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LIMIT CYCLES IN HIGHLY NON-LINEAR DIFFERENTIAL EQUATIONS

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This paper is concerned with both small-amplitude and large-amplitude limit cycle bifurcations of planar differential systems. The analysis is not restricted to minimal models with few non-linear terms, in fact, the novel approach adopted here is to consider differential equations containing highly non-linear terms in both the damping and restoring coefficients. The maximum number of limit cycles which may be bifurcated in a small region of the origin is given for certain classes of the more generalised mixed (Rayleigh–Liénard) oscillator equations of the form $\ddot{x} + (f(x) + h(\dot{x}))\dot{x} + g(x) = 0$. Certain mechanical systems are investigated.

1. INTRODUCTION

Limit cycles, or isolated periodic solutions, in planar differential systems commonly occur when modelling both the natural and technological sciences. The differential equations used to model physical systems can contain highly non-linear relationships often derived from experimental data. The problems of determining the maximum possible number and relative configurations of limit cycles for systems of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x, y), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x, y), \tag{1}$$

where P and Q are polynomials, dates back to the early investigations of Poincaré in the 1880's. In order to make progress with these questions, researchers have considered both local and global bifurcations. Unfortunately, the results in the global case number relatively few and it is only in recent years that many more results have been found when restricting the analysis to small-amplitude limit cycle bifurcations.

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Most of the early history in the theory of limit cycles was stimulated by practical problems displaying periodic behaviour. For example, the differential equation derived by Rayleigh [1] in 1877, related to the oscillation of a violin string, is given by

$$\frac{d^2x}{dt^2} + \varepsilon\left(\frac{1}{3}\left(\frac{dx}{dt}\right)^2 - 1\right)\frac{dx}{dt} + x = 0. \tag{2}$$

Let $\dot{x} = y$, then equation (2) can be written as a planar system of the form

$$\dot{x} = y, \quad \dot{y} = -x - \varepsilon\left(\frac{y^2}{3} - 1\right)y, \tag{3}$$

where $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$. Following the invention of the triode vacuum tube, which was able to produce stable self-excited oscillations of constant amplitude, van der Pol [2] obtained the following differential equation to describe this phenomenon:

$$\frac{d^2x}{dt^2} + \varepsilon(x^2 - 1)\frac{dx}{dt} + x = 0,$$

which may be transformed into a planar system of the form

$$\dot{x} = y, \quad \dot{y} = -x - \varepsilon(x^2 - 1)y. \tag{4}$$

Systems (3) and (4) can both display periodic behaviour.

Perhaps the most famous class of differential equations, which generalise equation (4), are those first investigated by Liénard in 1928:

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0,$$

where f and g are polynomials. In the phase plane, this becomes

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{5}$$

The equation can be used to model resistor–inductor–capacitor circuits with non-linear circuit elements. It can also be used to model certain mechanical systems, where $f(x)$ represents the damping coefficient and $g(x)$ the restoring force or stiffness. Liénard applied the change of variable $z = y - F(x)$, where $F(x) = \int_0^x f(s)\, ds$, to obtain an equivalent system in the so-called Liénard plane:

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x). \tag{6}$$

Let $H(i, j)$ denote the maximum number of global limit cycles, where i is the degree of f and j is the degree of g, and let ∂ denote the degree of a polynomial. The main results, in the global case, for systems (5) and (6) are summarised below.

1. In 1928, Liénard [3] proved that if $\partial g = 1$ and F is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geqslant a$, then equation (6) has a unique limit cycle.
2. In 1973, Rychkov [4] proved that if $\partial g = 1$ and F is an odd polynomial of degree five, then equation (6) has at most two limit cycles.

3. In 1977, Lins de Melo and Pugh [5] proved that $H(2,1)=1$. They also conjectured that $H(2m,1)=H(2m+1,1)=m$, where m is a natural number.
4. In 1988, Coppel [6] proved that $H(1,2)=1$.
5. In 1996, Dumortier and Chengzhi Li [7] proved that $H(1,3)=1$.
6. In 1997, Dumortier and Chengzhi Li [8] proved that $H(2,2)=1$.
7. In 1998, Khibnik *et al.* [9] gave partial results when $\partial f=2$ and $\partial g=3$.

