Journal of Sound and Vibration (1999) **224**(2), 267–281
Article No. jsvi.1999.2194, available online at http://www.idealibrary.com on IDEAL®

EXPLICIT FREQUENCY EQUATION AND MODE SHAPES OF A CANTILEVER BEAM COUPLED IN BENDING AND TORSION

J. R. BANERJEE

Department of Mechanical Engineering and Aeronautics, City University, Northampton Square, London EC1V 0HB, U.K.

(*Received* 11 *June* 1998, *and in final form* 14 *January* 1999)

Exact explicit analytical expressions which give the natural frequencies and mode shapes of a bending–torsion coupled beam with cantilever end condition are derived by rigorous application of the symbolic computing package REDUCE. The expressions are surprisingly concise and very simple to use. By way of illustration in this paper, they are used to determine the natural frequencies and mode shapes of a cantilever wing with substantial coupling between the bending and torsional modes of deformation. The results are compared with exact published results to confirm the correctness and accuracy of the expressions. The derived expressions can be used to solve bench-mark free vibration problems as an aid in validating the finite element and other approximate methods. They are also intended for future applications in aeroelastic and/or optimization studies. Computer implementation and a comparison of solution times show that there is more than a four-fold advantage in c.p.u. time when using the explicit expressions as opposed to the alternative numerical method involving determinant evaluation and matrix manipulation.

1. INTRODUCTION

Explicit analytical expressions for the frequency equations and mode shapes of a Bernoulli–Euler beam with various end conditions have been available in the literature for many years and can be found in standard texts, see for example, Table 7-1 on p. 277 of reference [1]. Similar expressions for the frequency equations and mode shapes of a Timoshenko beam [2, 3] and an axially loaded Timoshenko beam [4] have also become available relatively recently, although it should be recognized that the free vibration characteristics of such beams were investigated some years ago using the dynamic stiffness [5–8], finite element [9] or other methods [10], without resorting to the derivation of explicit frequency and mode shape formulae.

The reported investigations on Bernoulli–Euler, Timoshenko and axially loaded Timoshenko beams are all based on the assumption that the beam deflects only in

0022-460X/99/270267 + 15 $30.00/0

flexure and as a consequence, there is no coupling between the bending and torsional deformations of the beam cross-sections. Such an assumption imposes very serious restrictions on the free vibration analysis of beams for which the bending and torsional deformations are inherently coupled due to non-coincident mass and shear centres of the cross-sections. Examples include beams with "Angle", "Tee", "Channel", "Open Box", and "Aerofoil" cross-sections. The free vibration analysis of such beams is significantly more complicated than that of Bernoulli–Euler or Timoshenko beams (in flexure only), due mainly to the bending–torsion coupling effect which leads to the formulation of a higher order governing differential equation (usually the sixth order instead of the fourth). Investigators of the problem have generally relied either on the direct solution of the governing differential equation [11] and substitution of appropriate end-conditions for displacements and forces in the dynamic stiffness method [12–14], or on the traditional finite element and other approximate methods [15, 16]. The derivation of explicit expressions for the frequency equation and the corresponding mode shapes of a bending–torsion coupled beam is of quite considerable complexity. The difficulty would appear to arise from the complex nature of the problem which involves the algebraic expansion of determinants (for the frequency equations), together with matrix inversion and multiplication (for the mode shapes) of matrices whose elements are themselves complicated algebraic expressions involving transcendental functions. With the advent of, and advancement in, symbolic computing, it seems that this difficulty can be overcome. In the sequel, it has now become possible to handle problems in matrix algebra symbolically and to manipulate large expressions by simplifying them very considerably. The main purpose of this paper is to derive exact analytical expressions for the frequency equation and the corresponding mode shapes of a uniform bending–torsion coupled beam with cantilever end condition, using the symbolic computing package REDUCE [17, 18]. These expressions can be used to solve both free and forced vibration problems of bending–torsion coupled beams and can also be used to carry out bench mark studies to validate the finite element and other approximate methods. The explicit expressions are particularly useful in the context of aeroelastic analysis, and/or in optimization studies for which repetitive sensitive analyses are often required to establish design trends when principal beam parameters are varied. (Note that earlier discussions on the frequency and mode shape expressions of a bending–torsion coupled beam have been confined to the relatively trivial case where the beam is simply supported at both ends, so that the governing mode shapes are sine waves, see pp. 471–475 of reference [19]. In contrast, the present paper significantly advances the discussion through the introduction of algebraically very general mode shapes, where the analysis of response is far less transparent than in the earlier studies.) The use of the explicit expressions is shown to have more than four-fold advantage in c.p.u. time when compared with the alternative numerical method based on the determinant evaluation and matrix manipulation.

The theory developed in this paper is applied to a cantilever wing [20] with substantial coupling between the bending and torsional modes of deformation. The results are compared with those available in the literature [20, 21] and some conclusions are drawn.

2. THEORY

An important example of a bending–torsion coupled beam is an aircraft wing as shown in Figure 1. The wing has a length L and its mass and elastic axes, which are respectively the loci of the mass centre and the shear centre of the wing cross-sections, are shown in the figure, with x_α being the distance of separation between them (x_α is positive in the positive direction of X). In the right-handed co-ordinate system shown in Figure 1, the elastic axis (which is coincident with the Y-axis) is allowed to deflect out of plane by $h(y, t)$, whilst the cross-section is allowed to rotate (or twist) about OY by $\psi(y, t)$, where y and t denote distance from the origin and time respectively. Although the specific case of an aircraft wing is chosen as an example, the theory developed has much wider applications.

