Journal of Sound and Vibration (1999) **224**(2), 349–357
Article No. jsvi.1999.2170, available online at http://www.idealibrary.com on IDEAL®

DISCRETE TIME MODEL OF ACOUSTIC WAVES TRANSMITTED THROUGH LAYERS

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(*Received* 3 *April* 1998, *and in final form* 2 *February* 1999)

This paper describes a rapid solution of transmitted acoustic waves through a multiplicity of layers. The layers are considered lossless and have parallel interfaces, which are at normal incidence to the field. The force transfer across the layers is evaluated in discrete form using the z-transform technique. The discrete time response is developed and this is applied in the manner of a digital filter to a variety of excitation functions. The method has the advantage that the response to any real input can be evaluated.

1. INTRODUCTION

The transmission of acoustic waves through a multiplicity of layers has been analyzed by transmission line analogies. The transmission line model is normally Fourier transform representations of the piezoelectric element and therefore yields solutions in terms of angular frequency. Redwood [1] and Steutzer [2] have used equivalent circuits based on the Laplace transform and have derived the time domain transient response for certain basic excitation functions by inverse transformation. The mathematical treatment is complicated when more than two layers are considered and the technique is restricted to certain excitation functions. Also, Lewis [3] has found a closed form solution in the frequency domain for the transmitted signal, but he assumed the sound propagation time is constant for each layer.

In this paper, the similarly analogous electrical transmission line equivalent circuits can be employed to represent acoustic propagation through layers of non-piezoelectric material, plane-wave propagation being assumed throughout. The z-transform technique [4] is applied to the Laplace model to yield a discrete time model of transmitted acoustic wave through ten layers. Computer programs based on a model described by Ali [5, 6] have been developed to calculate the force transfer across these layers.

2. TRANSFER OF FORCE ACROSS AN ARBITRARY NUMBER OF LAYERS

When using the analogy between electrical and mechanical transmission lines [7] it is known that the transmission matrix of two transmission lines connected

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in tandem is equal to the product of the individual matrices [8]. This can be generalized to obtain the expression of the transmission matrix for any number of transmission lines connected in tandem (cascade). From this, the transfer of force across an arbitrary number of layers can be obtained.

The transfer of force from one material of acoustic impedance Z_c to another of acoustic impedance Z_r can be described in simple terms (analogous to voltage) as shown in Figure 1(a) with the equivalent circuit as shown in Figure 1(b). From this figure, the received force, F_0 is given in terms of the incident force, F_i, by the expression

$$F_0 = Z_r F_i/(Z_c + Z_r), \tag{1}$$

or

$$F_0 = F_i(1 + r_0)/2. \tag{2}$$

Here $r_0 = (Z_r - Z_c)/(Z_r + Z_c)$ is the reflection coefficient in terms of force at the boundary between the two media.

The transfer of force from a material of impedance Z_c, through a layer of impedance Z_1, to a material of impedance Z_r can be represented as shown in Figure 1(c) with the equivalent circuit as shown in Figure 1(d). In this diagram Z_{A1} and Z_{B1} represent the distributed components of the familiar "T" equivalent circuit of a transmission line, however in this case Z_{A1} and Z_{B1} have a mechanical interpretation,

$$Z_{A1} = 2Z_1/(\mathrm{e}^{(SX_1/V_1)} - \mathrm{e}^{(-SX_1/2V_1)}) \tag{3}$$

and

$$Z_{B1} = Z_1(\mathrm{e}^{(SX_1/2V_1)} - \mathrm{e}^{(-SX_1/2V_1)})/(\mathrm{e}^{(SX_1/2V_1)} + \mathrm{e}^{(-SX_1/2V_1)}). \tag{4}$$

