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FREE VIBRATION ANALYSIS OF TRUNCATED CONICAL SHELLS BY THE DIFFERENTIAL QUADRATURE METHOD

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1. INTRODUCTION

Conical shells are widely used in aeronautical, astronautical, civil, and chemical engineering and as a consequence, the vibration of conical shell structures has been extensively studied [1, 2]. It is difficult to obtain the closed-form solutions for shells with various boundary conditions. Therefore, numerical or approximate methods, such as finite-element method, finite-difference method, weighted residual method and Rayleigh–Ritz method, have been widely used to obtain the approximate solutions. These methods provide effective means for engineering analysis in most cases. However, their drawbacks in some specific situations, such as in the cases of stress concentration and high-frequency responses, are obvious. Therefore, new ways are still being developed [2].

The differential quadrature method (DQM) has proven to be quite an efficient numerical technique for the solution of partial differential equations [3–7]. Bert *et al.* [3] first used the DQM to solve problems in structural mechanics in 1988. Since then, the methods have been applied successfully to a variety of structural mechanics problems [4–12].

Noting the lack of published work on the vibration of truncated conical shells by using the DQM, the present paper presents the procedures in which the DQM is applied to the study of free vibration of truncated conical shells with a variety of boundary conditions based on Love's first approximation theory. Some results are compared with existing data to show the efficiency and accuracy of the method, and some new results are also given.

2. FORMULATIONS

As usual, the reference surface of the conical shell is taken to be at its mid-surface where an orthogonal co-ordinate system (θ, x, z) is located and r is the radius at co-ordinated point (θ, x, z), and the displacement fields of the conical shell in the θ, x and z directions are denoted by u, v, and w respectively. Assume that the displacements of the conical shell take the following form for free vibration

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analysis, namely

$$u(x,\ \theta,\ t) = U(x)\cos(n\theta + \omega\ t),$$

$$v(x,\ \theta,\ t) = V(x)\sin(n\theta + \omega\, t), \quad n = 0,\ 1,\ 2, \ldots \tag{1}$$

$$w(x,\ \theta,\ t) = W(x)\cos(n\theta + \omega t),$$

where n is the wave number in the circumferential direction, and ω is the circular frequency. Thus, the equation of motion of conical shells can be written in the following matrix form based on Love's first approximation theory, namely,

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \rho h \omega^2 \begin{Bmatrix} U \\ V \\ W \end{Bmatrix}, \tag{2}$$

where

$$L_{11} = A_{11}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A_{11}\sin\alpha}{r}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{A_{22}\sin^2\alpha}{r^2} - \frac{A_{66}n^2}{r^2},$$

$$L_{12} = \frac{n(A_{66} + A_{12})}{r}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{n\sin\alpha\ (A_{66} + A_{22})}{r^2},$$

$$L_{13} = \frac{A_{12}\cos\alpha}{r}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{A_{22}\sin\alpha\cos\alpha}{r^2},$$

$$L_{21} = -\frac{n(A_{12} + A_{66})}{r}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{n\sin\alpha(A_{22} + A_{66})}{r^2},$$

$$L_{22} = \left(A_{66} + \frac{D_{66}\cos^2\alpha}{r^2}\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\sin\alpha}{r}\left(A_{66} - \frac{D_{66}\cos^2\alpha}{r^2}\right)\frac{\mathrm{d}}{\mathrm{d}x} - \frac{A_{66}\sin^2\alpha}{r^2}$$
$$- \frac{n^2}{r^2}\left(A_{22} + \frac{D_{22}\cos^2\alpha}{r^2}\right),$$

$$L_{23} = \frac{n\cos\alpha(D_{12} + 2D_{66})}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{D_{22}n\sin\alpha\cos\alpha}{r^3}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{n\cos\alpha}{r^2}\left(A_{22} + \frac{n^2D_{22}}{r^2}\right),$$

$$L_{31} = -\frac{A_{12}\cos\alpha}{r}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{A_{22}\sin\alpha\cos\alpha}{r^2},$$

$$L_{32} = \frac{n\cos\alpha\,(D_{12}+2D_{66})}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{n\sin\alpha\cos\alpha(4D_{66}+2D_{12}+D_{22})}{r^3}\frac{\mathrm{d}}{\mathrm{d}x}$$