Using bending–torsion coupled beam theory, the governing partial differential equations of the motion of the wing shown in Figure 1 have been given amongst others, by Dokumaci [11], Hallauer and Liu [12] and Banerjee [14]. Using the notation of Figure 1, the equations are presented here as follows: (Note that in the derivation of these equations, St. Venant's torsion theory has been used so that the cross-section is allowed to warp without restraint, and also the effects of shear deformation and rotatory interia are assumed to be small and hence are not included in the derivation.)

$$\mathrm{EI}\, h'''' + m\ddot{h} - m x_\alpha \ddot{\psi} = 0 \tag{1}$$

and

$$\mathrm{GJ}\, \psi'' + m x_\alpha \ddot{h} - I_\alpha \ddot{\psi} = 0, \tag{2}$$

where EI and GJ are respectively the bending and torsional rigidities of the beam, m is the mass per unit length, I_α is the polar mass moment of inertia per unit length

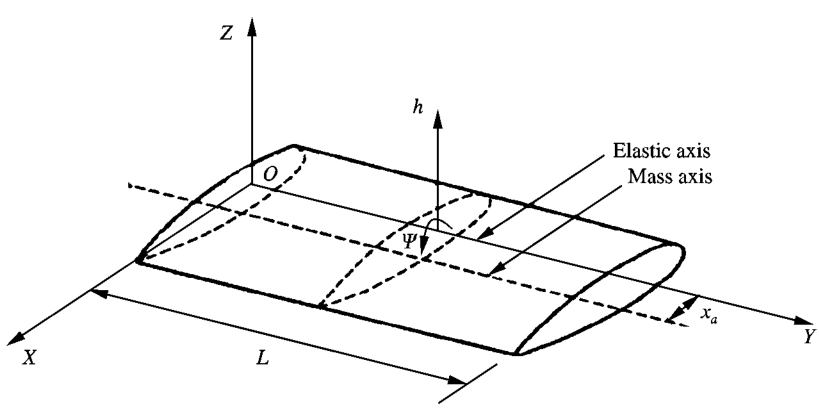

Figure 1. Co-ordinate system and notation for a bending–torsion coupled beam.

about the Y-axis (i.e., an axis through the shear centre) and primes and dots denote differentiation with respect to position y and time t respectively.

If a sinusoidal variation of h and ψ, with circular frequency ω, is assumed, then

$$h(y,t) = H(y)\sin\omega t, \quad \psi(y,t) = \Psi(y)\sin\omega t, \tag{3}$$

where $H(y)$ and $\Psi(y)$ are the amplitudes of the sinusoidally varying vertical displacement and torsional rotation respectively.

Substituting equation (3) into equations (1) and (2) gives

$$\mathrm{EI}\,H'''' - m\omega^2 H + m x_\alpha \omega^2 \Psi = 0, \tag{4}$$

$$\mathrm{GJ}\,\Psi'' + I_\alpha \omega^2 \Psi - \omega^2 m x_\alpha H = 0. \tag{5}$$

Equations (4) and (5) can be combined into one equation by eliminating either H or Ψ to give the sixth order differential equation as

$$W'''''' + (I_\alpha\omega^2/\mathrm{GJ})W'''' - (m\omega^2/\mathrm{EI})W'' - (m\omega^2/\mathrm{EI})(I_\alpha\omega^2/\mathrm{GJ})(1 - m x_\alpha^2/I_\alpha)W = 0, \tag{6}$$

where

$$W = H \text{ or } \Psi. \tag{7}$$

Introducing the non-dimensional length,

$$\xi = y/L \tag{8}$$

Equation (6) may be written in the non-dimensional form as

$$(D^6 + aD^4 - bD^2 - abc)W = 0, \tag{9}$$

where

$$a = I_\alpha\omega^2 L^2/\mathrm{GJ}, \quad b = m\omega^2 L^4/\mathrm{EI}, \quad c = 1 - m x_\alpha^2/I_\alpha \tag{10}$$

and

$$D = \mathrm{d}/\mathrm{d}\xi \tag{11}$$

The solution of the sixth order differential equation (9) is obtained as [14]

$$W(\xi) = C_1\cosh\alpha\xi + C_2\sinh\alpha\xi + C_3\cos\beta\xi + C_4\sin\beta\xi + C_5\cos\gamma\xi + C_6\sin\gamma\xi, \tag{12}$$

where C_1–C_6 are constants and

$$\alpha = [2(q/3)^{1/2}\cos(\phi/3) - a/3]^{1/2},$$

$$\beta = [2(q/3)^{1/2}\cos\{(\pi - \phi)/3\} + a/3]^{1/2},$$

$$\gamma = [2(q/3)^{1/2}\cos\{(\pi + \phi)/3\} + a/3]^{1/2}, \tag{13}$$

with

$$q = b + a^2/3$$

and

$$\phi = \cos^{-1}[(27abc - 9ab - 2a^3)/\{2(a^2 + 3b)^{3/2}\}]. \tag{14}$$

$W(\xi)$ in equation (12) represents the solution for both the bending displacement H and the torsional rotation Ψ with different constant values. Thus,

$$H(\xi) = A_1 \cosh\alpha\xi + A_2 \sinh\alpha\xi + A_3\cos\beta\xi + A_4\sin\beta\xi + A_5\cos\gamma\zeta + A_6\sin\gamma\xi, \tag{15}$$

$$\Psi(\xi) = B_1 \cosh\alpha\xi + B_2 \sinh\alpha\xi + B_3\cos\beta\xi + B_4\sin\beta\xi + B_5\cos\gamma\zeta + B_6\sin\gamma\xi, \tag{16}$$

where A_1–A_6 and B_1–B_6 are the two different sets of constants.

