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ON THE VIBRATION ANALYSIS OF MULTI-BRANCH TORSIONAL SYSTEMS

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An analytical method for the vibration analysis of unidirectional and multi-branched torsional systems is investigated. In the study, an alternative power-flow graph model and the model operation technology are introduced for constructing the analytical graph model of the torsional system, including internal and external damping properties. Then, the formula for the frequency response of each component can be derived directly according to the power-flow graph model of the system and the topology synthesis. Based on the method developed, the rotation response at the end of each shaft as well as the internal rotation and internal torque response of the system can be efficiently calculated by computer due to the summation expression in the derived formula. Since the derived formula is expressed in operation, it is very effective to calculate by computer. Finally, some different types of torsional systems are investigated in the examples to illustrate the effectiveness of this method on the vibration analysis of complex torsional systems.

1. INTRODUCTION

The analysis of torsional vibration plays a crucial roll in the design of shaft and rotor systems, which are widely used in the marine, aeronautical and mechanical engineering fields [1–3]. One of the typical objectives for the analysis of torsional vibration is to determine the internal torque and rotation response in each segment of the shaft subject to a fluctuation in the power drive. For the lumped model consdered, this problem can be formulated by using a set of coupled differential equations, which can be solved simultaneously. The impedance method was developed by analogy to the dynamics of the electrical impedance of the system, but the method is also required to solve coupled series equations in computing the Thevenin's equivalent voltage. Moreover, the transfer matrix method was proposed for the analysis of the unidirectional shaft system by numerical computation [4, 5]. However, it is difficult to use this method in the analysis of complex multi-branch torsional systems, which are very commonly used in practical applications. Based on the transfer matrix method, Munjal *et al*. [6] developed a method and some subprograms for the calculation of the velocity ratio using the analogous circuit model, but the processes are still

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complicated especially when used in multi-branch torsional system. Recently, a power-flow model has been introduced for the transmissibility analysis of isolation systems [7]. This model may be conveniently applied to the torsional analysis for unidirectional systems but is very difficult to apply to multi-branch systems due to the complexity of model construction for the total system.

In this paper, an alternative power-flow graph model is especially proposed for the torsional analysis of a complex multi-branch torsional system. Moreover, a modified model is introduced for easy connection between the branched shaft and main shaft. In order to express the formula in a standard form for a general multi-branched system, a model reduction technology is used to reduce the original complex model to a simple series connection type. Then the solution can be written down directly from the configuration of the system. Finally, some examples are provided for different types of torsional systems. The frequency responses of the internal torque and rotation angle of the systems are analyzed to show the performance of this method.

2. ANALYSIS FOR UNIDIRECTIONAL SYSTEMS

A general case of the unidirectionally torsional system with shafts connected in only one series is firstly considered as shown in Figure 1, in which the positive torques and positive rotation angles are indicated on the positive face by arrows pointing positively according to the right-hand screw rule. If a harmonic torque acts on the left of the disk 0, the response of the internal torque and rotation motion are all harmonic with the same frequency but having a different phase with respect to the excitation. Using the exponential form, the dynamic equations of the nth subsystem, including a massless shaft n and a lumped disk n, may be expressed by

$$\mathbf{\Gamma}_n = (\mathrm{j}b_n\omega + k_n)(\mathbf{\Theta}_{n-1} - \mathbf{\Theta}_n), \quad \mathbf{\Theta}_n = (1/-J_n\omega^2 + \mathrm{j}c_n\omega)(\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1}), \qquad (1, 2)$$

where J_n and c_n are the polar moment of inertia and the rotating damping of the disk n respectively, k_n and b_n are the torsional stiffness and the torsional damping of the shaft n respectively; $\mathbf{\Theta}_n$ is the complex amplitude of the rotation angle of the disk n; $\mathbf{\Gamma}_n$ is the complex amplitude of the internal torque in the shaft n. j is equal to $(-1)^{1/2}$. ω is the excitation frequency. The absolute value and the phase value of the complex amplitude correspond to the magnitude and phase difference of the variable to the excitation torque. The relationship between the variables of $\mathbf{\Theta}_n$, $\mathbf{\Theta}_{n-1}$ and $\mathbf{\Gamma}_n$ in equation (1) can be described by a

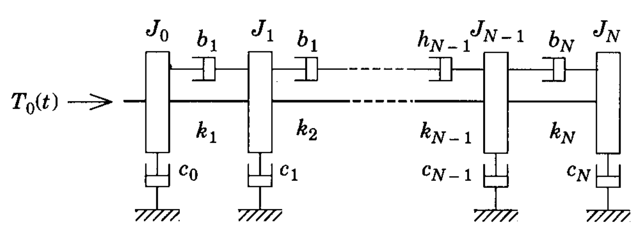

Figure 1. Generalized unidirectionally torsional system.