More recently, Giacomini and Neukirch [10, 11] have introduced a new method to investigate the limit cycles of Liénard systems when $\partial g=1$ and $F(x)$ is an odd polynomial. They are able to give algebraic approximations to the limit cycles and obtain information on the number and bifurcation sets of the periodic solutions. One of the most attractive properties of their method is that it is not perturbative in nature, and so it is not necessary to have large or small parameters in order to apply the theory. More recently, Llibre [12] has corrected one of the conjectures proposed by Giacomini and Neukirch, and Mickens [13] has shown that the well-known method of slowly varying amplitude and phase can be used to calculate the stability of the limit cycles as well as reproducing results from the Melnikov theory. The major problem with many perturbative methods is that in order to evaluate the integrals in the unperturbed system the restoring force must be at most cubic, and ideally linear. Hence, much interesting phenomena are missed. In contrast, a local method is presented here which allows systems to be highly non-linear in both the damping and restoring coefficients.

Although the Liénard equations appear simple enough, as shown above, the known global results on the maximum number of limit cycles are scant. By contrast, if the analysis is restricted to local bifurcations, then many more results may be obtained. It would be interesting, therefore, to investigate how the local and global results are related.

Suppose that the origin of system (5), and hence (6), is a fine focus. Thus, the origin is a centre for the linear system but not the non-linear system. Let $\hat{H}(i,j)$ be the maximum number of small-amplitude limit cycles that can bifurcate within a small neighbourhood of the origin for system (5), where i is the degree of f and j is the degree of g. Suppose that m and n are natural numbers. Blows *et al.* [14–16] have used inductive arguments in order to prove the results below.

1. If g is odd and $\partial f=2m$ or $2m+1$, then $\hat{H}=m$.
2. If f is even, $\partial f=2m$, then $\hat{H}=m$, whatever g is.
3. If f is odd, $\partial f=2m+1$ and $\partial g=2n+2$ or $2n+3$, then $\hat{H}=m+n$.
4. If $\partial f=2$, $g(x)=x+g_e(x)$, where g_e is even and $\partial g=2n$, then $\hat{H}=n$.

Recently, Christopher and Lynch [17–19] have developed a new algebraic method for determining the Liapunov quantities and this has allowed further computations which could not be carried out with the methods used in earlier papers. Let $\lfloor\,.\,\rfloor$ denote the 'integer part', then the new results are listed below.

5. If $\partial f=2$ and $\partial g=n$, then $\hat{H}=\lfloor(2n+1)/3\rfloor$.
6. If $\partial g=2$ and $\partial f=m$, then $\hat{H}=\lfloor(2m+1)/3\rfloor$.
7. If $\partial f=3$ and $\partial g=n$, then $\hat{H}=2\lfloor 3(n+2)/8\rfloor$, for all $1<n\leqslant 50$.
8 If $\partial g=3$ and $\partial f=m$, then $\hat{H}=2\lfloor 3(m+2)/8\rfloor$, for all $1<m\leqslant 50$.

TABLE 1

The values $\hat{H}(i, j)$ for varying degrees of f and g. The asterisk denotes those cases where one more limit cycle may appear when considering complex coefficients

Degree of f \ Degree of g	1	2	3	4	5	6	7	8	9	10	11	12	13	⋯	48	49	50
50	↑	↑	38*														
49	24	33	38														
48	24	32	36*														
⋮	⋮	⋮	⋮														
13	6	9	10*														
12	6	8	10														
11	5	7	8*														
10	5	7	8*														
9	4	6	8	9													
8	4	5	6*	9													
7	3	5	6	8													
6	3	4	6	7													
5	2	3	4	6	6												
4	2	3	4	4	6	7	8	9	9								
3	1	2	2	4	4	6	6	6*	8	8*	8*	10	10*	⋯	36*	38	38*
2	1	1	2	3	3	4	5	5	6	7	7	8	9	⋯	32	33	→
1	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→

Complementing these results is the calculation of $\hat{H}$ when f and g are of specified degrees. The results are presented in Table 1.

The ultimate aim is to establish a general formula for $\hat{H}(i, j)$ as a function of the degrees of f, g and, if possible, to relate these values to the results for the bifurcations of global limit cycles.