It can be readily verified by substituting equations (15) and (16) into equations (4) and (5) that constants A_1–A_6 and B_1–B_6 are related in the following way:

$$B_1 = k_\alpha A_1, \qquad B_3 = k_\beta A_3, \qquad B_5 = k_\gamma A_5,$$

$$B_2 = k_\alpha A_2, \qquad B_4 = k_\beta A_4, \qquad B_6 = k_\gamma A_6, \tag{17}$$

where

$$k_\alpha = (b - \alpha^4)/bx_\alpha, \qquad k_\beta = (b - \beta^4)/bx_\alpha, \qquad k_\gamma = (b - \gamma^4)/bx_\alpha. \tag{18}$$

The expressions for the bending rotation $\theta(\xi)$, the bending moment $M(\xi)$, the shear force $S(\xi)$ and the torque $T(\xi)$ can be obtained from equations (15) and (16) as [14]

$$\theta(\xi) = H'(\xi)/L = (1/L)\{A_1\alpha\sinh\alpha\xi + A_2\alpha\cosh\alpha\xi - A_3\beta\sin\beta\xi + A_4\beta\cos\beta\xi - A_5\gamma\sin\gamma\xi + A_6\gamma\cos\gamma\xi\}, \tag{19}$$

$$M(\xi) = -(\mathrm{EI}/L^2)H''(\xi) = -(\mathrm{EI}/L^2)\{A_1\alpha^2\cosh\alpha\xi + A_2\alpha^2\sinh\alpha\xi - A_3\beta^2\cos\beta\xi - A_4\beta^2\sin\beta\xi - A_5\gamma^2\cos\gamma\xi - A_6\gamma^2\sin\gamma\xi\}, \tag{20}$$

$$S(\xi) = -M'(\xi)/L = -(\mathrm{EI}/L^3)\{A_1\alpha^3\sinh\alpha\xi + A_2\alpha^3\cosh\alpha\xi + A_3\beta^3\sin\beta\xi - A_4\beta^3\cos\beta\xi + A_5\gamma^3\sin\gamma\xi - A_6\gamma^3\cos\gamma\xi\}, \tag{21}$$

$$T(\xi) = (\mathrm{GJ}/L)\Psi'(\xi) = (\mathrm{GJ}/L)\{B_1\alpha\sinh\alpha\xi + B_2\alpha\cosh\alpha\xi - B_3\beta\sin\beta\xi + B_4\beta\cos\beta\xi - B_5\gamma\sin\gamma\xi + B_6\gamma\cos\gamma\xi\}. \tag{22}$$

2.1. FREQUENCY EQUATION

The end conditions for the cantilever beam are as follows:

$$\text{at the built-in end (i.e., at } \xi = 0\text{): } H = 0,\ \theta = 0 \text{ and } \Psi = 0, \tag{23}$$

$$\text{at the free end (i.e., at } \xi = 1\text{): } S = 0,\ M = 0 \text{ and } T = 0. \tag{24}$$

Substituting equation (23) in equations (15)–(19), and (24) in equations (20)–(22) gives

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta & 0 & \gamma \\ k_\alpha & 0 & k_\beta & 0 & k_\gamma & 0 \\ -\alpha^3 S_{h\alpha} & -\alpha^3 C_{h\alpha} & -\beta^3 S_\beta & \beta^3 C_\beta & -\gamma^3 S_\gamma & \gamma^3 C_\gamma \\ \alpha^2 C_{h\alpha} & \alpha^2 S_{h\alpha} & -\beta^2 C_\beta & -\beta^2 S_\beta & -\gamma^2 C_\gamma & -\gamma^2 S_\gamma \\ \alpha k_\alpha S_{h\alpha} & \alpha k_\alpha C_{h\alpha} & -\beta k_\beta S_\beta & \beta k_\beta C_\beta & -\gamma k_\gamma S_\gamma & \gamma k_\gamma C_\gamma \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix} = 0, \tag{25}$$

where

$$C_{h\alpha} = \cosh\alpha, \qquad C_\beta = \cos\beta, \qquad C_\gamma = \cos\gamma,$$

$$S_{h\alpha} = \sinh\alpha, \qquad S_\beta = \sin\beta, \qquad S_\gamma = \sin\gamma. \tag{26}$$