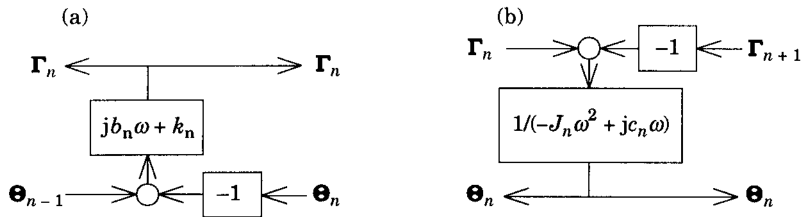

Figure 2. Power-flow model of the nth subsystem (a) for shaft n (b) for disk n.

two-way power-flow graph model as shown in Figure 2(a). From the graph model, one sees that $\mathbf{\Theta}_n$ and $\mathbf{\Theta}_{n-1}$ are the input from the left- and right-sides of the model individually and $\mathbf{\Gamma}_n$ is the output from the left- and right-sides of the graph model. In a similar way, equation (2) can be modelled as Figure 2(b), in which $\mathbf{\Gamma}_n$ and $\mathbf{\Gamma}_{n+1}$ are the input from the left- and right-sides, and $\mathbf{\Theta}_n$ is the output from the left- and right-sides of the model. One also finds that the input and output variables at the right-sides of Figure 2(a) match the output and input variables at the left-sides of Figure 2(b). So Figures 2(a) and (b) can be assembled to generate a combined power-flow graph model for the subsystem n, in which the variables $\mathbf{\Theta}_{n-1}$ and $\mathbf{\Gamma}_{n+1}$ are the input from the left- and right-sides of the model, and $\mathbf{\Gamma}_n$ and $\mathbf{\Theta}_n$ are the output from the left- and right-sides. In the same way, the power-flow graph model of the total system can be generated as a chain structure as shown in Figure 3, in which the variables I_n and S_n are defined as

$$S_n = jb_n\omega + k_n, \quad I_n = -J_n\omega^2 + jc_n\omega. \tag{3, 4}$$

by comparing Figures 3 and 1, one sees that the structure of the power-flow graph model is analogous to the physical model. The block I_n^{-1} of the graph model is analogous to the inertia and the rotating damping of the nth disk. The block S_n is analogous to the stiffness and torsional damping of the nth shaft. $2M$ flow loops are generated in the power-flow graph model of the total system. Moreover, only two flow loops pass through each shaft block S_n.

If the rotation angle $\mathbf{\Theta}_n$ is chosen as the output variable, only one forward flow path exits from the input variable $\mathbf{\Gamma}_0$ to the output. Then, the complex

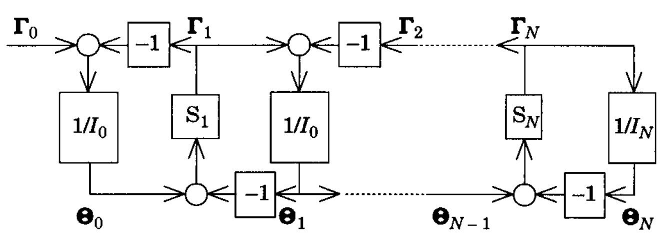

Figure 3. Power-flow model of Figure 1.

frequency function of the rotation angle can be calculated by the gain formula [7].

$$H_{T_0,\theta_n} = P_{T_0,\theta_n} D_{2n+2}/D_1 \tag{5}$$

where D_1 is the determinant of the total graph model. P_{T_0,θ_n} is the path gain of the forward path from $\boldsymbol{\Gamma}_0$ to $\boldsymbol{\Theta}_n$. D_{2n+1} is the cofactor of the forward path. Based on Figure 3, these parameters are given by

