrac{1}{4}\cos(3\omega t), \tag{17}$$

substituting it into (16) and applying the harmonic balance method leads to

$$2([\mathbf{K}] - \omega^2[\mathbf{M}])\{\mathbf{A}\} + 3[\mathbf{B}(\mathbf{A})]\{\mathbf{A}\} = \{\mathbf{0}\}. \tag{18}$$

For obtaining non-dimensional parameters one puts as in reference [1]:

$$w_i(x) = hw_i^*(x/L) = hw_i^*(x^*), \qquad \omega^2/\omega^{*2} = EI/\rho AL^4,$$
$$k_{ij}/k_{ij}^* = EIh^2/L^3, \qquad m_{ij}/m_{ij}^* = \rho Ah^2L, \qquad b_{ijkl}/b_{ijkl}^* = EIh^2/L^3, \tag{19}$$

where k_{ij}^*, m_{ij}^*, b_{ijkl}^* and ω^* are the non-dimensional generalised parameters. By substituting these notations in equation (18) one obtains the following non-linear algebraic equation [1]:

$$([\mathbf{K}^*] - \omega^{*2}[\mathbf{M}^*])\{\mathbf{A}\} + \tfrac{3}{2}[\mathbf{B}^*(\mathbf{A})]\{\mathbf{A}\} = \{\mathbf{0}\}. \tag{20}$$

Equation (20) is identical to that obtained in reference [1] for the non-linear free vibration of beams and plates using Hamilton's principle and integration over the range $[0, 2\pi/\omega]$. Numerical results obtained from equation (20) applied to beams in reference [1], have shown higher increase of curvatures near to the clamps at large deflections, compared with that predicted by the linear theory. This was in good agreement with experimental measurements reported in reference [3].

2.2. FORCED VIBRATION

By consideration of generalised forces, the model developed above is extended in this subsection to the forced case. Then, the expressions for the generalised forces are given in the case of a harmonic excitation force applied to the middle point of the beam or distributed over the whole beam length.

2.2.1. *General formulation*

Consider now the forced vibration case and assume that the structure is excited by the force $F(x, t)$ distributed over the range S (S is the length of the beam or a part of it). The physical force $F(x, t)$ excites the modes of the structures via a set of generalised forces $F_i(t)$ which depend on the expression for F, the excitation point for concentrated forces, the excitation length for distributed forces, and the mode considered. The generalised forces $F_i(t)$ are given by [34]:

$$F_i(t) = \int_S F(x, t) w_i(x) \, \mathrm{d}x, \tag{21}$$

in which w_i is the ith mode of the structure considered. Adding the forcing term $\{\mathbf{F}(t)\}$ to the right side of equation (13) leads to

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} + 2[\mathbf{B}(\mathbf{q})]\{\mathbf{q}\} = \{\mathbf{F}(t)\}. \tag{22}$$

Equation (22) represents a set of differential equations in which $\{\mathbf{F}(t)\}$ is a column vector of generalised forces. This equation appears as a generalisation to the non-linear case of the forced response equation, very well known in linear modal analysis theory [34] i.e.,

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} = \{\mathbf{F}(t)\}, \tag{23}$$

to which the term $2[\mathbf{B}(\mathbf{q})]\{\mathbf{q}\}$ corresponding to the non-linear geometrical rigidity is added. On the other hand, if only one mode is assumed, as will be considered in the next section, equation (22) reduces to

$$m_{11}\ddot{q}_1 + k_{11}q_1 + 2b_{1111}q_1^3 = F_1(t), \tag{24}$$

in which m_{11}, k_{11} and b_{1111} are the mass, rigidity and non-linearity terms corresponding to the first mode respectively. Putting

$$\omega_L^2 = k_{11}/m_{11}, \tag{25}$$

equation (24) can be written as

$$\ddot{q}_1 + \omega_L^2 q_1 = -2(b_{1111}/m_{11})q_1^3 + F_1(t)/m_{11}. \tag{26}$$

This equation is identical to that obained for the forced oscillation of a particle attached to a non-linear spring and is well known as the Duffing equation. For the solution of such an equation, many perturbation techniques are available which will be discussed in the next section.

It appears that the model developed above and summarised in equation (22) can be considered as a multidimensional form of the Duffing equation which is very often encountered in non-linear vibration analysis of structures having cubic non-linearities such as these considered here i.e., S–S and C–C beams [21]. The present non-linear model reduces to the classical linear modal analysis model for forced vibrations when the non-linear terms are neglected. It is also worth noticing here that the theory presented in [4] provides a means of calculating the cubic non-linearity coefficient of the approximate one-dimensional Duffing equation ($2b_{1111}/m_{11}$) for beams with various boundary conditions. This could allow numerical solutions to be obtained for engineering purposes, as will be shown later, which would be valid as far as the single mode assumption is valid.

2.2.2. *Solution based on the harmonic balance method*

Consider now the beam shown in Figure 1(a), excited by the concentrated harmonic force F^c applied at the point x_0, and the beam shown in Figure 1(b) excited by the distributed harmonic uniform force F^d, F^c and F^d are given by

$$F^c(x, t) = F^c \cos(\omega t)\delta(x - x_0), \qquad F^d(x, t) = F^d \cos(\omega t) \tag{27.1, 2}$$

in which δ is the Dirac function. The corresponding generalised forces $F_i^c(t)$ and $F_i^d(t)$ to be implemented in equation (24) for each case are given by

$$F_i^c(t) = F^c \cos(\omega t) w_i(x_0) = f_i^c \cos(\omega t), \tag{28.1}$$

$$F_i^d(t) = F_0^d \cos(\omega t) \int_0^L w_i(x)\, \mathrm{d}x = f_i^d \cos(\omega t). \tag{28.2}$$

Assuming a harmonic response $q_i = a_i \cos(\omega t)$ for $i = 1$ to n, substituting equation (28) in equation (22) and applying the harmonic balance method as in equations (16) and (17), one obtains

$$([\mathbf{K}] - \omega^2[\mathbf{M}])\{\mathbf{A}\} + \tfrac{3}{2}[\mathbf{B}(\mathbf{A})]\{\mathbf{A}\} = \{\mathbf{f}\}. \tag{29}$$

Using non-dimensional parameters, equation (29) can be written in non-dimensional form as:

$$([\mathbf{K}^*] - \omega^{*2}[\mathbf{M}^*])\{\mathbf{A}\} + \tfrac{3}{2}[\mathbf{B}^*(\mathbf{A})]\{\mathbf{A}\} = \{\mathbf{f}^*\}. \tag{30}$$

The dimensionless generalised forces f_i^{*c} and f_i^{*d} corresponding to the concentrated force at x_0 and the uniformly distributed force on the whole beam respectively are given by:

$$f_i^{*c} = F^c \frac{L^3}{EIh} w_i^*(x_0), \qquad f_i^{*d} = F^d \frac{L^4}{EIh} \int_0^1 w_i^*(x^*)\, \mathrm{d}x^* \tag{31.1, 2}$$

Expressions (31.1, 2) are used below for numerical calculation of the non-linear dynamic response of S–S and C–C beams and for comparison with previous results.