Equation (25) may be written in matrix form as

$$\mathbf{BA} = \mathbf{0}. \tag{27}$$

The necessary and sufficient condition for non-zero elements in the column vector **A** of equation (27) is that $\Delta = |\mathbf{B}|$ shall be zero, and the vanishing of Δ determines the natural frequencies of the system in the usual way. Thus, the frequency equation for the cantilever can be obtained for the non-trivial solution as

$$\Delta = |\mathbf{B}| = 0. \tag{28}$$

Expanding the 6×6 determinant Δ of **B** algebraically is quite a formidable task and became more feasible with the recent advances in symbolic computing. Thus most of the work reported here, was carried out using the software REDUCE [17, 18] in expanding the determinant $|\mathbf{B}|$, and more importantly in simplifying the expression for Δ. The final expression obtained for Δ is given below which is not necessarily in the shortest possible form, but is surprisingly concise.

$$\Delta = \nu_2(\lambda_1 + \eta_1 + \xi_1) + \nu_3(\lambda_2 + \eta_2 - \xi_2) - \nu_1(\lambda_3 + \eta_3 - \xi_3)$$

$$- \mu_1\varepsilon_1 - \mu_2\varepsilon_2 - \mu_3\varepsilon_3 + \delta_1 + \delta_2 + \delta_3, \tag{29}$$

where

$$\mu_1 = k_\alpha + k_\beta, \qquad \mu_2 = k_\beta + k_\gamma, \qquad \mu_3 = k_\gamma + k_\alpha, \tag{30}$$

$$\nu_1 = k_\alpha - k_\beta, \qquad \nu_2 = k_\beta - k_\gamma, \qquad \nu_3 = k_\gamma - k_\alpha, \tag{31}$$

$$\lambda_1 = \alpha^4(k_\beta C_\beta - k_\gamma C_\gamma), \qquad \lambda_2 = \beta^4(k_\gamma C_\gamma - k_\alpha C_{h\alpha}), \qquad \lambda_3 = \gamma^4(k_\beta C_\beta - k_\alpha C_{h\alpha}), \tag{32}$$

$$\xi_1 = \alpha^2 k_\alpha(\beta^2 C_\beta - \gamma^2 C_\gamma), \qquad \xi_2 = \beta^2 k_\beta(\alpha^2 C_{h\alpha} + \gamma^2 C_\gamma), \qquad \xi_3 = \gamma^2 k_\gamma(\alpha^2 C_{h\alpha} + \beta^2 C_\beta), \tag{33}$$

$$\eta_1 = \alpha^3 S_{h\alpha}(\beta k_\gamma S_\beta C_\gamma - \gamma k_\beta C_\beta S_\gamma),$$

$$\eta_2 = \beta^3 S_\beta(\gamma k_\alpha C_{h\alpha} S_\gamma + \alpha k_\gamma S_{h\alpha} C_\gamma),$$

$$\eta_3 = \gamma^3 S_\gamma(\alpha k_\beta S_{h\alpha} C_\beta + \beta k_\alpha C_{h\alpha} S_\beta), \tag{34}$$

$$\varepsilon_1 = \alpha\beta k_\gamma C_\gamma(\alpha\beta C_{h\alpha} C_\beta + \gamma^2 S_{h\alpha} S_\beta),$$

$$\varepsilon_2 = \beta\gamma k_\alpha C_{h\alpha}(\alpha^2 S_\beta S_\gamma - \beta\gamma C_\beta C_\gamma),$$

$$\varepsilon_3 = \gamma\alpha k_\beta C_\beta(\alpha\gamma C_{h\alpha} C_\gamma + \beta^2 S_{h\alpha} S_\gamma), \tag{35}$$

$$\delta_1 = 2\alpha^2 C_{h\alpha} C_\beta C_\gamma (\gamma^2 k_\beta^2 + \beta^2 k_\gamma^2),$$

$$\delta_2 = 2\alpha\beta\gamma k_\gamma S_\gamma (\alpha k_\beta C_{h\alpha} S_\beta + \beta k_\alpha S_{h\alpha} C_\beta),$$

$$\delta_3 = 2\beta\gamma^2 k_\alpha C_\gamma (\alpha k_\beta S_{h\alpha} S_\beta - \beta k_\alpha C_{h\alpha} C_\beta), \tag{36}$$

with α, β, γ and k_α, k_β, k_γ and $C_{h\alpha}$, C_β, C_γ, $S_{h\alpha}$, S_β, S_γ already defined in equations (13), (18) and (26) respectively. Note that it can be readily verified with the help of equations (10), (12) and (13) that the value of the determinant $\Delta = |\mathbf{B}|$ is zero when the frequency (ω) is zero. This known value of $\Delta = |\mathbf{B}| = 0$ at $\omega = 0$ (which corresponds to a beam with no inertial loading, i.e., at rest). can always be used to avoid any numerical problem of overflow at zero frequency when computing the value of Δ. Thus for any other (non-trivial) values of ω, the expression for Δ given by equation (29) can be used in locating the natural frequencies by successively tracking the changes of its sign.

2.2 MODE SHAPES

Once the natural frequencies ω_n are found from equation (28), the modal vector **A** (in which one element may be fixed arbitrarily) is found in the usual way, namely by deleting one row of the sixth order determinant and solving for the five remaining constants in terms of the arbitrarily chosen one.