$$P_{T_0,\theta_0} = L_1 L_3 L_5 \cdots L_{2n-1} I_n^{-1}, \tag{6}$$

$$\begin{aligned} D_{2n+2} = 1 &+ \sum_{i=2n+2}^{2N} L_i + \sum_{i_2=2n+4}^{2N} \sum_{i_1=2n+2}^{i_2-2} L_{i_2} L_{i_1} + \cdots \\ &+ \sum_{i_{N-n-1}=2N-2}^{2N} \sum_{i_{N-n-2}=2N-4}^{i_{N-n-1}-2} \cdots \sum_{i_2=2n+4}^{i_3-2} \sum_{i_1=2n+2}^{i_2-2} L_{i_{N-n-1}} L_{i_{N-n-2}} \cdots L_{i_2} L_{i_1} + \prod_{i=n+1}^{N} L_{2i}, \end{aligned} \tag{7}$$

$$\begin{aligned} D_1 = 1 &+ \sum_{i=1}^{2N} L_i + \sum_{i_2=3}^{2N} \sum_{i_1=1}^{i_2-2} L_{i_2} L_{i_1} + \cdots + \sum_{i_{N-1}=2N=3}^{2N} \sum_{i_{N-2}=2N-5}^{i_{N-n-1}-2} \cdots \sum_{i_2=3}^{i_3-2} \sum_{i_1=1}^{i_2-2} L_{i_{N-1}} L_{i_{N-2}} \\ &\cdots L_{i_2} L_{i_1} + \sum_{i_N=2N-1}^{2N} \sum_{i_{N-1}=2N-3}^{i_{N-n}-2} \cdots \sum_{i_2-3}^{i_3-2} \sum_{i_1=1}^{i_2-2} L_{i_N} L_{i_{N-1}} \cdots L_{i_2} L_{i_1}, \end{aligned} \tag{8}$$

where L_i is the negative loop gain of loop i given by

$$L_i = S_{(i+1)/2}/I_{(i-1)/2} \quad \text{for} \quad i = 1, 3, 5, \ldots, \quad L_i = S_{i/2}/I_{i/2} \quad \text{for} \quad i = 2, 4, 6, \ldots. \tag{9, 10}$$

If the torque $\boldsymbol{\Gamma}_n$ is chosen as the output variable, there is only one forward flow path from $\boldsymbol{\Gamma}_0$ to $\boldsymbol{\Gamma}_n$. Then, the complex frequency function of each output variable leads to

$$H_{T_0,T_n} = P_{T_0,T_n} D_{2n+1}/D_1, \tag{11}$$

where

$$P_{T_0,T_n} = L_1\, L_3\, L_5 \cdots L_{2n-1}, \tag{12}$$

$$
\begin{aligned}
D_{2n+1} = 1 &+ \sum_{i=2n+1}^{2N} L_i + \sum_{i_2=2n+3}^{2N} \sum_{i_1=2n+1}^{i_2-2} L_{i_2} L_{i_1} + \cdots + \sum_{i_{N-n-1}=2N-3}^{2N} \sum_{i_{N-n-2}=2N-5}^{i_{N-n-1}-2} \\
&\cdots \sum_{i_2=2n+3}^{i_3-2} \sum_{i_1=2n+1}^{i_2-2} L_{i_{N-n-1}} L_{i_{N-n-2}} \cdots L_{i_2} L_{i_1} + \sum_{i_{N-n}=2N-1}^{2N} \sum_{i_{N-n-1}-2N-3}^{i_{N-n}-2} \\
&\cdots \sum_{i_2=2n+3}^{i_3-2} \sum_{i_1=2n+1}^{i_2-2} L_{i_{N-n-1}} L_{i_{N-n-2}} \cdots L_{i_2} L_{i_1}.
\end{aligned}
\tag{13}
$$

3. ANALYSIS FOR MULTI-BRANCH SYSTEMS

The general case of a torsional system with one main series of shaft and one branch series of shaft as shown in Figure 4 is considered. If the two-way power-flow model as shown in Figure 2 is directly applied to the analysis of the branched system, it is difficult to connect the power-flow graph model of the branched subsystem to that of the main system because the input and output variables of both models with respect to the connected point are not compatible. For this reason, the dyanmic equation of the connected element is rewritten as

$$\tilde{\mathbf{\Gamma}}_{p,0} = \mathbf{\Gamma}_{p,1} + I_{p,0}\mathbf{\Theta}_{p,0}, \tag{14}$$

where

$$I_{p,0} = -J_{p,0}\omega^2 + \mathrm{j}c_{p,0}\omega. \tag{15}$$

$\mathbf{\Theta}_{p,i}$ and $\mathbf{\Gamma}_{p,i}$ are the complex amplitude of the rotational angle of the disk p,i and the internal torque on the shaft p,i respectively. $\tilde{\mathbf{\Gamma}}_{p,0}$ is the complex amplitude of the torque offered by the driven gear in the branched shaft. $\mathbf{\Gamma}_{p,0}$ is the complex amplitude of the torque acting on the driving gear of the main shaft transferred from the driven gear $p,0$. If the gear ratio R_p is defined by the ratio