Consider the generalised mixed (Rayleigh–Liénard) differential equations of the form

$$\frac{d^2x}{dt^2} + \varepsilon\left(a_0 + a_2x^2 + a_4x^4 + c_2\left(\frac{dx}{dt}\right)^2\right)\frac{dx}{dt} + b_1x + b_3x^3 = 0,$$

which may be written in the form

$$\dot{x} = y, \quad \dot{y} = -b_1x - b_3x^3 - \varepsilon(a_0 + a_2x^2 + a_4x^4 + c_2y^2)y. \tag{7}$$

Garcia-Margallo and Bejarano [20] considered system (7) and, by applying the generalised harmonic balance method using Jacobian elliptic functions, they showed that there are either zero, one or two limit cycles around the origin. More recently, the same authors [21] have shown that it is possible to have seven limit cycles; two nests of three surrounded by a larger limit cycle. The analysis not only works for small ε but also for $\varepsilon = 1$. By considering a local analysis, Lynch [22] investigated the same system and proved that a maximum of three small-amplitude limit cycles could be bifurcated from the origin. It is also possible to bifurcate three small-amplitude limit cycles simultaneously from each of two symmetric co-existing fine foci. In the case where system (7) is complex, it can be shown that four periodic orbits can bifurcate from each point [23].

In order to exemplify the usefulness of local methods consider the more generalised mixed (Rayleigh–Liénard) differential equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + (f(x) + h(\dot{x}))\frac{\mathrm{d}x}{\mathrm{d}t} + g(x) = 0,$$

which may be written in the form

$$\dot{x} = y, \quad \dot{y} = -g(x) - (f(x) + h(y))y. \tag{8}$$

Assume that the origin is a fine focus; let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$, $g(x) = x + b_2x^2 + b_3x^3 + \cdots + b_nx^n$ and $h(y) = c_1y + c_2y^2 + c_3y^3 + \cdots + c_py^p$, (when m, n and p are natural numbers). As far as the authors are aware, systems of form (8) with highly non-linear terms have not been investigated to-date.

In Section 2, the technique for generating small-amplitude limit cycles is briefly described and results are given for certain classes of systems of the form (8). Local and global bifurcations for certain mechanical systems are discussed in Section 3. The conclusions are presented in Section 4.

2. SMALL-AMPLITUDE LIMIT CYCLES OF THE MORE GENERALISED MIXED (RAYLEIGH–LIÉNARD) OSCILLATOR

The detailed method for obtaining the maximum number of small-amplitude limit cycles is well documented in the literature and for Liénard equations may be found in references [15, 24]. A brief summary of the method is given for completeness. It is well-known that there is a Liapunov function, $V(x, y)$ say, such that the rate of change of V along trajectories is given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \eta_2r^2 + \eta_4r^4 + \cdots + \eta_{2k}r^{2k} + \cdots.$$

The η_{2k} are polynomials in the coefficients of the system and are called the focal values. The sign of the first non-zero focal value determines the stability of the origin. The origin is a centre when all of these quantities are zero, in fact, one needs only consider the value of η_{2k} reduced modulo $(\eta_2, \eta_4, \ldots, \eta_{2k-2})$ in order to obtain the Liapunov quantity $L(k-1)$. One is then in a position to try to bifurcate as many small-amplitude limit cycles as possible. The coefficients in the Liapunov values, $L(r)$ say, are selected such that

$$|L(r)| \ll |L(r+1)| \quad \text{and} \quad \mathrm{L}(r)\, L(r+1) < 0,$$

for $r = 0, 1, \ldots, k-1$. At each stage the origin reverses stability and a limit cycle bifurcates in a small region of the critical point. If all of these conditions are satisfied, then there are exactly k small-amplitude limit cycles. Conversely, if $L(k) \neq 0$, then at most k limit cycles can bifurcate. Sometimes it is not possible to bifurcate the full complement of limit cycles as can be seen in the example given later in this section.

Using similar arguments to those appearing in references [15, 24], the authors have been able to prove the following results for system (8) when the origin is a fine focus.