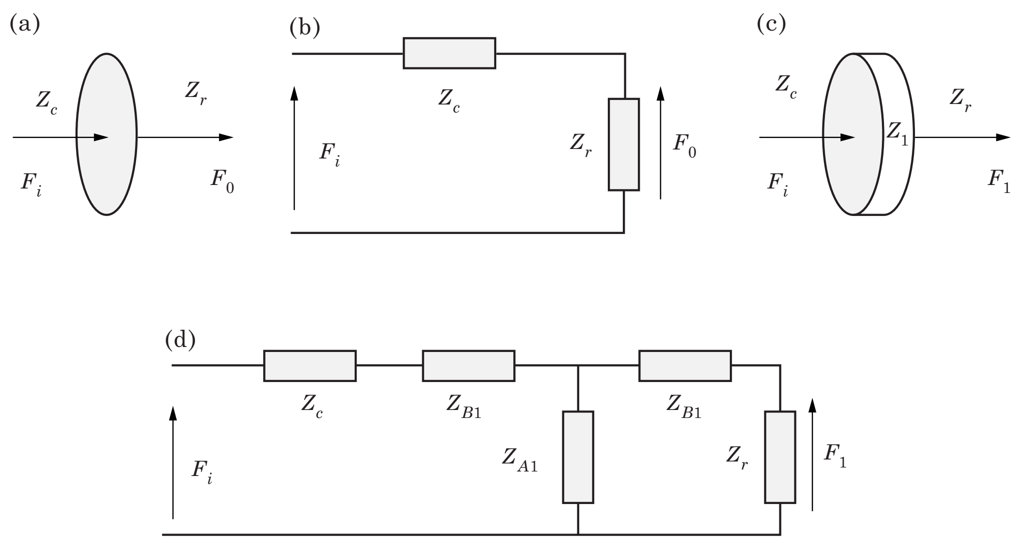

Figure 1. Representation of the transfer of force across a single boundary and a single layer (double boundary) (a) schematic representation of a boundary, (b) the equivalent circuit, (c) schematic representation of a layer, and (d) the equivalent circuit of a mechanical layer.

Here X_1 is the thickness of the acoustic layer, V_1 is the velocity of the compression wave in the layer material of acoustic impedance Z_1 and the Laplace variable S is the complex frequency. The term $X_1/V_1 = T_p$ represents the time delay for an acoustic wave to propagate from one face of the acoustic layer of impedance Z_1 to the other.

For digital computers it is convenient to consider continuous time as being sampled at regular intervals of T seconds, where T is the sampling period. It is clear that a number of discrete time intervals could approximately represent the time delay mT where m is an integer. In terms of the Laplace transform [9] the time delay e^{-SX^1/V^1} approximates to $e^{Sm^1T} = z^{-m^1}$, where the substitution $z = e^{ST}$ has been made and the approximate time delay is given in terms of the z-transform [10]. By using this substitution Z_{A1} and Z_{B1} can be written as

$$Z_{A1} = 2Z_1 z^{-ml}/(1 - z^{-m_1}), \quad Z_{B1} = Z_1(1 - z^{-m_1})/(1 + z^{-m_1}). \tag{5, 6}$$

By using standard circuit analysis techniques and with the appropriate z-transform expression substituted for Z_{A1} and Z_{B1}, the force transfer across one layer can be described as

$$F_1 = \frac{2Z_1 Z_r z^{-m_1} F_i}{(Z_1 + Z_c)(Z_1 + Z_r) + (Z_1 - Z_c)(Z_r - Z_1)z^{-2m_1}}, \tag{7}$$

or

$$F_1 = F_i(1 + r_0)(1 + r_1)z^{-m_1}/2(1 + r_0 r_1 z^{-2m_1}). \tag{8}$$

Equation (8) can be interpreted as a recurrence relationship between the sampled versions of $F_1(t)$ and $F_i(t)$. The relationship is

$$F_1(n) = 0{\cdot}5[(1 + r_0)(1 + r_1)F_i(n - m_1)] - 2r_0 r_1 F_1(n - 2m_1). \tag{9}$$