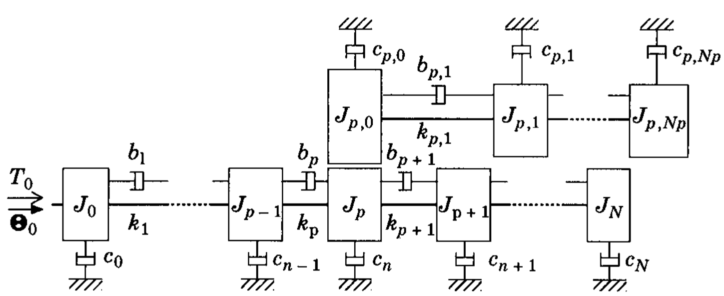

Figure 4. Generalized branched torsional system.

of the number of teeth of the driving gear p to that of the driven gear $p,0$, the relationship of the torque and rotation angle between both connected gears is given as

$$\mathbf{\Theta}_{p,0} = R_p \mathbf{\Theta}_p, \quad \mathbf{\Gamma}_{p,0} = R_p \tilde{\mathbf{\Gamma}}_{p,0}. \tag{16, 17}$$

Moreover, the dynamic equation of the driving gear is given as

$$\mathbf{\Theta}_p = (1/I_p)(\mathbf{\Gamma}_p - \mathbf{\Gamma}_{p+1} - \mathbf{\Gamma}_{p,0}). \tag{18}$$

Substituting equation (14) into equation (18) leads to

$$\mathbf{\Theta}_p = (1/I_{p,c}(\mathbf{\Gamma}_p - \mathbf{\Gamma}_{p+1} - R_p \mathbf{\Gamma}_{p,1})), \quad I_{p,c} = I_p + R_p^2 I_{p,0}, \tag{19, 20}$$

where $I_{p,c}$ is the equivalent inertia of the combination of the driving and driven gears. Based on equation (19), the power-flow graph model of the branched shaft can be connected to that of the main shaft to obtain the graph model for the total system as shown in Figure 5. Comparing Figures 5 and 4, one sees that the structure of the power-flow graph model of the torsional system is also similar to that of the physical system.

3.1. RESPONSE IN MAIN SHAFT

If the gain s directly applied to the calculation of the response for any internal torque or rotational angle, the formulation becomes complex and irregular especially for the system with many layers of branched series of shafts. A model reduction technology is introduced as follows to simplify the power-flow graph model of these complex systems to a standard structure. When the response of the internal torque and rotation angle in the main shaft is the objective to calculate, the transfer function from $\mathbf{\Theta}_p$ to $\mathbf{\Gamma}_{p,0}$ can be combined into the block $(I_{p,c})^{-1}$. The combined function of the branched shaft denoted by $(I'_p)^{-1}$ can be calculated by the gain formula given by

$$I'_p = I_{p,c} D_{p,1}/D_{p,2}, \tag{21}$$

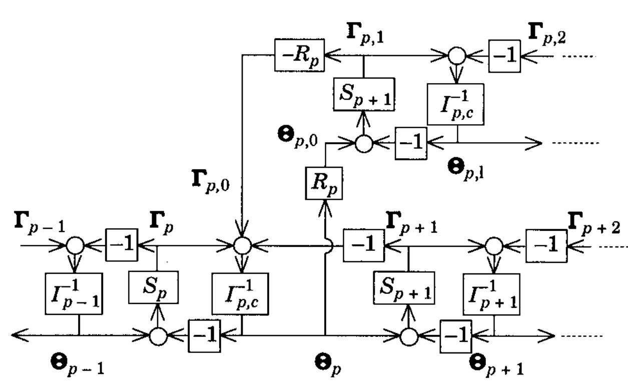

Figure 5. Power-flow model of Figure 4.