1. If g is odd, f is odd, $\partial f = 2m + 1$ and h is odd, $\partial h = 2p + 1$, then $\hat{H} = m + p$.
2. If f is even, $\partial f = 2m$ and h is odd, then $\hat{H} = m$, whatever g is.
3. If g is odd, f is odd and h is even, $\partial h = 2p + 2$, then $\hat{H} = p$.

The methods of proof involve inductive arguments very similar to those used in the earlier papers and will not be listed here.

Consider the system

$$\dot{x} = y, \quad \dot{y} = -g(x) - (f(x) + c_2 y^2)y, \tag{9}$$

with f even, $\partial f = 2m$ and g odd, $\partial g = 2m - 1$. System (9) is similar to system (7) with more non-linear terms. Let $\hat{H}(m)$ denote the maximum number of small-amplitude limit cycles which can be bifurcated from the origin, then the following results have been proven for system (9):

4. $\hat{H}(1) = 2$, $\hat{H}(2) = 3$, $\hat{H}(3) = 6$, $\hat{H}(4) = 7$ and $\hat{H}(5) = 10$.

The proof for the case when $m = 5$ will be given, the other results may be obtained using similar arguments.

Lemma 1. *The first* 12 *Liapunov quantities for system* (9) *when* $m = 5$ *are as follows*:

$$L(0) = -a_0;$$

$$L(1) = -3c_2 - a_2;$$

$$L(2) = -3b_3c_2 - a_4;$$

$$L(3) = -15b_5c_2 - 6c_2^3 + 5a_6;$$

$$L(4) = 72b_3c_2^3 - 105b_7c_2 - 35a_8;$$

$$L(5) = -84c_2^5 + 90c_2^3b_3^2 + 196c_2^3b_5 - 315c_2b_9 - 105a_{10};$$

$$L(6) = c_2^3(-555b_3c_2^2 + 440b_3b_5 + 504b_7);$$

$$L(7) = c_2^3(56448c_2^4 - 117000b_3^2b_5 + 2565b_3^2c_2^2 + 63700b_5^2 - 158760b_5c_2^2 + 158760b_9);$$

$$L(8) = -b_3c_2^3(-1055950b_5c_2^2 + 225939c_2^4 + 321750b_3^2c_2^2 - 750000b_3^2b_5 + 1225000b_5);$$

$$L(9) = -c_2^3(2905146000b_3^2c_2^4 + 1603525000000b_5^3 - 2070983775000b_5^2c_2^2 + 890432956350b_5c_2^4 - 127481090247c_2^6);$$

$$L(10) = b_3c_2^3(-1662338750000b_5^3 + 2143093106250b_5^2c_2^2 - 914876988300b_5c_2^4 + 129198395007c_2^6);$$

$$L(11) = c_2^7(-99496813683170271962000b_5^2 + 84987917227714837678890b_5c_2^2 - 17992465848130721750991c_2^4).$$

In the reduction phase of the computation substitute for a_2 *from* $L(1) = 0$, a_4 *from* $L(2) = 0$, a_6 *from* $L(3) = 0$, a_8 *from* $L(4) = 0$, a_{10} *from* $L(5) = 0$, b_7 *from* $L(6) = 0$ *and*

b_9 *from* $L(7) = 0$. *Assume that* b_3 *and* c_2 *are non-zero in the remaining substitutions and let* $L(8) = L(9) = L(10) = 0$.

Theorem 1. *At most* 10 *small-amplitude limit cycles may be bifurcated from the origin of system* (9) *when* $m = 5$.

Proof. Set $L(0) = L(1) = \cdots = L(6) = 0$. Now $L(6) = 0$ if either (i) $c_2 = 0$ or (ii) $\alpha = 440b_3b_5 - 555c_2^2 + 504b_7 = 0$. If $c_2 = 0$, then $a_2 = a_4 = a_6 = a_8 = a_{10} = 0$ and the origin is a centre by the divergence criteria. Suppose that $c_2 \neq 0$, then $L(6)$ is zero if condition (ii) is satisfied.