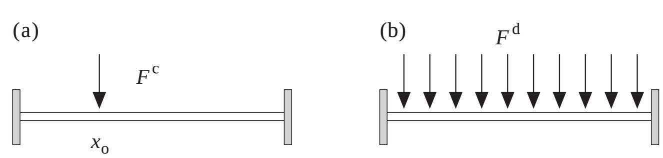

Figure 1. Details of applied forces: (a) concentrated; (b) distributed.

3. 1-D NON-LINEAR FREQUENCY RESPONSE FUNCTIONS

The single mode assumption, denoted in the present paper as the 1-D mode, consists of neglecting all the co-ordinates except a single "resonant" co-ordinate. Thus, it reduces the multi-degree-of-freedom system to a single one. It has been shown in previous studies that such an assumption may not be very accurate, with regard to some effects in nonlinear vibration of structures such as the increase of curvature near the clamps of a C–C beam for example [1]. In spite of this, the single mode approach still remains very often used in the literature [9–19]. This is due to the great simplification it introduces in the theory on one hand, and on the other hand because the error it introduces in the estimation of the non-linear frequency remains small, as will be shown later. By assuming one mode, one obtains the forced Duffing equation (24). The exact solution of this equation exists only when the excitation force has a specified form of an elliptic function [9]. In the case of harmonic excitation, very often considered in the literature, various techniques such as approximation with elliptic functions [9], the multiple scale method [20, 21, 32] or the harmonic balance method [10, 11, 14] have been used.

In this section, various solutions based on the single mode assumption, obtained from the model presented above and from other methods based on different perturbation procedures for the forced case, and on the exact elliptic solution in the free case are examined and compared. The objective is to analyse and to improve the accuracy of the amplitude–frequency relationships available for both free and forced simply supported and clamped–clamped beams.

3.1. NON-LINEAR FORCED VIBRATION

As has been shown above, the model developed in the present work, represented by the multidimensional Duffing equation (22) reduces to the classical Duffing equation (26) when only one mode is considered. If the HBM is applied to equation (22), it leads to the non-linear algebraic system (30). Applying the single mode assumption to equation (30) ($a_1 \neq 0$, $a_i = 0$ for $i > 1$), leads to

$$(\omega^*/\omega_L^*)^2 = 1 + \tfrac{3}{2}(b_{1111}^*/k_{11}^*)a_1^2 - (1/k_{11}^*)f_1^*/a_1, \tag{32}$$

in which $\omega_L^{*2} = k_{11}^*/m_{11}^*$.

Equation (32) can also be obtained by applying the HBM to the one mode Duffing equation (26). In the remainder of this paper, equation (32) will be referred to as the 1-D non-linear frequency response function (1-D NFRF) and will be compared with published results. In order to make a quantitative comparison with results obtained in reference [21], based on the multiple scale method, equation (32) can be transformed by putting $\epsilon\alpha = 2b_{1111}^*/m_{11}^*$, $\epsilon k = f_1^*/m_{11}^*$. Developing the expression obtained for ω^*/ω_L^* and neglecting the ϵ^2 terms leads to

$$\omega^* = \omega_L^* + \tfrac{3}{8}(\epsilon\alpha/\omega_L^*)a_1^2 - \epsilon k/2a_1\omega_L^*. \tag{33}$$

This equation is identical to (4.1.19, $\mu = 0$) obtained in reference [21]. Specifying the parameters m_{11}^*, k_{11}^*, b_{1111}^* and f_1^* for a given structure, gives the analytical frequency–amplitude relationship for each set of boundary conditions. For a S–S

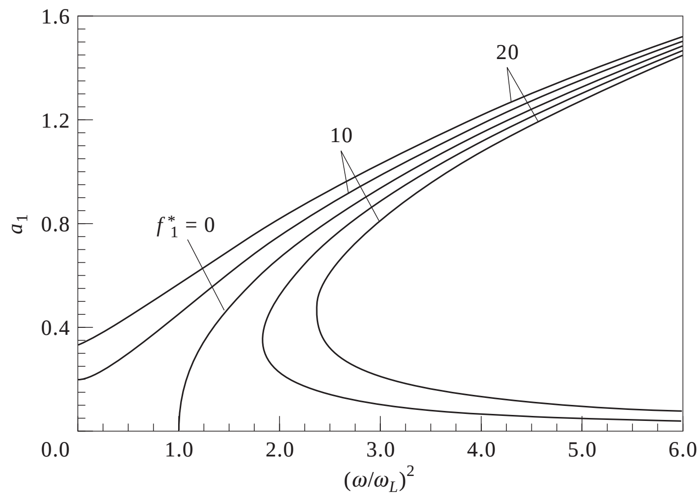

Figure 2. Forced vibration of a S–S beam (equation (34)).

beam, the coefficients b^*_{1111} and k^*_{11} can be analytically computed. (Appendix A.1.) Equation (32) becomes

$$(\omega^*/\omega^*_L)^2 = 1 + \tfrac{9}{4}\,a_1^2 - (2/\pi^4)f^*_1/a_1. \tag{34}$$

For a C–C beam, the coefficients of equation (32) are numerically computed (Appendix A.2.), which leads to:

$$(\omega^*/\omega^*_L)^2 = 1 + 1{\cdot}36065a_1^2 - (1/k^*_{11})f^*_1/a_1. \tag{35}$$