Thus, if A_1 is chosen to be the one in terms of which the remaining constants A_2–A_6 are to be expressed, as in the present case, the matrix equations (25), will take the following reduced order form (Note that terms relating to A_1 are taken to the right-hand side.):

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ \alpha & 0 & \beta & 0 & \gamma \\ 0 & k_\beta & 0 & k_\gamma & 0 \\ -\alpha^3 C_{h\alpha} & -\beta^3 S_\beta & \beta^3 C_\beta & -\gamma^3 S_\gamma & \gamma^3 C_\gamma \\ \alpha^2 S_{h\alpha} & -\beta^2 C_\beta & -\beta^2 S_\beta & -\gamma^2 C_\gamma & -\gamma^2 S_\gamma \end{bmatrix} \begin{bmatrix} A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -k_\alpha \\ \alpha^3 S_{h\alpha} \\ -\alpha^2 C_{h\alpha} \end{bmatrix} A_1. \tag{37}$$

The symbolic computing package REDUCE [17, 18] was further used to solve the above system of equations giving the following mode shape coefficients in terms of A_1:

$$A_2 = A_1[\beta\gamma(\phi_1 + \gamma^2 v_1 \sigma_3 - \beta^2 v_3 \sigma_2)/\chi],$$

$$A_3 = A_1[v_3/v_2],$$

$$A_4 = A_1[\gamma\alpha(-\phi_2 - \gamma^2 v_1 \kappa_3 + \alpha^2 v_2 \kappa_1)/\chi],$$

$$A_5 = A_1[v_1/v_2],$$

$$A_6 = A_1[\alpha\beta(\phi_3 + \beta^2 v_3 \kappa_2 - \alpha^2 v_2 \sigma_1)/\chi], \tag{38}$$

where v_1, v_2 and v_3 have already been defined in equations (31) and the following further variables are introduced to compute the parameters within the square brackets:

$$\zeta_1 = \alpha S_{h\alpha} + \beta S_\beta, \qquad \zeta_2 = \beta S_\beta - \gamma S_\gamma, \qquad \zeta_3 = \gamma S_\gamma + \alpha S_{h\alpha}, \tag{39}$$

$$\tau_1 = \alpha^2 C_{h\alpha} + \beta^2 C_\beta, \qquad \tau_2 = \beta^2 C_\beta - \gamma^2 C_\gamma, \qquad \tau_3 = \gamma^2 C_\gamma + \alpha^2 C_{h\alpha}, \tag{40}$$

$$\sigma_1 = \alpha^2 - \alpha\beta S_{h\alpha} S_\beta + \beta^2 C_{h\alpha} C_\beta, \qquad \sigma_2 = \beta^2 - \beta\gamma S_\beta S_\gamma - \gamma^2 C_\beta C_\gamma,$$

$$\sigma_3 = \gamma^2 - \beta\gamma S_\beta S_\gamma - \beta^2 C_\beta C_\gamma, \tag{41}$$

$$\kappa_1 = \alpha^2 - \alpha\gamma S_{h\alpha} S_\gamma + \gamma^2 C_{h\alpha} C_\gamma, \qquad \kappa_2 = \beta^2 + \alpha\beta S_{h\alpha} S_\beta + \alpha^2 C_{h\alpha} C_\beta,$$

$$\kappa_3 = \gamma^2 + \alpha\gamma S_{h\alpha} S_\gamma + \alpha^2 C_{h\alpha} C_\gamma, \tag{42}$$

$$\phi_1 = \alpha^2 v_2(\tau_2 C_{h\alpha} - \alpha\zeta_2 S_{h\alpha}), \qquad \phi_2 = \beta^2 v_3(\tau_3 C_\beta + \beta\zeta_3 S_\beta),$$

$$\phi_3 = \gamma^2 v_1(\tau_1 C_\gamma + \gamma\zeta_1 S_\gamma) \tag{43}$$

and

$$\chi = \alpha\beta\gamma v_2(\alpha^2\zeta_2 C_{h\alpha} - \beta^2\zeta_3 C_\beta + \gamma^2\zeta_1 C_\gamma). \tag{44}$$

Note that α, β, γ, k_α, k_β, k_γ, $C_{h\alpha}$, $S_{h\alpha}$, C_β, S_β, C_γ and S_γ appearing in equations (39)–(44) are given by equations (13), (18) and (26) but must be calculated for the particular natural frequency ω_n at which the mode shape is required.

Thus, the mode shape of the bending–torsion coupled beam with cantilever end condition is given in explicit form by rewriting equations (15) and (16) with the help of equations (17), (18) in the form

$$H(\xi) = A_1(\cosh\alpha\xi + R_1\sinh\alpha\xi + R_2\cos\beta\xi + R_3\sin\beta\xi + R_4\cos\gamma\xi + R_5\sin\gamma\xi), \tag{45}$$

$$\Psi(\xi) = A_1(k_\alpha\cosh\alpha\xi + R_1 k_\alpha\sinh\alpha\xi + R_2 k_\beta\cos\beta\xi + R_3 k_\beta\sin\beta\xi + R_4 k_\gamma\cos\gamma\xi + R_5 k_\gamma\sin\gamma\xi), \tag{46}$$

where the ratios R_1, R_2, R_3, R_4 and R_5 are respectively A_2/A_1, A_3/A_1, A_4/A_1, A_5/A_1 and A_6/A_1, and follow from equations (38).