Similarly, the expression for two layers is

$$F_2 = \frac{F_i(1 + r_0)(1 + r_1)(1 + r_2)z^{-(m_1+m_2)}}{2(1 + r_1 r_2 z^{-2m_2} + r_0[r_1 + r_2 z^{-2m_2}]z^{-2m_1})}. \tag{10}$$

Equation (10) can be interpreted as a recurrence relationship which represents a digital filter relating the nth sample of the transmitted force to the nth and previous samples of the input force waveform and previously calculated force output values:

$$\begin{aligned} F_2(n) = 0{\cdot}5[(1 + r_0)(1 + r_1)(1 + r_2)F_i(n - (m_1 + m_2))] - r_1 r_2 F_2(n - 2m_2) \\ - r_0 r_1 F_2(n - 2m_1) - r_0 r_2 F_2(n - 2(m_1 + m_2)). \end{aligned} \tag{11}$$

The digital filters representing the transfer of force across three and four layers can be written as

$$F_3(n) = 0{\cdot}5[(1+r_0)(1+r_1)(1+r_2)(1+r_3)F_i(n-(m_1+m_2+m_3))] \\ - r_2r_3F_3(n-2m_3) - r_1r_2F_3(n-2m_2) - r_1r_3F_3(n-2(m_2+m_3)) \\ - r_0r_1F_3(n-2m_1) - r_0r_1r_2r_3F_3(n-2(m_1+m_3)) \\ - r_0r_2F_3(n-2(m_1+m_2)) - r_0r_3F_3(n-2(m_1+m_2+m_3)), \quad (12)$$

$$F_4(n) = 0{\cdot}5[(1+r_0)(1+r_1)(1+r_2)(1+r_3)(1+r_4)F_i(n-(m_1+m_2+m_4))] \\ - r_3r_4F_4(n-2m_4) - r_2r_3F_4(n-2m_3) - r_2r_4F_4(n-2m_2) \\ - r_1r_2r_3r_4F_4(n-2(m_2+m_4)) - r_1r_3F_4(n-2(m_2+m_3)) \\ - r_1r_4F_4(n-2(m_2+m_3+m_4)) - r_0r_1F_4(n-2m_1) \\ - r_0r_1r_3r_4F_4(n-2(m_1+m_4)) - r_0r_1r_2r_3F_4(n-2(m_1+m_3)) \\ - r_0r_1r_2r_4F_4(n-2(m_1+m_3+m_4)) - r_0r_2F_4(n-2(m_1+m_2)) \\ - r_0r_2r_3r_4F_4(n-2(m_1+m_2+m_4)) - r_0r_3F_4(n-2(m_1+m_2+m_3)) \\ - r_0r_4F_4(n-2(m_1+m_2+m_3+m_4)), \quad (13)$$

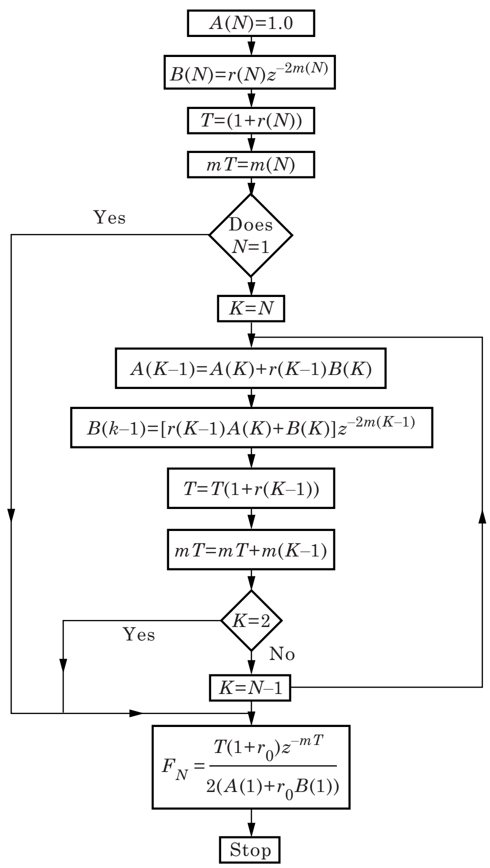

Figure 2. Recursive algorithm flowchart for the force transfer across an arbitrary number of layers. N number of layers; $A(N)$, $B(N)$ vectors; r_1, r_2, $\cdots$, r_N, the reflection coefficients refer to the interface between the layers; m_1, m_2, $\cdots$, m_N integer numbers.