Equations (34) and (35) corresponding to the 1-D NFRF for S–S and C–C beams respectively are represented in Figures 2 and 3 for various values of the dimensionless excitation amplitude f^*_1. The corresponding numerical values are

Table 1

Forced vibration frequency ratio ω/ω_L for a simply supported beam under a uniform harmonic distributed force $\mathscr{F}^d = 2$

$\mathscr{A}^G = \sqrt{12 a_i w_i^*(1/2)}$	Multi-D present model† (30)	$\mathscr{A} = a_1\sqrt{12}$	1-D present model (40)	Elliptic solution [9, 29]	F.E.M. + linearisation [29]
−1·000048	1·7889745	−1	1·7853571	1·7852	1·7856
2·000017	0·87122632	2	0·8660254	0·8472	0·8460
−2·000019	1·6578010	−2	1·6583124	1·6557	1·6512
3·000032	1·4247162	3	1·4215602	1·4003	1·3760
−3·000027	1·8298262	−3	1·8314384	1·8217	1·8002
4·000007	1·8734095	4	1·8708287	1·8413	1·7846
−4·000012	2·1194099	−4	2·1213203	2·1013	2·0495
5·000003	2·3016790	5	2·2994565	2·2606	2·1619
−5·000042	2·4654062	−5	2·4672859	2·4361	2·3432

† f^*_is are given in Appendix A.

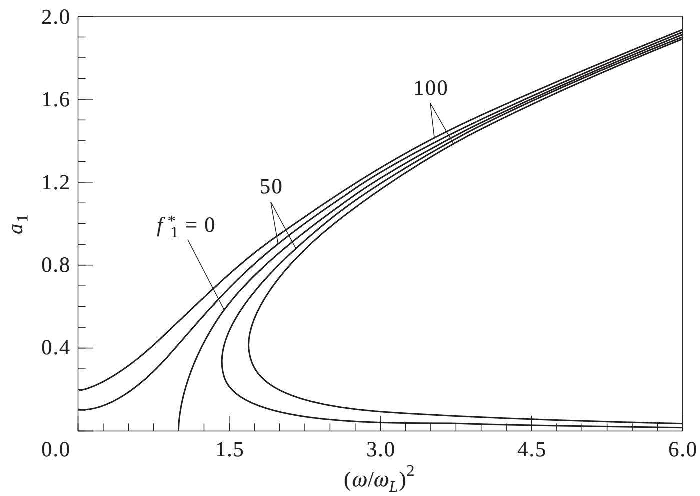

Figure 3. Forced vibration of a C–C beam (equation (35)).

summarised in Tables 1–3 and compared with published results as will be discussed later. The qualitative behaviour obtained in Figures 2 and 3 is characteristic for non-linear frequency response functions of systems with a cubic non-linearity. It includes multivalued regions corresponding to the jump phenomena occurring in non-linear frequency response testing. This behaviour has been investigated both theoretically and experimentally by many authors.

3.1.1. *Comparison with numerical results*

Published numerical results for the non-linear forced response of beams subjected to a harmonic excitation are quite rare. In this subsection, the cases of harmonic uniform distributed and concentrated force are considered and a comparison is made between numerical results available, based on the FEM [29] and on an elliptic solution [9]. In the works mentioned above, the deflection was written in the form

$$W(x, t) = R\mathscr{A}w(x)q(t), \tag{36}$$

in which $\mathscr{A}$ is the maximum amplitude parameter, R is the radius of gyration, $R = \sqrt{I/A}$, $w(x)$ and $q(t)$ are spatial and time functions normalised in such a manner that $w_{max} = q_{max} = 1$. According to these notations, the dimensional maximum amplitude w_{max}, obtained in the middle of the beam, is given by: $w_{max} = \mathscr{A}R$. In the present work, according to the notation developed in reference [1], $W(x, t) = ha_i w_i^*(x^*)\cos(\omega t)$. So, when only symmetric modes are excited, w_{max} is obtained at the middle of the beam and is given by $w_{max} = ha_i w_i^*(1/2)$. If only one mode is considered $w_{max} = ha_1 w_1^*(1/2)$. The comparison of these equations leads to $\mathscr{A}R = ha_i w_i^*(1/2)$.

For a beam with a rectangular cross-section, $(h^2A/I = 12)$, the amplitude $\mathscr{A}$ is given in the 1-D case by

$$\mathscr{A} = (h/R)a_1 w_1^*(1/2) = \sqrt{12}a_1 w_1^*(1/2) \tag{37}$$

Substituting the expressions for a_1 and f_1^* given in equations (37) and (31) into the 1-D NFRF (32) leads to

$$(\omega^*/\omega_L^*)^2 = 1 + \tfrac{1}{8}(b_{1111}^*/k_{11}^*)\mathscr{A}^2/[w_1^*(1/2)]^2 - \mathscr{F}/\mathscr{A} \tag{38}$$

where $\mathscr{F}$ is defined in the case of a distributed and a concentrated harmonic force respectively by:

$$\mathscr{F}^d = F^d \frac{1}{k_{11}^*}\frac{L^4}{EIR} w_1^*(1/2)\int_0^1 w_1^*(x^*)\,\mathrm{d}x^*, \qquad \mathscr{F}^c = F^c \frac{1}{k_{11}^*}\frac{L^3}{EIR} w_1^*(1/2) w_1^*(x_0), \tag{39.1, 2}$$

in which F^d and F^c are related to the numerical non-dimensional excitation parameters f_i^{*c} and f_i^{*d} by equations (31.1, 2). Computing the coefficients of equation (38) and equation (39) (for $x_0^* = 1/2$) for each boundary condition case leads to analytical formulations as:

simply supported beam

$$\left(\frac{\omega^*}{\omega_L^*}\right)^2 = 1 + \tfrac{3}{16}\mathscr{A}^2 - \frac{\mathscr{F}}{\mathscr{A}}, \qquad \mathscr{F}^d = \frac{4}{\pi^5}\frac{L^4}{EIR}F^d, \qquad \mathscr{F}^c = \frac{2}{\pi^4}\frac{L^3}{EIR}F^c. \tag{40}$$

clamped–clamped beam

$$\left(\frac{\omega^*}{\omega_L^*}\right)^2 = 1 + 0{\cdot}0449558\mathscr{A}^2 - \mathscr{F}/\mathscr{A},$$

$$\mathscr{F}^d = 0{\cdot}00263608(L^4/EIR)F^d, \qquad \mathscr{F}^c = 0{\cdot}0053873(L^3/EIR)F^c. \tag{41}$$

Equation (40) is identical to that generally obtained [14]. However, for equation (41), there are small discrepancies with some authors. This is due to the different choices of the spatial function. For example, Dumir and Bhaskar [14] used a transverse displacement function $w(x) = \mathscr{A}\sin^2(\pi x/L)$ for a C–C beam and obtained $(\omega^*/\omega_L^*)^2 = 1 + 0{\cdot}046875\mathscr{A}^2 - \mathscr{F}/\mathscr{A}$, $\mathscr{F}^d = 0{\cdot}002566(L^4/EIR)F^d$, $\mathscr{F}^c = 0{\cdot}005133(L^3/EIR)F^c$.