2.3. DEGENERATE CASE ($x_\alpha = 0$)

The degenerate case of the above theory reduces to the Bernoulli–Euler theory when the term x_α (i.e., the distance between the mass and elastic axes) which (inertially) couples the bending displacement and torsional rotation is set to zero. This leads to separate (uncoupled) bending and torsional frequency equations and mode shapes of a (Bernoulli–Euler) cantilever beam as follows.

It is evident from equation (10) that $c = 1$ when x_α is zero. Thus the governing differential equation (9) for the degenerate case becomes

$$(D^6 + aD^4 - bD^2 - ab)W = 0. \tag{47}$$

Using simple factorization rules for differential operators with constant coefficients, equation (47) can be written as

$$(D^4 - b)(D^2 + a)W = 0. \tag{48}$$

The above splits into two independent differential equations, one corresponding to bending displacement (H) and the other corresponding to torsional rotation (Ψ) as follows:

$$(D^4 - b)H = 0 \tag{49}$$

and

$$(D^2 + a)\Psi = 0. \tag{50}$$

The solutions of the differential equations (49) and (50) are respectively given by [1]

$$H(\xi) = A_1 \cosh \alpha\xi + A_2 \sinh \alpha\xi + A_3 \cos \alpha\xi + A_4 \sin \alpha\xi \tag{51}$$

and

$$\Psi(\xi) = B_1 \cos \gamma\xi + B_2 \sin \gamma\xi, \tag{52}$$

where

$$\alpha = (b)^{1/4} = L(m\omega^2/\mathrm{EI})^{1/4} \tag{53}$$

and

$$\gamma = \sqrt{a} = \omega L \sqrt{I_\alpha/\mathrm{GJ}}. \tag{54}$$

The derivation of the frequency equations and mode shapes is now a standard procedure [1] which can be accomplished by applying the boundary conditions of equations (23) and (24) for a cantilever to the general solutions for bending displacements and torsional rotations of equations (51) and (52) and to the corresponding expressions for bending slope, bending moment, shear force and torque given by equations (19)–(22). For completeness, the expressions for the frequency equations and mode shapes of the degenerate case leading to the Bernoulli–Euler beam in bending and torsional natural vibration are respectively given below.

2.3.1. *Bending vibration*

Frequency equation:

$$\cosh\alpha\cos\alpha + 1 = 0 \tag{55}$$

from which the values of α yield the natural frequencies (see equation (53)) in free bending vibration.

Mode shapes:

$$H(\xi) = A_1[(\cosh\alpha\xi - \cos\alpha\xi) + R_1(\sinh\alpha\xi - \sin\alpha\xi)], \tag{56}$$

where

$$R_1 = (\sin\alpha - \sinh\alpha)/(\cos\alpha + \cosh\alpha) = -(\cos\alpha + \cosh\alpha)/(\sin\alpha + \sinh\alpha). \tag{57}$$

Note that the values of α in equations (56) and (57) must be calculated at the natural frequencies for which the mode shapes are required (see equation (53)).

2.3.2. *Torsional vibration*

Frequency equation:

$$\omega_n = \frac{(2n-1)\pi}{2L}\sqrt{\mathrm{GJ}/I_\alpha}, \tag{58}$$

where $n = 1, 2, 3, 4, \ldots$ denotes the order of the torsional natural frequency of the cantilever beam.

Mode shapes:

$$\Psi_n = B_2 \sin\gamma_n\xi, \tag{59}$$

The value of γ_n in equation (59) must be calculated using the natural frequency ω_n in place of ω in equation (54).

It is worth noting that for this degenerate case the three roots α, β and γ, given by equations (13) for the general case, reduce to two coincident roots $\alpha(=\beta)$ and the third root being different is γ. A proof that the condition for $\alpha = \beta$ is $c = 1$ (i.e. $x_\alpha = 0$), is given in Appendix A.

3. DISCUSSION OF RESULTS

An illustrative example on the application of the frequency equation and mode shapes derived above is chosen to be that of an aircraft wing with cantilever end-condition, as discussed in reference [20, 21]. The data used for the wing are: (i) $EI = 9.75 \times 10^6$ Nm^2, (ii) $GJ = 9.88 \times 10^5$ Nm^2, (iii) $m = 35.75$ kg/m, (iv) $I_\alpha = 8.65$ kgm, (v) $x_\alpha = 0.18$ m and (vi) $L = 6$ m. The determinant Δ of the matrix **B** of equation (25) was computed both numerically and using the analytical expression of equation (29), for a range of frequencies. Both sets of results were found to agree up to machine accuracy. The plot of Δ against frequency (ω) is shown in Figure 2. The first two natural frequencies are identified as 49.6 and 97.0 rad/s which agree completely with the exact dynamic stiffness results of reference [21]. The mode shapes for the two natural frequencies were next computed by using the analytical expressions of equations (45) and (46). These were further checked to machine accuracy by solving the system of equations in equation (37) numerically, using the computational steps of matrix inversion and multiplication. These modes are shown in Figure 3 and are in complete agreement with the modes shown in reference [21]. In order to demonstrate the substantial computational advantage of the proposed method, the determinant Δ was computed both numerically and analytically for a large number of iterations, each performed at a different frequency. The recorded elapsed c.p.u. time on a SUN (Ultra-1) workstation is shown in Table 1. It is clearly evident that programming the explicit expression for Δ has more than four-fold advantage over the numerical method.