$$-\frac{n\cos\alpha}{r^2}\left(A_{22}+\frac{n^2D_{22}}{r^2}\right)+\frac{n\cos\alpha\sin^2\alpha(4D_{66}+2D_{12}+2D_{22})}{r^4},$$

$$L_{33} = -D_{11}\frac{\mathrm{d}^4}{\mathrm{d}x^4} - \frac{2D_{11}\sin\alpha}{r}\frac{\mathrm{d}^3}{\mathrm{d}x^3} + \frac{D_{22}\sin^2\alpha + n^2(2D_{12}+4D_{66})}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

$$-\left[\frac{D_{22}\sin^3\alpha}{r^3}+\frac{n^2\sin\alpha(4D_{66}+2D_{12})}{r^3}\right]\frac{\mathrm{d}}{\mathrm{d}x} - \frac{A_{22}\cos^2\alpha}{r^2} - \frac{n^4D_{22}}{r^4}$$

$$+\frac{n^2\sin^2\alpha\,(4D_{66}+2D_{12}+2D_{22})}{r^4}$$

in which

$$A_{11}=\frac{Eh}{1-\mu^2},\qquad A_{12}=A_{21}=\frac{\mu Eh}{1-\mu^2},\qquad A_{66}=\frac{Eh}{2(1-\mu)},$$

$$D_{11}=\frac{Eh^3}{12(1-\mu^2)},\qquad D_{12}=D_{21}=\frac{\mu Eh^3}{12(1-\mu^2)},\qquad D_{66}=\frac{Eh^3}{24(1-\mu)},$$

and E, μ, α, and h are Young's modulus, the Poisson ratio, the semi-vertex angle, and the shell thickness respectively.

Let B_{ij}, C_{ij}, E_{ij}, and F_{ij} denote the weighting coefficients of the first, second, third and fourth order derivative with respect to the variable of x, and δ_{ij} be the Kronecker delta. Details on the determination of the weighting coefficients and on the differential quadrature (DQ) method can be found, for example, in references [3–6]. In the DQ method, the equation of motion can be rewritten in the following matrix form:

$$\begin{bmatrix}[P_{11}]\,[P_{12}]\,[P_{13}]\\ [P_{21}]\,[P_{22}]\,[P_{23}]\\ [P_{31}]\,[P_{32}]\,[P_{33}]\end{bmatrix}\begin{Bmatrix}\{U\}\\ \{V\}\\ \{W\}\end{Bmatrix} = \rho h\omega^2\begin{Bmatrix}\{U\}\\ \{V\}\\ \{W\}\end{Bmatrix}, \tag{3}$$

where

$$[P_{11}]_{ij} = A_{11}C_{ij} + \frac{A_{11}\sin\alpha}{r_i}B_{ij} - \left(\frac{A_{22}\sin^2\alpha}{r_i^2}+\frac{A_{66}n^2}{r_i^2}\right)\delta_{ij},$$

$$[P_{12}]_{ij} = \frac{n(A_{66}+A_{12})}{r_i}B_{ij} - \frac{n\sin\alpha\,(A_{66}+A_{22})}{r_i^2}\delta_{ij},$$

$$[P_{13}]_{ij} = \frac{A_{12}\cos\alpha}{r_i} B_{ij} - \frac{A_{22}\sin\alpha\cos\alpha}{r_i^2}\delta_{ij},$$

$$[P_{21}]_{ij} = -\frac{n(A_{12}+A_{66})}{r_i} B_{ij} - \frac{n\sin\alpha(A_{22}+A_{66})}{r_i^2}\delta_{ij},$$

$$[P_{22}]_{ij} = \left(A_{66} + \frac{D_{66}\cos^2\alpha}{r_i^2}\right) C_{ij} + \frac{\sin\alpha}{r_i}\left(A_{66} - \frac{D_{66}\cos^2\alpha}{r_i^2}\right) B_{ij}$$

$$- \left[\frac{A_{66}\sin^2\alpha}{r_i^2} + \frac{n^2}{r_i^2}\left(A_{22} + \frac{D_{22}\cos^2\alpha}{r_i^2}\right)\right]\delta_{ij},$$