Tables 1 and 2 show the frequency ratios for a S–S beam under a harmonic distribution force and a harmonic concentrated force at the centre respectively. Comparison is made between results obtained by the multidimensional model presented in section 2 using six basic functions, results obtained using the 1-D NFRF given by equation (40) and results obtained in references [9] and [29] using an elliptic solution and the finite element method respectively.

Table 3 shows the frequency ratios for a C–C beam under a harmonic distributed force. It is seen clearly that there are small discrepancies between numerical results obtained by the multidimensional model and the 1-D solution (41) at the maximum of amplitude. This is due to the contributions of higher modes for which more details will be given and discussed in part II of this series of papers.

3.2. NON-LINEAR FREE VIBRATIONS

In this subsection, non-linear free vibration of beams with various boundary conditions is considered. Attention is focused on the amplitude frequency dependence, i.e., the backbone curve, obtained using different techniques of modelling and solution. The validity of the solutions obtained is discussed. A mathematical technique, based on the Padé approximants is proposed in order to increase the range of amplitude in which the power series expansion obtained from the exact elliptical solution can be used. Then, a survey is made for comparison purposes of the numerical results available in the literature for S–S and C–C beams.

3.2.1. *Exact solution and first approximation*

Applying the one mode assumption to equation (12) and putting $\omega t = \tau$, leads to

$$m_{11}\omega^2 q_{1,\tau\tau} + k_{11} q_1 + 2b_{1111} q_1^3 = 0. \tag{42}$$

This equation can be written as:

$$q_{1,\tau\tau} + (\omega_L/\omega)^2[q_1 + \beta q_1^3] = 0, \tag{43}$$

where $\beta = 2b_{1111}/k_{11}$.

One assumes that the amplitude of $q_1(\tau)$ is equal to a_1 at $\tau = 0$ and $q_{1,\tau}(0) = 0$. It is well known [9–11] that the exact mathematical solution of (43) can be given in term of the Jacobean elliptic function Cn. The exact form of this solution is:

$$q_1(\tau) = a_1 \,\mathrm{Cn}\,(\gamma\tau, k), \qquad \gamma^2 = (\omega_L/\omega)^2[1 + \beta a_1^2], \qquad k^2 = (\omega_L/\omega)^2(\beta/2\gamma^2)a_1^2, \tag{44.1–3}$$

in which k is the modulus of the elliptic function and γ may be taken as the "circular frequency". The elliptic function Cn is periodic with a period of 4K(k),

TABLE 2

Forced vibration frequency ratio ω/ω_L for a simply supported beam under a concentrated harmonic force at the centre, $\mathscr{F}^c = 0{\cdot}5\pi$

$\mathscr{A}^G = \sqrt{12 a_i w_i^*}(1/2)$	Multi-D present model† (30)	$\mathscr{A} = a_1\sqrt{12}$	1-D present model (40)	Elliptic solution [9, 29]	F.E.M. + linearisation‡ [29]
−1·00002	1·6529432	−1	1·6608119	1·6607	1·6425
3·000025	1·4623273	3	1·4710205	1·4519	1·2717
−3·000003	1·7974729	−3	1·7921954	1·7815	1·6326
5·000023	2·3112625	5	2·3180467	2·2801	2·9621
−5·000015	2·4559827	−5	2·4498284	2·4179	2·1165

† f_i^* is given in Appendix A3; ‡ the effects of longitudinal and rotary inertia are considered.

TABLE 3

Forced vibration frequency ratio ω/ω_L for a clamped–clamped beam under uniform harmonic distributed force, $\mathscr{F}^d = 1$

$\mathscr{A}^G = \sqrt{12 a_i w_i^*(1/2)}$	Multi-D present model† (30)	$\mathscr{A} = \sqrt{12 a_1 w_1^*(1/2)}$	1-D present model (41)	Elliptic solution [9, 29]	F.E.M. + linearisation [29]
1·00000	0·3034762	1	0·2120278	0·2118	0·2091
−1·00373	1·4495721	−1	1·4300195	1·4307	1·4297
2·00001	0·8300614	2	0·8245139	0·8279	1·8203
−2·00417	1·3023173	−2	1·2960799	1·2987	1·2936
3·00003	1·0340077	3	1·0350212	1·0401	1·0239
−3·00425	1·3208783	−3	1·3183078	1·3232	1·3099
4·00004	1·2066750	4	1·2121439	1·2183	1·1888
−4·00105	1·4017724	−4	1·4033149	1·4101	1·3836
5·00004	1·3763271	5	1·3870454	1·3938	1·3457
−5·00823	1·5184193	−5	1·5244327	1·5322	1·4874

† f_i^* is given in Appendix A3.

where K is the complete elliptic integral of the first kind. The period on τ of $q_1(\tau)$ is $T = (4/\gamma)\mathrm{K}(k)$ with

$$\mathrm{K}(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2 \theta}}. \tag{45}$$

The assumption that the solution $q_1(\tau)$ is periodic with a period of 2π gives $\gamma = (2/\pi)\mathrm{K}(k)$. The final form of the exact solution of (43) is

$$q_1(\tau) = a_1 \,\mathrm{Cn}\,(\gamma\tau, k), \qquad (\omega/\omega_L)^2 = (\pi^2/4)(1 + \beta a_1^2)1/[\mathrm{K}(k)]^2, \tag{46.1, 2}$$

$$k^2 = \beta a_1^2/(2 + 2\beta a_1^2) \quad \text{where} \quad \beta = 2b_{1111}/k_{11}, \qquad \gamma = (\omega_L/\omega)\sqrt{1 + \beta a_1^2}. \tag{46.3, 4}$$

In contrast to the harmonic balance method the computations presented in this section are exact. However, these formulations are intractable and unenlightening. Using a perturbation method these formulations will be replaced by more useful equations.