The Liapunov quantity $L(8)$ is equal to zero if either (iii) $b_3 = 0$ or (iv) $-1055950b_5c_2^2 + 225939c_2^4 + 321750b_3^2c_2 - 750000b_3^2b_5 + 1225000b_5 = 0$. Suppose that condition (iii) holds, then

$$L(9) = c_2^3(-116875b_5^3 + 151725b_5^2c_2^2 - 65250b_5c_2^4 + 9288c_2^6),$$

$$L(10) = 0$$

and

$$L(11) = b_5^3c_2(317829875b_5^2 - 294878790b_5c_2^2 + 68313816c_2^4).$$

Since $L(11) \neq 0$, there can be at most 10 small-amplitude limit cycles.

Take $b_3 \neq 0$ and suppose instead that condition (iv) holds. Using Groebner bases on the REDUCE mathematical package; $L(8) = L(9) = L(10) = 0$ if and only if $b_3 = c_2 = 0$. Thus, there are at most 10 limit cycles if $b_3 \neq 0$. □

Select $c_2, b_5, b_3, b_9, b_7, a_{10}, a_8, a_6, a_4, a_2$ and a_0 such that

$$|L(11)| \ll 1 \quad \text{and} \quad L(9)L(11) < 0,$$

and

$$|L(r)| \ll |L(r+1)| \text{ and } L(r)L(r+1) < 0,$$

where $r = 0, 1, \cdots, 8$. The perturbations are chosen one by one so that the origin reverses stability ten times and the limit cycles which bifurcate persist.

3. LIMIT CYCLES IN CERTAIN HIGHLY NON-LINEAR MECHANICAL SYSTEMS

The theory of local and global limit cycle bifurcations will now be applied to two mechanical systems. The problem of surge oscillations in axial flow compressors was investigated by Greitzer [25] in 1976, and wing rock oscillations in aircraft flight dynamics were discussed by Hsu and Lan [26] in 1985. Minimal models for both of these mechanical systems were applied by Ananthkrishnan *et al.* [27] in 1998. All of the analysis for these two systems to-date has been concentrated on large-amplitude limit cycle bifurcations. The question of the maximum possible number of limit cycles does not appear to have been addressed.

Consider the reduced two-dimensional differential equation derived by Greitzer [25] to model the non-dimensional mass flow through an axial flow compressor :

$$\frac{d^2x}{dt^2} + (\lambda - g'(x))\frac{dx}{dt} + (x + g(x)) = 0,$$

or in the phase plane

$$\dot{x} = y \quad \dot{y} = -x - g(x) - (\lambda + g'(x))y, \tag{10}$$

where g is odd and the restoring term contains the compressor characteristic which relates the pressure rise across the compressor to the mass flow through it. Experimental studies have shown that the compressor characteristic is a highly non-linear relationship. Note that the damping coefficient is a derivative of the restoring term, and therefore, is an even function. This theory may be important when predicting surge in modern aircraft gas turbine engines.

Suppose that $\partial g = 2n + 1$ and let $\hat{H}(n)$ denote the maximum number of small-amplitude limit cycles that can be bifurcated from the origin. Using the results from section two, it is known that $\hat{H}(n) = n$ for system (10); this naturally leads to the folowing conjecture.

Conjecture 1. System (10) has at most n limit cycles surrounding the origin when $\partial g = 2n + 1$.

Two types of global limit cycle bifurcations were discussed in reference [27], these were normal Hopf and primary Hopf-secondary saddle-node bifurcations, respectively. Only stable limit cycles are of interest when modelling physical systems. Consider systems of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu),$$

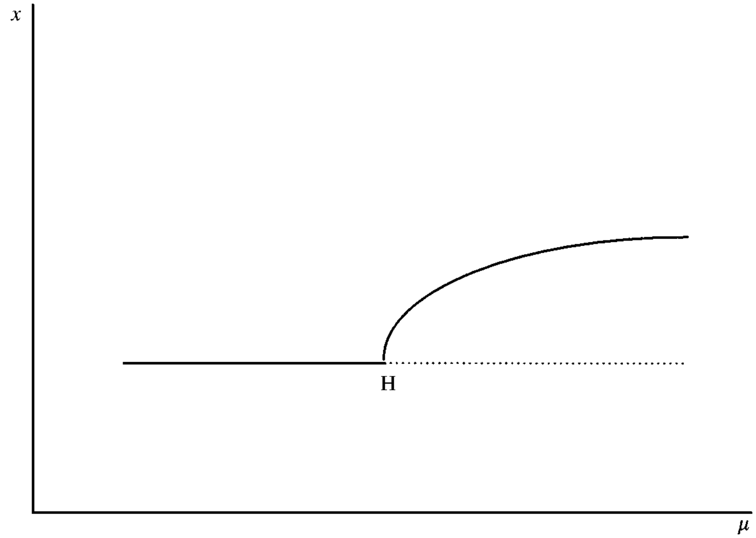

Figure 1. A typical Hopf bifurcation diagram. The solid curves represent stable equilibria and the dashed curves represent unstable equilibria. The point H is the Hopf bifurcation point.