TABLE 1
The materials data

Code number	Materials	Acoustic impedance ($\times 10^6$ kg/m^2 s)	Velocity of longitudinal waves (m/s)
M1	Quartz (fused)	13·13	5970
M2	Glass (Crown)	14·15	5660
M3	Aluminum	17·20	6374
M4	Titanium	27·28	6050
M5	Tungsten	28·80	2400
M6	Brass	36·94	4372
M7	Nickel silver	41·34	4796
M8	Hardened tool steel	45·64	5874
M9	Stainless steel	46·16	5980
M10	Devcon	57·00	2980

where r_0, r_1, r_2, r_3 and r_4 are the reflection coefficients. The recursive algorithm flowchart described in Figure 2 is essentially a recursive digital filter. This algorithm can easily be implemented on a computer using polynomial algebra routines. It is clear that a recursive algorithm can be used to obtain the expression for the force transfer across an arbitrary number of layers.

3. RESULTS

The computer implementation of the discrete time model of transmitted acoustic waves was investigated by studying the performance of the digital filter algorithm described in Figure 2. A digital Dirac impulse of unit amplitude was input to the filter. It is useful to use a Dirac type impulse waveform since the Fourier transform of a Dirac function is flat for all frequencies, so it is possible to test the response of the filter to all frequencies simultaneously.

Table 1 shows the materials layers data that have been used in this technique and these materials are arranged in this table according to the impedance materials. Each layer was assumed to have the same thickness of 1 mm, but the sound propagation time is different for each layer according to the velocity of longitudinal waves. The ten layered acoustic structure is assumed to be bounded on either side by materials of different characteristic impedance and infinite in extent. The layers are surrounded on the front face by Perspex material and by brass material on the back face.

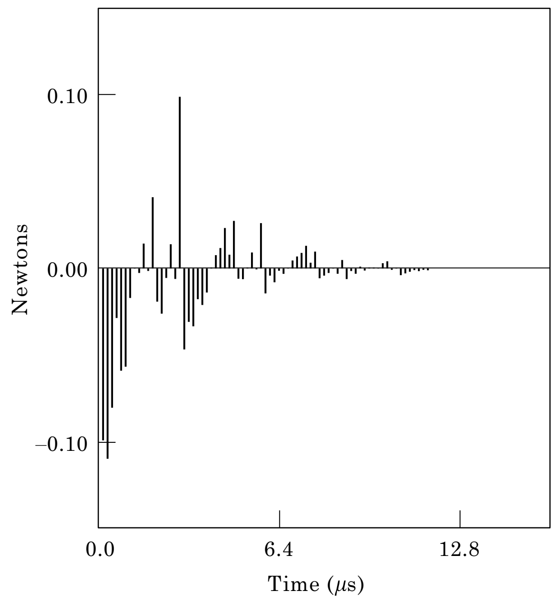

Figure 3. Time domain impulse response for a pressure wave transmitted through ten layers of materials.