$$[P_{23}]_{ij} = \frac{n\cos\alpha(D_{12}+2D_{66})}{r_i^2} C_{ij} + \frac{D_{22}\, n\sin\alpha\cos\alpha}{r_i^3} B_{ij}$$

$$- \frac{n\cos\alpha}{r_i^2}\left(A_{22} + \frac{n^2 D_{22}}{r_i^2}\right)\delta_{ij},$$

$$[P_{31}]_{ij} = -\frac{A_{12}\cos\alpha}{r_i} B_{ij} - \frac{A_{22}\sin\alpha\cos\alpha}{r_i^2}\delta_{ij},$$

$$[P_{32}]_{ij} = \frac{n\cos\alpha(D_{12}+2D_{66})}{r_i^2} C_{ij} - \frac{n\sin\alpha\cos\alpha(4D_{66}+2D_{12}+D_{22})}{r_i^3} B_{ij}$$

$$- \left[\frac{n\cos\alpha}{r_i^2}\left(A_{22} + \frac{n^2 D_{22}}{r_i^2}\right) - \frac{n\cos\alpha\sin^2\alpha(4D_{66}+2D_{12}+2D_{22})}{r_i^4}\right]\delta_{ij},$$

$$[P_{33}]_{ij} = -D_{11}F_{ij} - \frac{2D_{11}\sin\alpha}{r_i} E_{ij} + \frac{D_{22}\sin^2\alpha + n^2(2D_{12}+4D_{66})}{r_i^2} C_{ij}$$

$$- \left[\frac{D_{22}\sin^3\alpha}{r_i^3} + \frac{n^2\sin\alpha(4D_{66}+2D_{12})}{r_i^3}\right] B_{ij}$$

$$- \left[\frac{A_{22}\cos^2\alpha}{r_i^2} + \frac{n^4 D_{22}}{r_i^4} - \frac{n^2\sin^2\alpha(4D_{66}+2D_{12}+2D_{22})}{r_i^4}\right]\delta_{ij}$$

$$i, j = 1,2, \ldots, m,$$

in which m is the number of grid point and is assumed to be the same for all three displacements for convenience.

In terms of the DQ method, various boundary conditions can be expressed as follows:

1. At a clamped edge (denoted by C), $U_k = V_k = W_k = \sum_{j=1}^{m} B_{kj} W_j = 0.$

2. At a simply supported edge (denoted by S);

$$U_k = W_k = 0,$$

$$A_{11} \sum_{j=1}^{m} B_{kj} U_j + \frac{A_{12}}{r_{ik}} (nV_k + U_k \sin\alpha + W_k \cos\alpha) = 0,$$

$$-D_{11} \sum_{j=1}^{m} C_{kj} W_j + \frac{D_{12}}{r_k^2} \left(n^2 W_k + nV_k \cos\alpha - r_k \sin\alpha \sum_{j=1}^{m} B_{kj} W_j \right) = 0.$$

3. At a free edge (denoted by F),

$$A_{66} \left(\sum_{j=1}^{m} B_{kj} V_j - \frac{nU_k + V_k \sin\alpha}{r_k} \right) = 0.$$

$$-\frac{A_{66} n}{r_k} U_k - \left(\frac{A_{66} \sin\alpha}{r_k} + \frac{2D_{66} \sin\alpha \cos^2\alpha}{r_k^3} \right) V_k + \left(A_{66} + \frac{D_{66} \cos^2\alpha}{r_k^2} \right) \sum_{j=1}^{m} B_{kj} V_j$$

$$-\frac{2D_{66} n \sin\alpha \cos\alpha}{r_k^3} W_k + \frac{2D_{66} \cos\alpha}{r_k^2} \sum_{j=1}^{m} B_{kj} W_j = 0.$$

$$-D_{11} \sum_{j=1}^{m} C_{kj} W_j + \frac{D_{12}}{r_k^2} (n^2 W_k + nV_k \cos\alpha - r_k \sin\alpha \sum_{j=1}^{m} B_{kj} W_j) = 0.$$

$$\frac{n(D_{12} + 2D_{66}) \cos\alpha}{r_k^2} \sum_{j=1}^{m} B_{kj} V_j - \frac{n(D_{12} + D_{22} + 4D_{66}) \sin\alpha \cos\alpha}{r_k^3} V_k - D_{11} \sum_{j=1}^{m} E_{kj} W_j$$

$$-\frac{D_{11} \sin\alpha}{r_k} \sum_{j=1}^{m} C_{kj} W_j + \left[\frac{n^2 (D_{12} + 4D_{66})}{r_k^2} + \frac{D_{22} \sin^2\alpha}{r_k^2} \right] \sum_{j=1}^{m} B_{kj} W_j$$

$$-\frac{n^2 (D_{12} + D_{22} + 4D_{66}) \sin\alpha}{r_k^3} W_k = 0,$$

where k takes the value of 1 for the small edge and of m for the large edge respectively.