For small values of a_1, the modulus k is also small. $\mathrm{Cn}\,(\gamma\tau, k)$ can be approximated by [38]

$$\mathrm{Cn}\,(\gamma\tau, k) = \cos\,(\pi\gamma\tau/2K) = \cos\,(\tau) = \cos\,(\omega t). \tag{47}$$

Expanding the elliptic function $\mathrm{K}(k)$ as a power series of a_1 equation (46.2) gives

$$(\omega/\omega_L)^2 = 1 + \tfrac{3}{4}\,\beta a_1^2 - \tfrac{3}{128}\,\beta^2 a_1^4 + \mathcal{O}(a_1^6) \tag{48}$$

Finally, the first approximation of the exact solution, obtained by truncating the above series at the second order is given by

$$q_1(t) = a_1 \cos\,(\omega t), \qquad (\omega/\omega_L)^2 = 1 + \tfrac{3}{2}(b_{1111}/k_{11})a_1^2 \tag{49.1, 2}$$

It appears that the solution obtained in the present subsection from the first approximation of the elliptic function solution, i.e., equation (49.2), is identical to that obtained in the present work in equation (32), based on the harmonic balance method applied to the 1-D free vibrations.

3.2.2. *Improvement of the solution based on the Padé approximants*

There are various methods for increasing the accuracy of the frequency–amplitude relationship (49.2). For example the multiscale method [20, 21], the intrinsic balance harmonic technique [15] or the iterative method proposed in [13] may be used. These methods give the frequency in the form of a power series expansion of a perturbed parameter related to the amplitude of vibration. The objective of the present subsection is to use a power series expansion of the exact solution (46.2) in order to show that *one must be extremely careful in the choice of the polynomial approximation because of the divergence of the solution obtained outside the zone of convergence*. The power series expansion of the exact solution (46.2) at higher orders can be easily done using the symbolic manipulation programs "Maple". But the validity of this expansion is limited by the radius of convergence ($\mathcal{R} = 1/\sqrt{\beta}$) due to the existence of the singularity at $|a_1| = 1/\sqrt{\beta}$.

Thus, the solution obtained is valid only in the zone of convergence. In the remainder of this section, the power series expansion of the exact solution is given up to the order 22. The limitation due to the divergence of the series outside the zone of convergence is discussed. The power series expansion of the exact solution (46.2) given by the symbolic manipulation program up to the order 22 is:

$$\begin{aligned}\left(\frac{\omega}{\omega_L}\right)^2 = {} & 1 + \tfrac{3}{4}\beta a_1^2 - \tfrac{3}{128}\beta^2 a_1^4 + \tfrac{9}{512}\beta^3 a_1^6 - \tfrac{1779}{131072}\beta^4 a_1^8 + \tfrac{5643}{524288}\beta^5 a_1^{10} \\ & - \tfrac{146661}{16777216}\beta^6 a_1^{12} + \tfrac{486603}{67108864}\beta^7 a_1^{14} - \tfrac{841910643}{137438953472}\beta^8 a_1^{16} + \tfrac{2890461807}{549755813888}\beta^9 a_1^{18} \\ & - \tfrac{80479468611}{17592186044416}\beta^{10} a_1^{20} + \tfrac{283412131281}{70368744177664}\beta^{11} a_1^{22} + \mathcal{O}(a_1^{24}). \end{aligned} \tag{50}$$

This result is the same as that obtained by Ben Saadi [39] using the Poincaré–Lindstedt method. The latter method necessitated the solution of a differential equation at each order of a_1.

Representation at a higher order of (46.2) can be easily achieved and the approximation at lower orders is given by truncation of (50) at the desired order. For numerical tests, it is necessary to specify the parameter β. This parameter depends on the type of structure and boundary conditions considered. In Figure 4, is presented a comparison of the exact solution (46.2) computed numerically and different truncated solutions of equation (50) in the case of a S–S beam ($\beta = 3$). (For clarity, only the orders 2, 4, 6, 20 and 22 are represented.)

It can be seen clearly that, increasing the order of the series does not increase the validity of the solution because of the divergence beyond the radius of convergence. It appears also in this case that, the second order approximated solution (49.2) is *the best one* because it remains very close to the exact solution over a large range of amplitudes.

This test shows that the users of perturbation methods have to be careful concerning the divergence of the solution obtained at higher orders. Fortunately, there are some powerful mathematical methods for recovering a more accurate approximation of the exact solution using only a few terms of the expansion series. These methods are crucial since they justify the use of perturbation methods, which otherwise would be effective only for local analyses. For this purpose, the so called Padé approximants [40, 41] are used in this subsection. These approximants have been tested for increasing the range of validity of the solution obtained by the Asymptotic–Numerical Method [42, 43]. They have been also used for improving the perturbed solution obtained by the Poincaré–Lindstedt method for Duffing and Van der Pol equations [39]. Other procedures, like the projection technique and Euler's transformation have been tested and gave a very large zone of validity of the solution [39]. All of the above mentioned studies showed the effectiveness of this technique in increasing the zone of validity of solutions obtained by perturbation methods. The Padé approximant, denoted $P[M, N]$, is the quotient of two polynomials of degree M and N

respectively. The coefficients of these polynomials are chosen, so that the first $(M + N + 1)$ terms in the Taylor series expansion of $P[M, N](a_1)$ match the first $(M + N + 1)$ terms of the power series (50). The construction of $P[M, N]$ involves only algebraic operations or a simple sequence of matrix operations [39–43]. Each choice of M, degree of the numerator and N, degree of the denominator, leads to an approximant. The major difficulty in applying this technique is how to direct the choice in order to obtain the best approximants. This needs the use of a criterion for the choice depending on the shape of the solution. A criterion which has worked well here is the choice of $P[M, N]$ such that $M - N = 2$. Presented here are just two approximants giving a very good accuracy, as will be shown later, from comparison with the exact solution at large amplitudes.

$$(\omega/\omega_L)^2 = P[4, 2](a_1) = \tfrac{1}{32}(128 + 192\beta a_1^2 + 69\beta^2 a_1^4)/(4 + 3\beta a_1^2), \qquad (51.1)$$

$$(\omega/\omega_L)^2 = P[6, 4](a_1) = \frac{1}{4}\frac{4096 + 9216\beta a_1^2 + 6748\beta^2 a_1^4 + 1605\beta^3 a_1^6}{1024 + 1536\beta a_1^2 + 559\beta^2 a_1^4}. \qquad (51.2)$$

These relationships can be used for determining the amplitude–frequency dependence at large vibration amplitudes for various beam boundary conditions. The parameter β, which depends on the case considered, has to be computed and replaced in equations (51.1) and (51.2). These frequency–amplitude relationships can be easily incorporated in various computing programes for large vibration amplitudes.