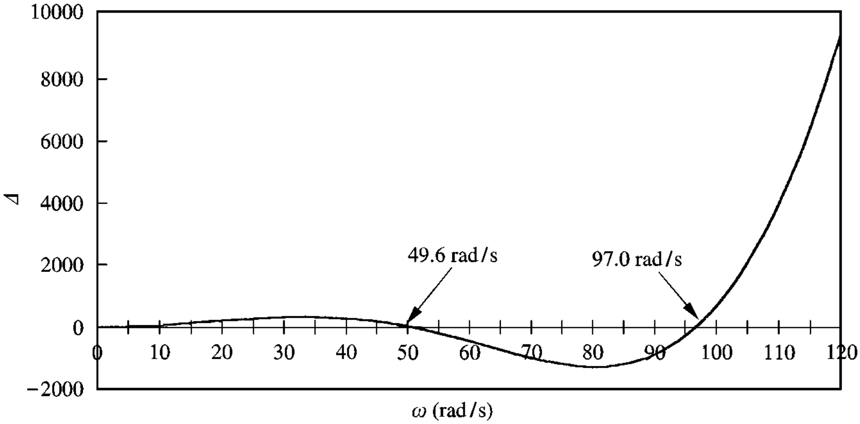

Figure 2. The variation of Δ against frequency (ω).

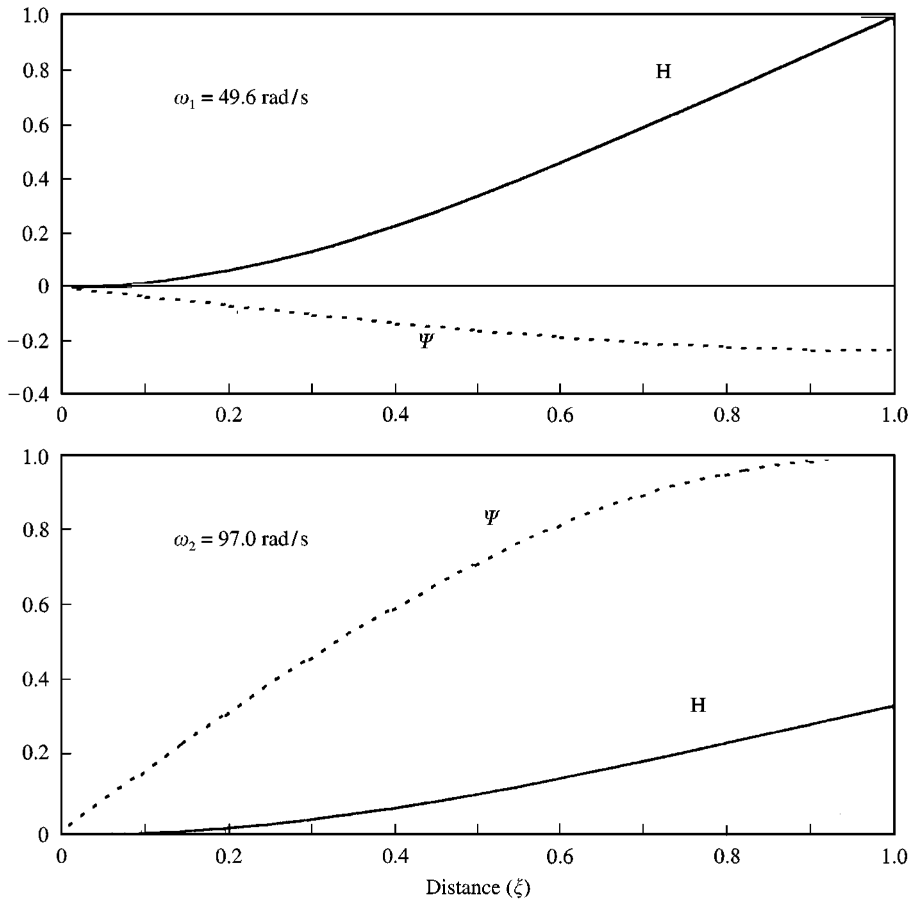

Figure 3. Coupled bending–torsional natural frequencies and mode shapes of an aircraft wing: ———— bending displacement (H); - - - - - torsional rotation (Ψ).

TABLE 1

c.p.u. time on a SUN (Ultra-1) computer using Fortran

Number of iterations (number of frequencies)	c.p.u. time (s) Numerical method	Explicit expression
500	0·076	0·018
1000	0·149	0·035
2500	0·367	0·086
5000	0·761	0·177

4. CONCLUSIONS

Exact frequency equation and mode shape expressions for a bending–torsion coupled beam with cantilever end condition have been derived using the symbolic computing package REDUCE. The correctness of the expressions has been checked by numerical results which agree completely with exact published results. The expressions developed can be used to solve bench-mark problems as an aid in validating the finite element and other approximate methods. They can be further utilized in aeroelastic and/or in optimization studies. Programming the explicit expressions has a substantial advantage in c.p.u. time over numerical methods, and a typical gain of four-fold computational efficiency is realized.

ACKNOWLEDGMENT

The author is grateful to his friend Adam Sobey for many useful discussions on the subject.