where

$$D_{p,2} = 1 + \sum_{i=2}^{2N_p} L_{p,i} + \sum_{i_2=4}^{2N_p} \sum_{i_1=2}^{i_2-2} L_{p,i_2} L_{p,i_1} + \cdots + \sum_{i_{N_p-1}=2N_p-2}^{2N_p} \sum_{i_{N_p-2}=2N_p-4}^{i_{N_p-1}-2}$$

$$\cdots \sum_{i_2=4}^{i_3-2} \sum_{i_1=2}^{i_2-2} L_{p,i_{N_p-n-1}} L_{p,i_{N_p-n-2}} \cdots L_{p,i_2} L_{p,i_1} + \prod_{i=1}^{N_p} L_{p,2i}, \tag{22}$$

$$D_{p,1} = 1 + \sum_{i=1}^{2N_p} L_{p,i} + \sum_{I_2=3}^{2N_p} \sum_{i_1=1}^{i_2-2} L_{p,i_2} L_{p,i_1} + \cdots + \sum_{i_{N_p-1}=2N_p-3}^{2N_p} \sum_{i_{N_p-2}2N_p-5}^{i_{N_p-1}-2}$$

$$\cdots \sum_{i_2=3}^{i_3-2} \sum_{i_1=1}^{i_2-2} L_{p,i_{N_p-1}} L_{p,i_{N_p-2}} \cdots L_{p,i_2} L_{p,i_1} + \sum_{i_{N_p}=2N_p-1}^{2N_p} \sum_{i_{N_p-1}=2N_p-3}^{i_{N_p}-2}$$

$$\cdots \sum_{i_2=3}^{i_3-2} \sum_{i_1=1}^{i_2-2} L_{p,i_{N_p}} L_{p,i_{N_p}} \cdots L_{p,i_2} L_{p,i_1} \tag{23}$$

where

$$L_{p,1} = R_p^2 S_{p,1}/I_{p,c}, \quad L_{p,i} = S_{p,i/2}/I_{p,i/2} \quad \text{for} \quad i = 2, 4, 6, \ldots, \tag{24, 25}$$

$$L_{p,i} = S_{p,(i+1)/2}/I_{p,(i-1)/2} \quad \text{for} \quad i = 3, 5, \ldots. \tag{26}$$

Then, the complex frequency response of the internal torque or the rotational angle in the main shaft can be calculated by equation (5) or (11), in which only the item I_p replaced by I'_p is required. If there are subbranches of shafts connected to the branch shaft, the power-flow graph model of the subbranches of shafts can be reduced and combined into the graph model of the branch shaft also using the model reduction scheme previously derived. Then the formula expressed in equation (21) can be used for the model reduction for the branch shaft, in which only the inertia functions for their connected disks are replaced by the combined inertia functions. When more than one branch shaft is connected to a disk of the main shaft, the model reduction work can be done in sequence.

3.2. RESPONSE IN BRANCH SHAFT

When the analysis of the response of the internal torque and rotation angle in the branch shaft is the objective, the power-flow graph model for the main shaft after the connected disk can be combined into the connected block $(I_{p,c})^{-1}$ denoted by $(I'_{p,0})^{-1}$, which can be calculated by

$$I'_{p,0} = I_{p,c} D_{2p+1}/D_{2p+2}, \tag{27}$$

where D_{2p+1} is the determinant of the power-flow graph model of the main shaft after disk p. D_{2p+2} is the cofactor of the path just passing $(I_{p,c})^{-1}$ in the graph model with respect to the main shaft after disk p. Then the power-flow graph model of the total system will become a structure of connected series very similar to the unidirectional shaft system. Then, the response of the internal torque and rotation angle in the branched shaft leads to

$$H_{T_0, T_{p,n}} = P_{T_0, T_{p,n}} D_{p,2n+1}/D_1, \quad H_{T_0, \theta_{p,n}} = P_{T_0, \theta_{p,n}} D_{p,2n+2}/D_1, \tag{28, 29}$$

where

$$P_{T_0, T_{p,n}} = L_1\, L_3\, L_5 \cdots L_{2p-1}(R_p S_{p,1}/I'_{p,0}) L_{p,3} L_{p,5} \cdots L_{p,2n-1}, \tag{30}$$

$$P_{T_0, \theta_{p,n}} = L_1\, L_3\, L_5 \cdots L_{2p-1}(R_p S_{p,1}/I'_{p,0}) L_{p,3} L_{p,5} \cdots L_{p,2n-1} I_{p,n}^{-1}, \tag{31}$$