The simulations were carried out with an effective sampling frequency of 6·25 MHz. Figure 3 shows the impulse time response function for the ten layers arrangement respectively as described in Table 1. Once the time domain response was obtained, a fast Fourier transform was performed on the data. The resulting frequency response curve is plotted in Figure 4. From this figure, it can be seen

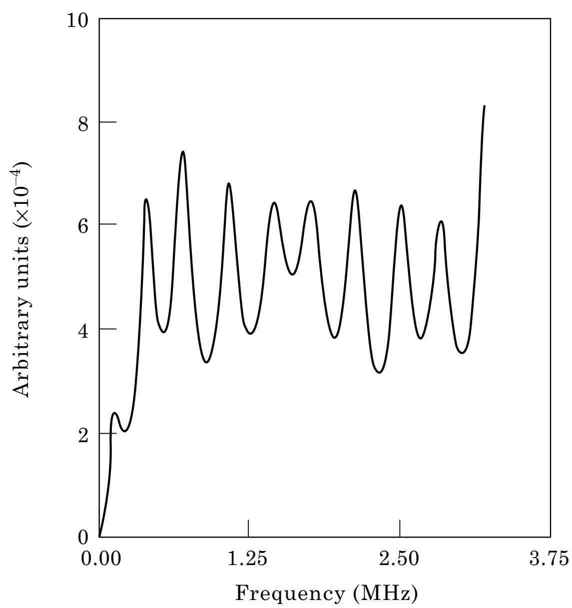

Figure 4. Impulse frequency response, derived from the time domain impulse response of Figure 3.

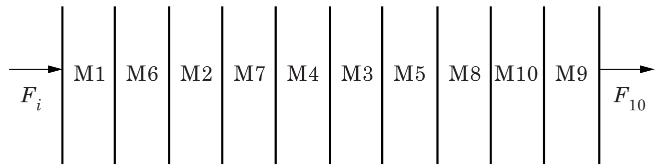

Figure 5. A ten layered acoustic structure having an arbitrary impedance profile as shown. An incident pressure wave (F_i) travelling from left to right sets up transmitted waves (F_{10}).

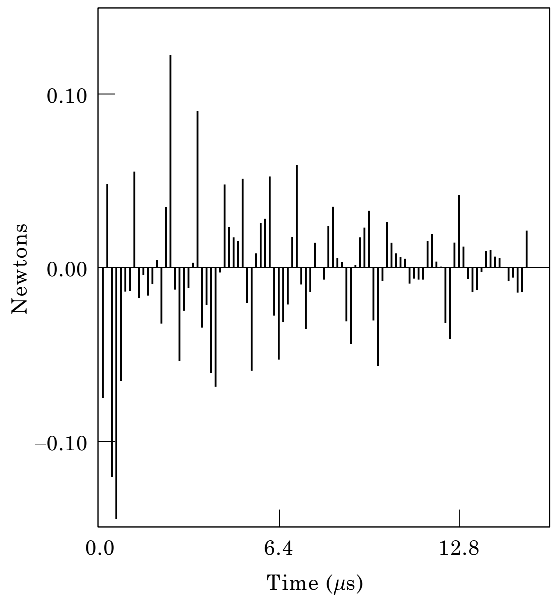

Figure 6. Transmitted time sequence for pressure wave through the arbitrary acoustic impedance layers as shown in Figure 5.

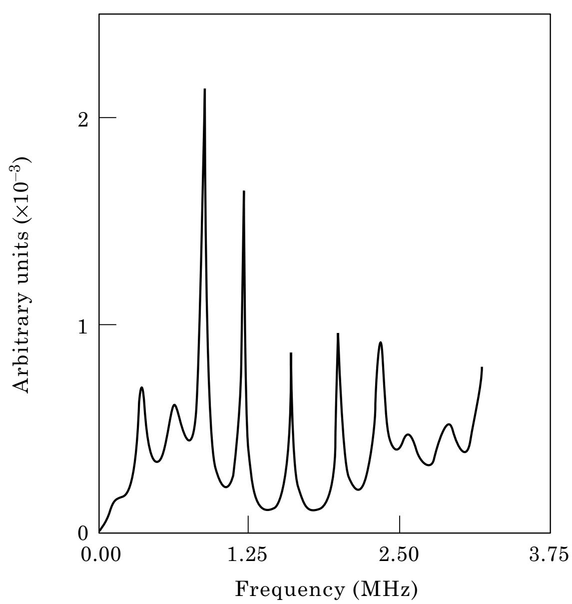

Figure 7. The frequency domain spectrum of Figure 6.