REFERENCES

1. F. S. TSE, I. E. MORSE and R. T. HINKLE 1978 *Mechanical Vibrations : Theory and Applications*, London: Allyn and Bacon, Inc; second edition.
2. A. M. HORR and L. C. SCHMIDT 1995 *Computers and Structures* **55**, 405–412. Closed-form solution for the Timoshenko beam theory using a computer-based mathematical package.
3. M. W. D. WHITE and G. R. HEPPLER 1995 *Journal of Applied Mechanics* **62**, 193–199. Vibration modes and frequencies of Timoshenko beams with attached rigid bodies.
4. S. H. FARGHALY and M. G. SHEBL 1995 *Journal of Sound and Vibration* **180**, 205–227. Exact frequency and mode shape formulae for studying vibration and stability of Timoshenko beam system.
5. F. Y. CHEUNG 1970 *Journal of Structural Division, ASCE* **96**, 551–571. Vibration of Timoshenko beams and frameworks.
6. T. M. WANG and T. A. KINSMAN 1971 *Journal of Sound and Vibration* **14**, 215–227. Vibration of frame structures according to the Timoshenko theory.
7. W. P. HOWSON and F. W. WILLIAMS 1973 *Journal of Sound and Vibration* **26**, 503–515. Natural frequencies of frames with axially loaded Timoshenko members.
8. F. Y. CHENG and W. H. TSENG 1973 *Journal of Structural Division, ASCE* **99**, 527–549. Dynamic matrix of Timoshenko beam columns.
9. J. B. KOSMATKA, 1995 *Computers and Structures* **57**, 141–149. An improved 2-node finite element for stability and natural frequencies of axially loaded Timoshenko beams.
10. R. D. HENSHELL and G. B. WARBURTON 1969 *International Journal for Numerical Methods in Engineering* **1**, 47–66. Transmission of vibration in beam systems.
11. E. DOKUMACI 1987 *Journal of Sound and Vibration* **119**, 443–449. An exact solution for coupled bending and torsion vibrations of uniform beams having single cross-sectional symmetry.
12. W. L. HALLAUER and R. Y. L. LIU 1982 *Journal of Sound and Vibration* **85**, 105–113. Beam bending-torsion dynamic stiffness method for calculation of exact vibration modes.
13. P. O. FRIBERG 1983 *International Journal for Numerical Methods in Engineering* **19**, 479–493. Coupled vibration of beams—An exact dynamic element stiffness matrix.
14. J. R. BANERJEE 1989 *International Journal for Numerical Methods in Engineering* **28**, 1283–1298. Coupled bending–torsional dynamic stiffness matrix for beam elements.
15. C. MEI 1970 *International Journal of Mechanical Sciences* **12**, 883–891. Coupled vibrations of thin-walled beams of open section using the finite element method.
16. M. FALCO and M. GASPARETTO 1973 *Meccanica* **8**, 181–189. Flexural-torsional vibrations of thin-walled beams.

17. J. FITCH 1985 *Journal of Symbolic Computing* **1**, 211–227. Solving algebraic problems using REDUCE.
18. G. RAYNA 1986 *REDUCE Software for Algebraic Computation*, New York: Springer.
19. S. P. TIMOSHENKO, D. H. YOUNG and W. WEAVER 1974 *Vibration Problems in Engineering*, New York : Wiley; fourth edition.
20. M. GOLAND 1945 *Journal of Applied Mechanics* **12**, A197–A208. Flutter of a uniform cantilever wing.
21. J. R. BANERJEE 1991 *Advances in Engineering Software* **13**, 17–24. A FORTRAN routine for computation of coupled bending–torsional dynamic stiffness matrix of beam elements.

APPENDIX A: A PROOF FOR THE CONDITION THAT $c = 1$ FOR $\alpha = \beta$

Equating $\alpha = \beta$ given by equations (13) gives

$$\cos(\phi/3) - \cos(\pi/3 - \phi/3) = a/\sqrt{3q}. \tag{A1}$$

Using simple trigonometric rules, the above equation becomes

$$\sin(\pi/6 - \phi/3) = a/\sqrt{3q}. \tag{A2}$$

Noting that $\tan x = \sin x/(1 - \sin^2 x)^{1/2}$ and making use of the relationship $3q = a^2 + 3b$ of equation (14), it can be shown with the help of equation (A2) that

$$\tan(\pi/6 - \phi/3) = a/\sqrt{3b}. \tag{A3}$$

An expansion of the above equation gives

$$\tan(\phi/3) = \sqrt{3}(\sqrt{b} - a)/(a + 3\sqrt{b}) \tag{A4}$$

Using the trigonometric relationship $\cos x = 1/(1 + \tan^2 x)^{1/2}$, equation (A4) gives

$$\cos(\phi/3) = (a + 3\sqrt{b})/2\sqrt{3q} \tag{A5}$$

Now $\cos\phi$ can be related to $\cos(\phi/3)$ as follows:

$$\cos\phi = 4\cos^3(\phi/3) - 3\cos(\phi/3). \tag{A6}$$

Substituting for $\cos(\phi/3)$ from equation (A5) into equation (A6) gives

$$\cos\phi = (18ab - 2a^3)/\{2(a^2 + 3b)^{3/2}\}. \tag{A7}$$

Comparison of equation (A7) with equation (14) suggests that for the above equation to be valid c must be equal to one. Thus the condition for $\alpha = \beta$ is $c = 1$ which correspond to $x_\alpha = 0$, i.e. the case for the Bernoulli–Euler beam.