$$D_{p,2n+1} = 1 + \sum_{i=2n+1}^{2N_p} L_{p,i} + \sum_{i_2=2n+3}^{2N_p} \sum_{i_1=2n+1}^{i_2-2} L_{p,i_2} L_{p,i_1} + \cdots + \sum_{i_{N_p-n}=2N_p-1}^{2N_p} \sum_{i_{N_p-n-1}=N_p-3}^{i_{N_p-n}-2} \cdots \sum_{I_2=2n+3}^{i_3-2} \sum_{i_1=2n+1}^{i_2-2} L_{p,i_{N_p-n}} L_{p,i_{N_p-n}} \cdots L_{p,i_2} L_{p,i_1} \tag{32}$$

$$D_{p,2n+2} = 1 + \sum_{i=2n+2}^{2N_p} L_{p,i} + \sum_{i_2=2n+4}^{2N_p} \sum_{i_1=2n+2}^{i_2-2} L_{p,i_2} L_{p,i_1} + \cdots + \sum_{i_{N_p-n-1}=2N_p-2}^{2N_p} \sum_{i_{N_p-n-2}=N_p-4}^{i_{N_p-n-1}-2} \cdots \sum_{I_2=2n+4}^{i_3-2} \sum_{i_1=2n+2}^{i_2-2} L_{p,i_{N_p-n-1}} L_{p,i_{N_p-n-2}} \cdots L_{p,i_2} L_{p,i_1} + \prod_{i=n+1}^{N_p} L_{p,2i}. \tag{33}$$

To simplify the expression of D_1, the loop gain can be redefined as

$$L'_i = L_i \quad \text{for} \quad i = 1, 2, 3, \ldots, 2p; \quad L'_{2p+1} = R_p^2 S_{p,1}/I'_{p,0}, \tag{34, 35}$$

$$L'_{2p+i} = L_{p,i} \quad \text{for} \quad i = 2, 3, 4, \ldots, 2N_p. \tag{36}$$

Then,

$$D_1 = 1 + \sum_{i=1}^{2p+2N_p} L'_i + \sum_{I_2=3}^{2p+2N_p} \sum_{i_1=1}^{i_2-2} L'_{i_2} L'_{i_1} + \cdots + \sum_{i_{p+N_p-1}=2p+2N-3}^{2p+2N_p} \sum_{i_{p+N_p-2}=2p+2N-5}^{i_{N-n-1}-2}$$

$$\cdots \sum_{i_2=3}^{i_3-2} \sum_{i_1=1}^{i_2-2} L'_{i_{p+N_p-1}} L'_{i_{p+N_p-2}} \cdots L'_{i_2} L'_{i_1} + \sum_{i_{p+N_p}=2+2N_p-1}^{2p+2N_p} \sum_{i_{p+N_p-1}=2p+2N-3}^{i_{p+N_p}-2}$$

$$\cdots \sum_{i_2=3}^{i_3-2} \sum_{i_1=1}^{i_2-2} L'_{i_{p+N_p}} L'_{i_{p+N_p-1}} \cdots L'_{i_2} L'_{i_1}. \tag{37}$$

4. EXAMPLES AND DISCUSSIONS

Two torsional systems as shown in Figures 6 and 7 are considered in the analysis to illustrate the performance of the present method. The frequency response of T_3 and θ_4 with respect to input from T_0 in Figure 6 are first calculated. From the derived formula shown in equations (5) and (11), the

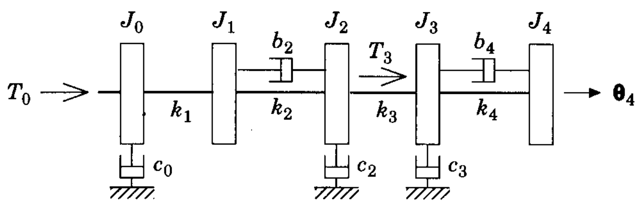

Figure 6. Unidirectionally torsional system.