3.2.3. *Numerical results for free vibrations*

Simply supported beam

For the application of equations (51.1, 2) to a given structure, one has to specify the parameter β. For S–S beams $\beta = 3$, which gives:

$$(\omega/\omega_L)^2 = P[4, 2](a_1) = (1 + \tfrac{9}{2}a_1^2 + \tfrac{621}{128}a_1^4)/(1 + \tfrac{9}{4}a_1^2), \qquad (52.1)$$

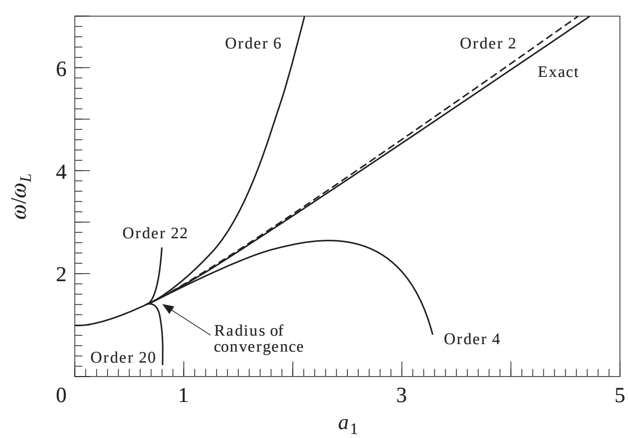

Figure 4. Comparison of the exact solution (46.2) and the power series expansion of the assumed one mode solution of free vibration of a simply supported beam, $\beta = 3$.

$$(\omega/\omega_L)^2 = P[6,4](a_1) = \frac{1 + \frac{27}{4}a_1^2 + \frac{15183}{1024}a_1^4 + \frac{43335}{4096}a_1^6}{1 + \frac{9}{2}a_1^2 + \frac{5031}{1024}a_1^4}. \qquad (52.2)$$

It has been shown in Figure 4, that the series expansion (50) diverges from the exact solution for values of a_1 greater than the radius of convergence, $\mathscr{R} = 1/\sqrt{3}$, except if the series is truncated at the second order. It will be demonstrated here that, even with $P[4, 2]$, which needs only the coefficients of (50) up to a_1^6, the approximated solution practically coincides with the exact solution very far from this limit. In order to show the very small discrepancies, the numerical results are summarised in Table 4. It appears that, for an amplitude up to 10, the difference between the exact solution and the approximations does not exceed 0·07% for $P[4, 2]$, and 0·003% for $P[6, 4]$.

The formulae (51.1) and (51.2) can be used to approximate the non-linear amplitude frequency dependence of free vibrations with a very good agreement with the exact 1-D solution expressed in terms of elliptic integrals. This technique appears to be very useful in increasing the zone of validity of the perturbed series approximations.

Now, in order to make comparisons with results obtained by various authors, one has to adapt the usual notations of the maximum amplitude ratio as was done in the forced case. For a simply supported beam $\mathscr{A}^2 = 12a_1^2$ and $\beta = 3$, equations (46) become:

$$\omega/\omega_L = (\pi/2)\sqrt{1 + \mathscr{A}^2/4}(1/\mathrm{K}(k)), \qquad k^2 = 0{\cdot}\mathscr{A}^2/(4 + \mathscr{A}^2). \qquad (53)$$

$$(\omega/\omega_L)^2 = 1 + \tfrac{3}{16}\mathscr{A}^2, \qquad (\omega/\omega_L)^2 = P[4,2](\mathscr{A}) = (1 + \tfrac{3}{8}\mathscr{A}^2 + \tfrac{69}{2048}\mathscr{A}^4)/1 + \tfrac{3}{16}\mathscr{A}^2, \qquad (54.1, 2)$$

TABLE 4

Comparison of the forced vibration frequency ratio ω/ω_L for the exact elliptic solution (46.2) *with the improved approximate solutions obtained by Padé approximants $P[4, 2]$ and $P[6, 4]$. Case of a S–S beam, $\beta = 3$*

a_1	Exact solution (46.2)	Padé approx. $P[4, 2]$ (52.1)	Padé approx. $P[6, 4]$ (52.2)	Order 2 (49.2)
0·2	1·043882323178	1·0438824	1·0438823	1·0440307
0·4	1·164483218851	1·1644868	1·1644832	1·1661904
0·6	1·339703729830	1·3397374	1·3397039	1·3453624
0·8	1·550554296213	1·5506741	1·5505555	1·5620499
1	1·784419122151	1·7846838	1·7844228	1·8027756
1·5	2·425402399966	2·4261814	2·4254185	2·4622145
2	3·107093328080	3·1084562	3·1071263	3·1622777
2·5	3·807969392722	3·8099164	3·8080203	3·8810437
3	4·519202506028	4·5217205	4·5192713	4·6097722
3·5	5·236606555137	5·2396823	5·2366928	5·3443896
4	5·957966013310	5·9615888	5·9580694	6·0827625
4·5	6·682005983284	6·6861674	6·6821262	6·8236720
5	7·407943518308	7·4126371	7·4080803	7·5663730
10	14·70974280983	14·719595	14·710038	15·033296

TABLE 5

Frequency ratios (ω/ω_L) of the non-linear free vibration of a S–S beam at various amplitudes