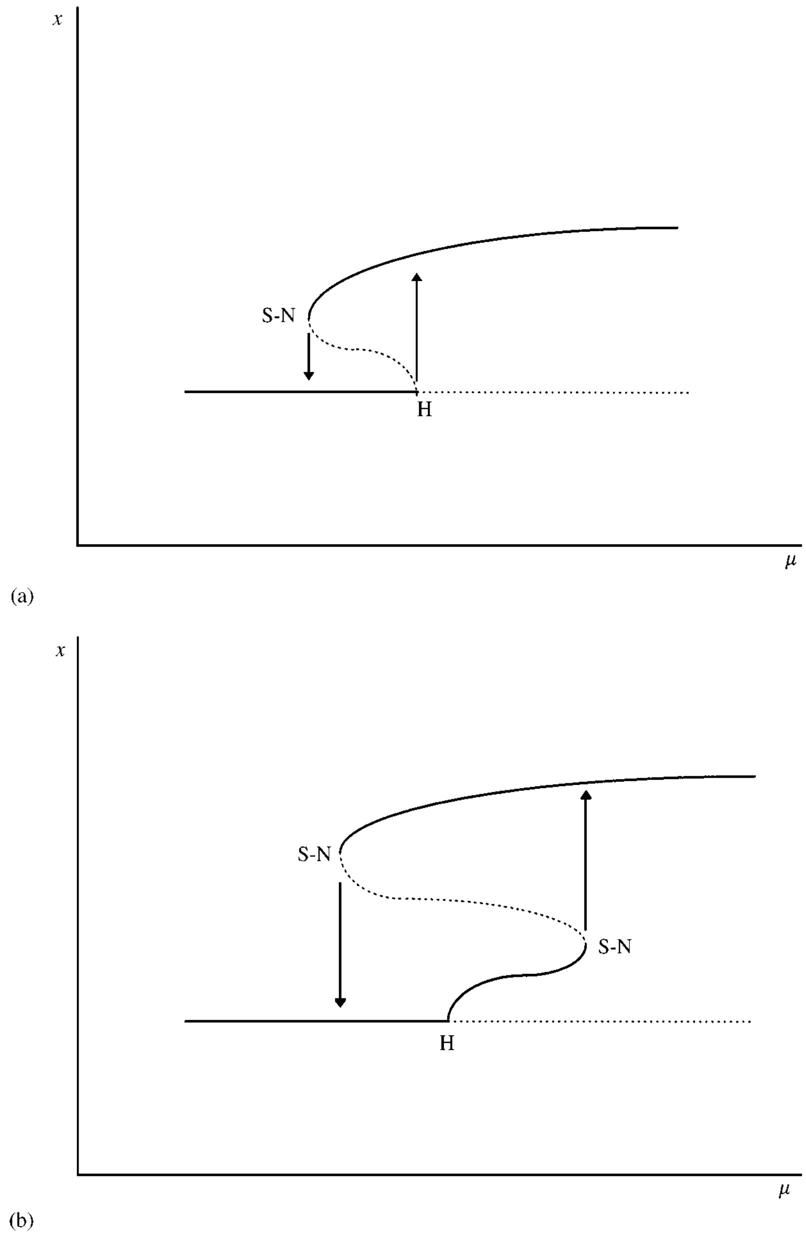

Figure 2. The two ways in which large-amplitude limit cycles may be bifurcated. The solid curves represent the stable equilibria and the dashed curves the unstable equilibria. The point H is a Hopf bifurcation point and SN denotes the saddle-node bifurcation point.

where μ is a scalar parameter. The analysis involved in Hopf bifurcations is well documented in the literature, see reference [28] for instance. A typical bifurcation diagram is given in Figure 1.