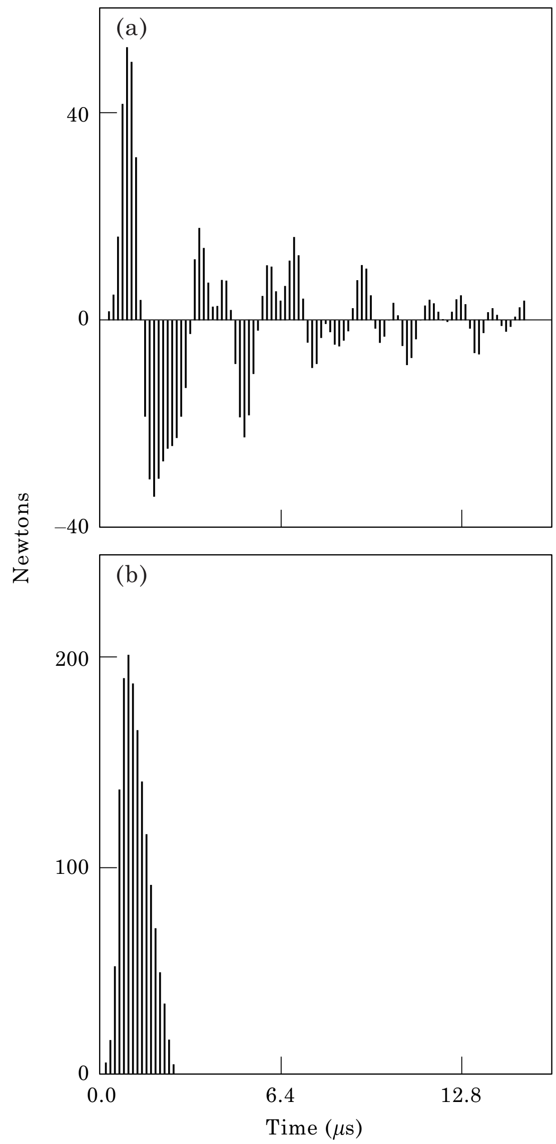

Figure 8. Calculated impulse response of transmitted wave through the arbitrary acoustic impedance layer described in Figure 5. The exciting waveform is shown in (b).

that the frequency response is periodic with a period equal to the ratio of the velocity of sound propagation to the thickness of the layer. Figure 5 shows a multilayered structure having arbitrary impedance materials. An incident impulse pressure wave travelling from left to right has transmitted a time domain signal given in Figure 6. The corresponding frequency response curve is shown in Figure 7. It is clear from these figures that the magnitude and shape of the time and frequency domain strongly depends on the acoustic impedance variation of the adjacent layers. Lewis [3] used a matrix model for analyzing the transmitted acoustic waves from layered structures of biological tissue. The matrix analysis has a closed form solution in the frequency domain for the transmitted signal. It has been shown that the results of this paper agree very well with Lewis' results, however, the present work extends the analysis of multilayered acoustic structures with different sound propagation time for each layer.

In order to illustrate the great versatility of the technique, the response has also been calculated for a finite duration function. The result is shown in Figure 8 for the system described in Figure 4. It is clear from all of these results that calculation using the technique proposed in this paper yields an accurate determination of the transmission of pressure waves through an arbitrary acoustic impedance variation of the layers.

4. CONCLUSIONS

A discrete time model, which accurately describes the pressure time response of multilayered acoustic structure, has been developed. The z-transform has been applied to the Laplace model to yield the impulse response of force transfer across arbitrary acoustic layers. The method has the advantage that the response for transmission of acoustic waves through a multiplicity of practical material layers and arbitrary input function can be estimated rapidly.

The multilayered acoustic structures with small and high impedance variations have been analyzed. The model could be extended to incorporate the effects of absorption of the layers.

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