	$\mathscr{A}$		
	1	2	3
Elliptic solution (53)	1·089158179	1·317776064	1·625676616
Padé approx. P[6, 4] (54.3)	1·089158179	1·317776212	1·625678474
Padé approx. P[4, 2] (54.2)	1·0891586	1·3178039	1·6258383
Order 2 (54.1)	1·089724736	1·322875656	1·639359631
Ritz method [25]	1·0897	1·3229	1·6394
F.E.M. + D.N.I.M. [22]	1·0892	1·3178	1·6257
F.E.M. + linearization [27]	1·0888	1·3119	1·6022
F.E.M. + linearization [16]	1·0889	1·3183	1·6260

$$(\omega/\omega_L)^2 = P[6,4](\mathscr{A}) = (1 + \tfrac{9}{16}\mathscr{A}^2 + \tfrac{1687}{16384}\mathscr{A}^4 + \tfrac{1605}{262144}\mathscr{A}^6)/(1 + \tfrac{3}{8}\mathscr{A}^2 + \tfrac{559}{16384}\mathscr{A}^4) \tag{54.3}$$

In the case of simply supported beams, the numerical resolution of the non-linear algebraic system (20) developed in section 2 of the present paper gives exactly the same results as that obtained by order 2 (54.1). In Table 5, comparison is made with the present results and those obtained by various authors. It appears that, the agreement with these results is excellent.

Clamped–clamped beams

In the case of a clamped–clamped beam,

$$\left(\frac{\omega_L}{\omega}\right)^2 = \frac{\pi^2}{4}\left(1 + \frac{\beta\mathscr{A}^2}{12w_1^*(1/2)^2}\right)\frac{1}{\mathrm{K}(k)^2}, \qquad k^2 = \frac{\beta\mathscr{A}^2}{24w_1^*(1/2)^2 + 2\beta\mathscr{A}^2} \tag{55.1, 2}$$

In this case, $\beta = 1{\cdot}81420731$ and $w_1^*(1/2) = 1{\cdot}588146262$. The Padé approximants (51.1, 2) lead to

$$(\omega/\omega_L)^2 = 1 + 0{\cdot}0449558213\mathscr{A}^2, \tag{56.1}$$

$$(\omega_L/\omega)^2 = P[4,2](\mathscr{A}) = \frac{1 + 0{\cdot}0899116426\mathscr{A}^2 + 0{\cdot}0019368164\mathscr{A}^4}{1 + 0{\cdot}0449558213\mathscr{A}^2} \tag{56.2}$$

$$(\omega_L/\omega)^2 = P[6,4](\mathscr{A}) = \frac{1 + 0{\cdot}13486746\mathscr{A}^2 + 0{\cdot}00591922\mathscr{A}^4 + 0{\cdot}000084389\mathscr{A}^6}{1 + 0{\cdot}08991164\mathscr{A}^2 + 0{\cdot}001961377\mathscr{A}^4}. \tag{56.3}$$

In Table 6 the frequency ratios are given at various amplitudes for a clamped–clamped beam. It may be noticed from this table that the results obtained by using the present method match very well with those obtained by the Ritz method [25] and by the finite element method and a direct integration method

TABLE 6

Frequency ratio (ω/ω_L) *at various amplitudes for the non-linear free vibration of a clamped–clamped beam. Comparison with results given by formulations presented here and by various authors*

$\mathscr{A}^G$	Multi-D present model (20)	$\mathscr{A}$	Elliptic solution Eq. (55)	Padé approx. $P[6, 4]$ (56.3)	Padé approx. $P[4, 2]$ (56.2)	Order 2 (56.1)	Ritz method [25]	F.E.M. + D.N.I.M.† [22]	Perturbation method [19]	F.E.M. + linearizat. [28]	Analytical method [17]
0·200042	1·0008990	0·2	1·0008986	1·0008986	1·0008986	1·0008987	1·0009	1·0009	1·0009	1·0012	–
0·400037	1·0035899	0·4	1·0035889	1·0035889	1·0035889	1·0035900	1·0036	1·0036	1·0036	1·0048	–
0·60004	1·0080568	0·6	1·0080542	1·0080542	1·0080542	1·0080596	1·0080	1·0080	1·0080	1·0107	–
0·800004	1·0142720	0·8	1·0142673	1·0142673	1·0142673	1·0142838	1·0142	1·0142	1·0143	1·0190	–
1·000034	1·0222035	1	1·0221914	1·0221914	1·0221914	1·0222308	1·0222	1·0221	1·0222	1·0295	1·0628
1·50004	1·0492201	1·5	1·0491727	1·0491727	1·0491727	1·0493572	1·0492	–	1·0494	1·0650	1·1322
2·00003	1·0857842	2	1·0856705	1·0856705	1·0856709	1·0861967	1·0858	1·0854	1·0863	1·1127	1·2140
2·500016	1·1308605	2·5	1·1306645	1·1306645	1·1306662	1·1318012	1·1308	–	1·1319	1·1708	1·3017
3·000007	1·1833628	3	1·1831037	1·1831037	1·1831087	1·1851592	1·1833	1·1825	1·1853	1·2377	1·3904
3·500001	1·2422372	3·5	1·2419860	1·2419861	1·2419983	1·2452746	1·2422	–	1·2455	1·3119	1·4786
4·00001	1·3065226	4	1·3064031	1·3064032	1·3064281	1·3112182	1·3064	1·3055	1·3115	1·3920	1·5635
4·500018	1·3753698	4·5	1·3755569	1·3755572	1·3756015	1·3821560	1·3751	–	1·3825	1·4770	1·6418
5·00004	1·4480584	5	1·4487605	1·4487611	1·4488323	1·4573591	1·4476	1·4474	1·4577	1·5659	–
–	–	10	2·3106712	2·3106847	2·3113526	2·3442658	–	–	–	–	–
–	–	20	4·2723167	4·2723793	4·2746387	4·3568714	–	–	–	–	–

† Finite element method and direct numerical integration method in time.

[22]. All of these results agree very well with each other except those obtained in [17] which are overestimated.