Applying the appropriate boundary conditions to Eq. (3) and solving the corresponding eigenvalue problem numerically yield the frequencies of conical shells with various boundary conditions (nine cases). There are several ways to apply for the boundary conditions [3–7]. One way [5] similar to the δ-approach [3, 5] is used here. The slight difference between these two methods is that for the method used in the present analysis, the two boundary conditions in the wall thickness direction (z direction) are applied at the boundary point, and then, one of the boundary conditions is used to replace the equation established at the inner point adjacent to the boundary point; while for the δ-approach, the two boundary conditions are applied at the boundary point as well as at the inner point separated by a small distance δ.

3. RESULTS AND CONCLUSIONS

A small computer program is written which can be used to obtain the frequency of conical shells with various boundary conditions, various lengths and vertex angles. To demonstrate the accuracy and efficiency of the DQ method, a few cases are analyzed ($m = 13$) and compared with existing results. The results are listed in Table 1 and shown in Figures 1 and 2. The frequency parameter $\tilde{\omega}$ is defined as

$$\tilde{\omega} = \sqrt{E/\rho(1-\mu^2)}(\omega/a)$$

where ρ is the mass density and a is the radius of the large edge. For the notation on the boundary conditions, the first letter denotes the small boundary condition and the second letter denotes the large boundary conditions. For example, F–C means the shell is free at the small edge and fixed at the large edge.

Table 1 shows the comparison of the frequency parameters for the truncated conical shells with three different boundary conditions, namely, F–F, S–S and C–C. Overall excellent agreement between the DQ data and existing results by Irie [1] is observed. In the DQ analysis, it is observed that little CPU time (less than 30 s if 13

Table 1

Comparison of the frequency parameter for the truncated conical shells with three different boundary conditions (C–C, S–S, and F–F) ($\mu = 0{\cdot}3$, $h/a = 0{\cdot}01$, $\alpha = 45°$, *and* $l \sin\alpha = 0{\cdot}50$)

Circumferential wave number	F–F		S–S		C–C	
n	DQM	Irie [1]	DQM	Irie [1]	DQM	Irie [1]
2	0·0132	0·0135	0·6319	0·6310	0·6696	0·6696
3	0·0340	0·0343	0·5063	0·5065	0·5428	0·5430
4	0·0620	0·0622	0·3944	0·3947	0·4566	0·4570
5	0·0963	0·0964	0·3341	0·3348	0·4089	0·4095
6	0·1361	0·1361	0·3239	0·3248	0·3962	0·3970
7	0·1809	0·1809	0·3513	0·3524	0·4142	0·4151
8	0·2309	0·3211	0·4022	0·4033	0·4567	0·4577
9	0·2866	0·2871	0·4673	0·4684	0·5175	0·5186