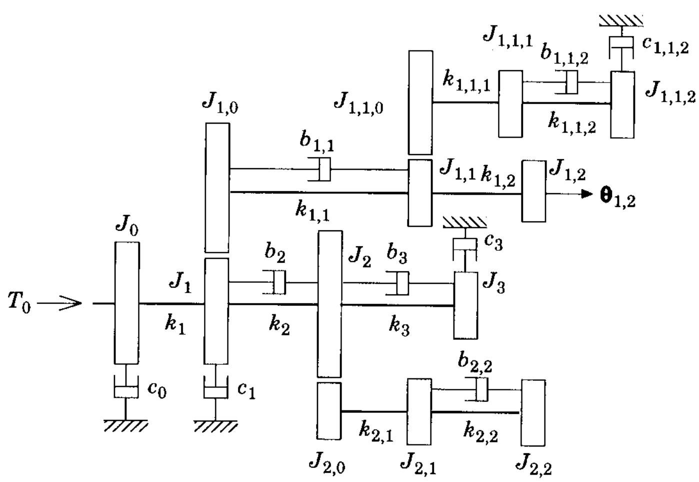

Figure 7. Multi-branch torsional system.

complex frequency function of both responses can be expressed as

$$H_{T_0,\theta_4} = L_1 L_3 L_5 L_7/(-J_4\omega^2)D_1, \quad H_{T_0,T_3} = (L_1 L_3 L_5)(1 + L_7 + L_8)/D_1, \tag{38, 39}$$

$$D_1 = 1 + \sum_{i=1}^{8} L_i + \sum_{i_2=3}^{8} L_{i_2} \sum_{i_1=1}^{i_1=i_2-2} L_{i_1} + \sum_{i_3=5}^{8} L_{i_3} \sum_{I_2=3}^{i_2=i_3-2} L_{i_2} \sum_{i_1=1}^{i_1=i_2-2} L_{i_1}$$
$$+ \sum_{i_4=7}^{8} L_{i_4} \sum_{i_3=5}^{i_3=i_4-2} L_{i_3} \sum_{i_2=3}^{i_2=i_3-2} L_{i_2} \sum_{i_1=1}^{i_1=i_2-2} L_{i_1}, \tag{40}$$

where

$$\begin{aligned} &L_1 = k_1/(-J_0\omega^2 + \mathrm{j}c_0\omega), \quad L_2 = k_1/-J_1\omega^2, \quad L_3 = (\mathrm{j}b_2\omega + k_2)/-J_1\omega^2, \\ &L_4 = (\mathrm{j}b_2\omega + k_2)/(-J_2\omega^2 + \mathrm{j}c_2\omega), \quad L_5 = k_3/(-J_2\omega^2 + \mathrm{j}c_2\omega), \\ &L_6 = k_3/(-J_3\omega^2 + \mathrm{j}c_3\omega), \quad L_7 = (\mathrm{j}b_4\omega + k_4)/(-J_3\omega^2 + \mathrm{j}c_3\omega), \\ &L_8 = (\mathrm{j}b_4\omega + k_4)/-J_4\omega^2. \end{aligned} \tag{41}$$

The multi-branch torsional system as shown in Figure 7 is considered in the second example. If the frequency response of $\theta_{1,2}$ with respect to the input from T_0 is computed, the part of the power-flow graph model with respect to the branch from $J_{1,1,0}$ to $J_{1,1,2}$ and $J_{2,0}$ to $J_{2,2}$ can first be reduced and combined into the graph model of $J_{1,1}$ and J_2. Then the part of the graph model with respect to J_1 to J_3 can be reduced. The reduced model for the analysis is shown in Figure 8. From the method derived in this paper, we can write the complex frequency function of $\theta_{1,2}$ as

$$H_{T_0,\theta_{1,2}} = L'_1 L'_3 L'_5/(-J_3\omega^2)R_1 D_1, \tag{42}$$

where

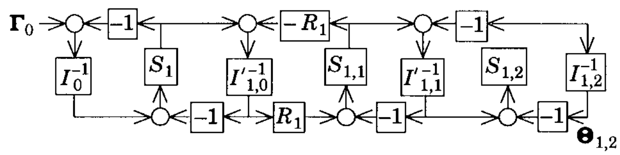

Figure 8. Reduced power-flow model for calculating $H_{T_0,\theta_{1,2}}$.