4. CONCLUSIONS

A semi-analytical approach to the non-linear dynamic response problem has been developed, based on Lagrange's principle and the harmonic balance method. The theory effectively reduces the dynamic problem to a set of non-linear algebraic equations depending on classical rigidity and mass matrices, and a fourth order tensor due to the non-linearity. A mathematical formulation has been presented for free and forced vibration of beams. By choosing the convenient basic functions in each case, numerical results can be obtained for various forcing systems and boundary conditions. This approach has been applied to determine the amplitude–frequency dependence (i.e., the backbone curve) for non-linear free and forced vibration of S–S and C–C beams. The effectiveness of this method will be extensively presented for various forcing and boundary conditions of beams in part II.

It is well known that when the non-linear coupling is weak, it is possible to obtain quite accurate response curves by neglecting the non-linear coupling terms and considering each mode individually for response prediction. For that, the 1-D analysis is largely presented here for free and forced cases. The dynamic problem is reduced to a 1-D Duffing equation which has been extensively studied in the literature. However, many authors use perturbation methods to increase the accuracy of their results. These techniques must be used with some caution because the validity of the solution is limited by a radius of convergence. A mathematical technique based on Padé approximants is proposed in order to increase the range of amplitudes in which power series expansion can be used. Some formulations of the non-linear frequency response functions are given for S–S and C–C beams. It is shown that the numerical results obtained by the formulations presented here coincide perfectly with the exact solution obtained by elliptic functions in the free case. This concept is very helpful when the higher order of accuracy is needed for very large amplitudes.

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APPENDIX A

A.1. SIMPLY SUPPORTED BEAMS

The linear mode shapes of a simply supported beam are given by: $w_i(x) = h \sin(i\pi x/L) = h \sin(i\pi x^*) = hw_i^*(x^*)$. Using equations (11) and (19) one can easily obtain [6]

$$m^*_{ij} = \delta ij/2, \qquad k^*_{ij} = i^2 j^2 \pi^4\, \delta ij/2, \qquad b^*_{ijkl} = \alpha_{ijkl}\pi^4\, \delta ij\, \delta kl/4,$$

where $\alpha = Ah^2/4I$ and δij is Kronecker's symbol ($\alpha = 3$ for a beam of rectangular cross-section).

A.2. CLAMPED–CLAMPED BEAM

As presented in references [1, 4], the chosen basic functions $w_i(x)$ were the linear clamped–clamped functions.

$$w_i(x) = \frac{\cosh(v_i x/L) - \cos(v_i x/L)}{\cosh(v_i) - \cos(v_i)} - \frac{\sinh(v_i x/L) - \sin(v_i x/L)}{\sinh(v_i) - \sin(v_i)},$$

where the constants v_i are the eigenvalue parameters for a C–C beam given by solving numerically the equation $\cosh(v_i)\cos(v_i) = 1$. The functions $w_i(x)$ were normalised in such a manner that

$$m^*_{ij} = \int_0^1 w^*_i(x) w^*_j(x)\,\mathrm{d}x = \delta_{ij}.$$

In this case the coefficients of equation (32) are given numerically by:

$$m^*_{11} = 1,\ k^*_{11} = 500{\cdot}5639,\ b^*_{1111} = 454{\cdot}06334,\ w^*_1(1/2) = 1{\cdot}588146262,\ (\alpha = 3).$$

A.3. CONCENTRATED AND UNIFORMLY DISTRIBUTED FORCES, f^*_i

For numerical resolution of equation (30) and comparisons with the work of various authors, the dimensionless generalised forces f^*_i are

$$f^{*c}_i = F^c(k^*_{11}/\sqrt{12} w^*_1(1/2) w^*_1(x_0)) w^*_i(x_0),$$

$$f^{*d}_i = F^d(k^*_{11}/\sqrt{12} w^*_1(1/2) \int_0^1 w^*_1(x)\,\mathrm{d}x) \int_0^1 w^*{}_i(x)\,\mathrm{d}x,$$

where F^c and F^d are given.

APPENDIX B: NOTATION

$u(x, t)$, $W(x, t)$	axial and transverse displacements at point x on the beam
ϵ_x, K_x	axial strain, curvature
N_x, M	axial resultant force, bending moment
E, h	Young's modulus of a beam, the thickness of the beam
ρ	mass per unit length of the beam
L, A, I	length, area and second moment of area of cross-section of the beam
$W(x, t)$	transverse displacement at point x on the beam
V_b, V_a, V	bending, axial and total strain energy respectively, $V = V_b + V_a$
T	kinetic energy
q_i	generalised co-ordinate $q_i(t) = a_i \cos(\omega t)$
$w_i(x)$	ith mode of the beam
$\{\mathbf{A}\}$	column matrix of basic function contributions to the forced response $\{\mathbf{A}\}^{\mathrm{T}} = \{a_1, \ldots, a_n\}$

ω_L	linear natural frequency corresponding to the one mode assumed
k_{ij}, m_{ij}, b_{ijkl}	general term of the rigidity tensor, the mass tensor and the non-linearity tensor, respectively
k_{11}, m_{11}, b_{1111}	rigidity, mass and non-linearity parameters corresponding to the one mode assumed, respectively
$[\mathbf{K}]$, $[\mathbf{M}]$, $[\mathbf{B}]$	rigidity, mass and non-linearity matrices, respectively
k^*_{ij}, m^*_{ij}, b^*_{ijkl}, f^*_i	general term of the non-dimensional rigidity tensor, mass tensor, non-linearity tensor and excitation, respectively
ω, ω^*	frequency and non-dimensional frequency parameter respectively
$\mathrm{F}(x, t)$, S	exciting force, range of application of the exciting force
$\{\mathbf{F}(t)\}$	column matrix of generalised forces $F_i(t)$
$\{\mathbf{f}\}$	column matrix of generalised force amplitudes in the harmonic excitation case: $\{\mathbf{F}(t)\} = \{\mathbf{f}\}\cos(\omega t)$ in the one point x excitation case: $\{\mathbf{f}\} = F_c\{\mathbf{w}_i(x)\}$ in the distributed excitation case: $\{\mathbf{f}\} = F_d\{\mathbf{w}_i(x)\}$
$\mathscr{A}$	maximum amplitude parameter
R	radius of gyration
$\mathscr{F}^d$, $\mathscr{F}^c$	distributed force and the concentrated force at x_0, respectively
Cn, K	Jacobean elliptic function and complete elliptic integral of the first kind
$\mathscr{R}$	radius of convergence of the expansion series
$P[M, N]$	Padé approximant