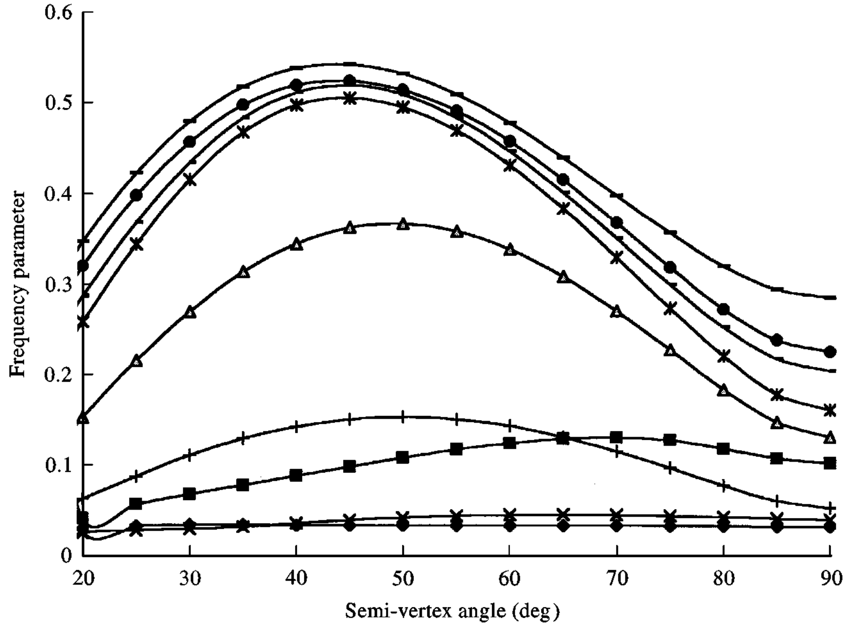

Figure 1. Variation of frequency parameter with semi-vertex angles.

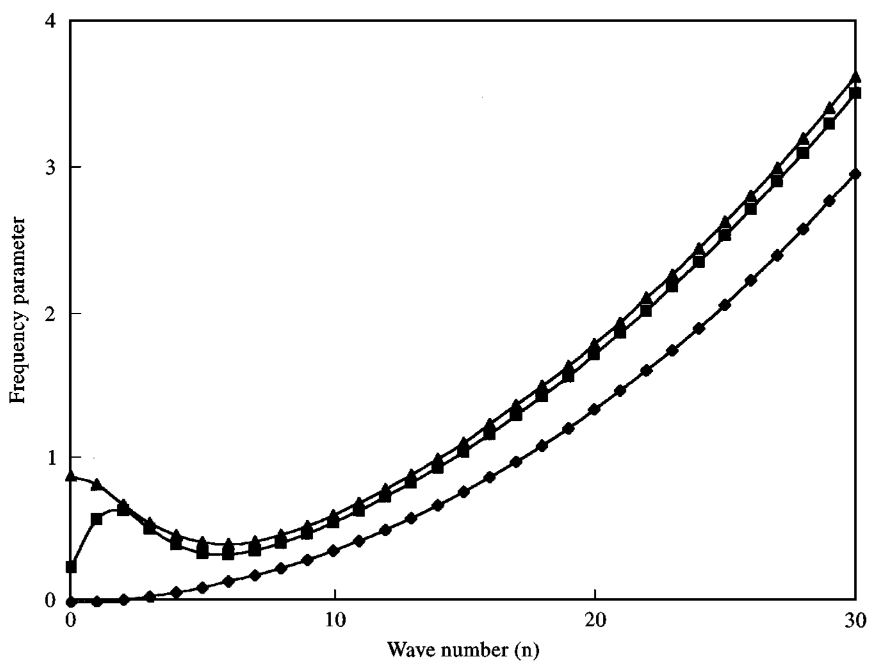

Figure 2. Effect of boundary conditions on the frequency parameters ($\mu = 0{\cdot}3$).

grid points are used) is needed to get the results. Therefore, the method is good for use in engineering practice.

Figure 1 shows the variation of frequency parameter with the semi-vertex angle for nine different boundary conditions. The dimensions of the conical shell are

$h/a = 0{\cdot}01$ and $l \sin \alpha = 0{\cdot}50$. l is the meridional length of the shell and $\mu = 0{\cdot}3$. As was expected, the sequence of the boundary conditions for the frequency parameter (wave number n is 3) from the smallest to the largest is F–F, S–F, F–S, C–F, F–C, S–S, C–S, S–C, and C–C at $\alpha = 60°$. However, this tendency is not valid for the entire range of the semi-vertex angle considered, as is clearly shown in Figure 1.

The variation of frequency parameter with the wave number is shown in Figure 2 for three different boundary conditions, namely, S–S (denoted by filled rectangle), C–C (denoted by the filled triangle), and F–F. The dimensions of the conical shell are $h/\alpha = 0{\cdot}01$, $\alpha = 45°$, and $l \sin \alpha = 0{\cdot}50$. It can be seen that the frequency parameter of the F–F shell is lower than that for the shells with the other two boundary conditions. For higher wave numbers, the difference in frequency parameter between the other two boundary conditions (S–S and C–C) becomes smaller.

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