$$
\begin{aligned}
D_1 &= 1+\sum_{i=1}^{6} L'_i+\sum_{i_2=3}^{6} L'_{i_2}\left(\sum_{i_1=1}^{i_1=i_2-2} L'_{i_1}\right)+\sum_{i_3=5}^{6} L'_{i_3}\left(\sum_{i_2=3}^{i_2=i_3-2} L'_{i_2}\left(\sum_{i_1=1}^{i_1=i_2-2} L'_{i_1}\right)\right),\\
L'_1 &= k_1/(-J_0\omega^2+\mathrm{j}c_0\omega),\quad L'_2=k_1/I'_{1,0},\quad L'_3=R_1^2(\mathrm{j}b_{1,1}+K_{1,1})/I'_{1,0},\\
L'_4 &= R_1^2(\mathrm{j}b_{1,1}+k_{1,1})/I'_{1,1},\quad L'_5=k_{1,2}/I'_{1,1},\quad L'_6=K_{1,2}/-J_{1,2}\omega^2,\\
I'_{1,0} &= -(J_1+R_1^2J_{1,0})\omega^2 D_3/D_4,\\
D_3 &= 1+\sum_{i=3}^{6} L_i+\sum_{i_2=5}^{6} L_{i_2}\left(\sum_{i_1=3}^{i_1=i_2-2} L_{i_1}\right),\quad D_4=1+\sum_{i=4}^{6} L_i+L_4L_6,\\
L_3 &= (\mathrm{j}b_2\omega+k_{2,1})/[-(J_1+R_1^2J_{1,0})\omega^2+\mathrm{j}c_1\omega],\quad L_4=\mathrm{j}b_2\omega+k_2/I'_2,\\
L_5 &= (\mathrm{j}b_3\omega+k_3)/I'_2,\quad L_6=ib_3\omega+k_3/(-J_3\omega^2+\mathrm{j}c_3\omega),\\
I'_{1,1} &= -(J_{1,1}+R_{1,1}^2J_{1,1,0})\omega^2 D_{1,1,1}/D_{1,1,2},\\
D_{1,1,1} &= 1+\sum_{i=1}^{4} L_{1,1,i}+\sum_{i_2=3}^{4} L_{1,1,i_2}\left(\sum_{i_1=1}^{i_1=i_2-2} L_{1,1,i_1}\right),\\
D_{1,1,2} &= 1+\sum_{i=2}^{4} L_{1,1,i}+L_{1,1,2}L_{1,1,4},\quad L_{1,1,1}=R_{1,1}^2k_{1,1,1}/-(J_{1,1}+R_{1,1}^2J_{1,1,0})\omega^2,\\
L_{1,1,2} &= k_{1,1,1}/-J_{1,1,1}\omega^2,\quad L_{1,1,3}=(k_{1,1,2}+\mathrm{j}b_{1,1,2}\omega)/-J_{1,1,1}\omega^2,\\
L_{1,1,4} &= (k_{1,1,2}+\mathrm{j}b_{1,1,2}\omega)/(-J_{1,1,2}\omega^2+\mathrm{j}c_{1,1,2}\omega),\quad I'_2=-(J_2+R_2^2J_{2,0})\omega^2D_{2,1}/D_{2,2},\\
D_{2,1} &= 1+\sum_{i=1}^{4} L_{2,i}+\sum_{i_2=3}^{4} L_{2,i_2}\left(\sum_{i_1=1}^{i_1=i_2-2} L_{2,i_1}\right),\quad D_{2,2}=1+\sum_{i=2}^{4} L_{2,i}+L_{2,2}L_{2,4},\\
L_{2,1} &= R_1^2k_{2,1}/-(J_2+R_1^2J_{2,0})\omega^2,\quad L_{2,2}=k_{2,1}/-J_{2,1}\omega^2,\\
L_{2,3} &= (k_{2,2}+\mathrm{j}c_{2,2}\omega)/-J_{2,1}\omega^2,\quad L_{2,4}=(k_{2,2}+\mathrm{j}c_{2,2}\omega)/-J_{2,2}\omega^2. \qquad (43)
\end{aligned}
$$

5. CONCLUSIONS

An analytical method for the calculation of the dynamic response of internal torque and rotary angle in unidirectional and multi-branch torsional systems has been proposed. Using the proposed power-flow model, the torsional system can be expressed as a two way graph model analogous to the configuration of the physical system, in which the internal damping for the shaft twist and external damping for the rotational disk are all considered. Then the frequency response of the internal torque and rotation angle in the shaft can be calculated directly from the derived formula. Since the derived formula is expressed in summation operation, it is very effective to calculate by computer. Although the graph model for the multi-branch torsional system may be very complex, the model reduction method presented in this paper can be applied to reduce this complex model to a unidirectional strip model very similar to that of a unidirectionally torsional system. Then the calculation work for the multi-branched torsional system can be reduced to a standard and simplified formula. Finally, the frequency response of internal torque and rotary angle in the shaft of a unidirectional and three shafts system are analyzed in the examples to show the effectiveness of this scheme. By using this method, similar analysis can be extended for more complex systems.

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