There are two possible ways in which stable large-amplitude limit cycles may be created, the details may be found in reference [27]. The bifurcation diagrams are given in Figure 2. In the first case, the system jumps from the stable critical point at

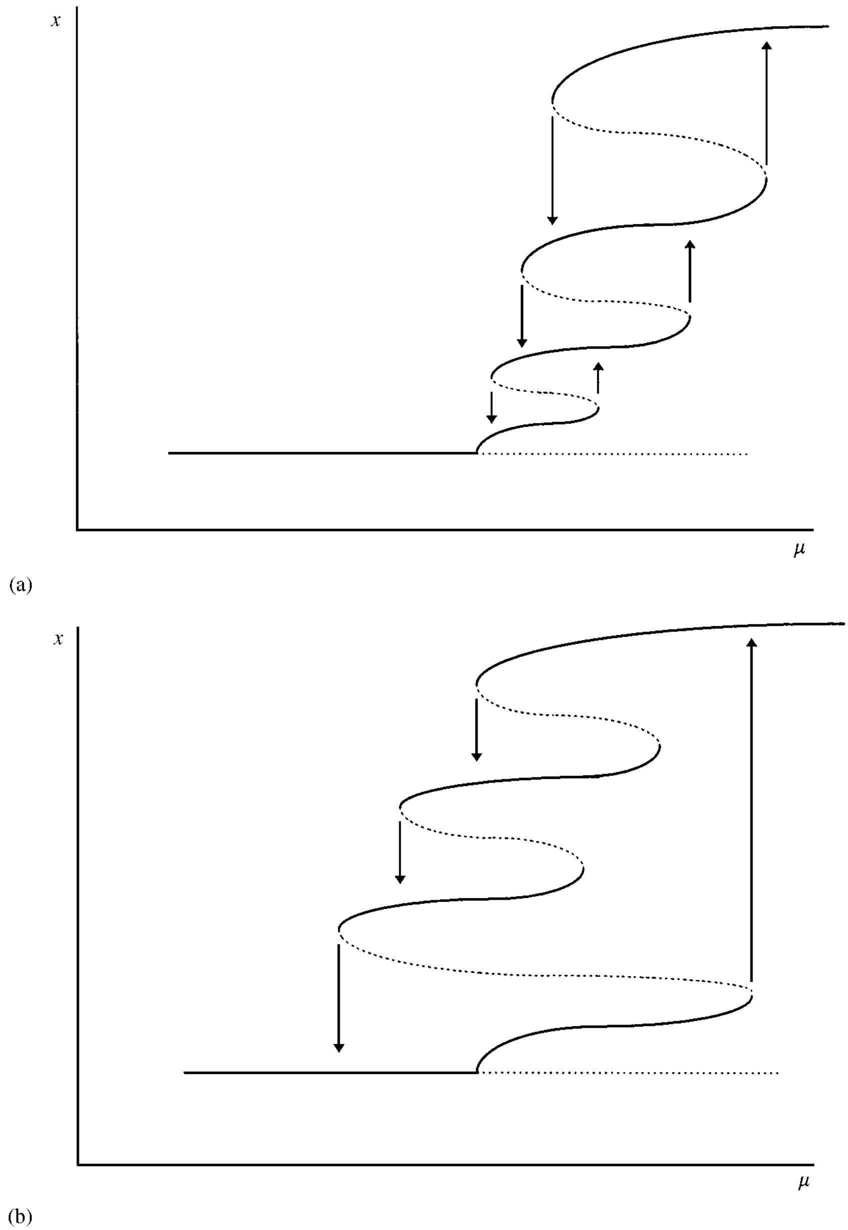

Figure 3. Possible large-amplitude limit cycle bifurcation diagrams for differential equaions with highly non-linear terms.

the origin to a stable large-amplitude limit cycle as μ increases. If μ is then decreased, the reverse jump occurs at a saddle-node bifurcation point and the system displays hysteresis as depicted by the arrows. In the second case, there is a normal Hopf bifurcation followed by a saddle-node bifurcation to a large-amplitude limit cycle as μ is increased. When μ is decreased there is a jump back to the original stable critical point and a hysteresis loop is created.

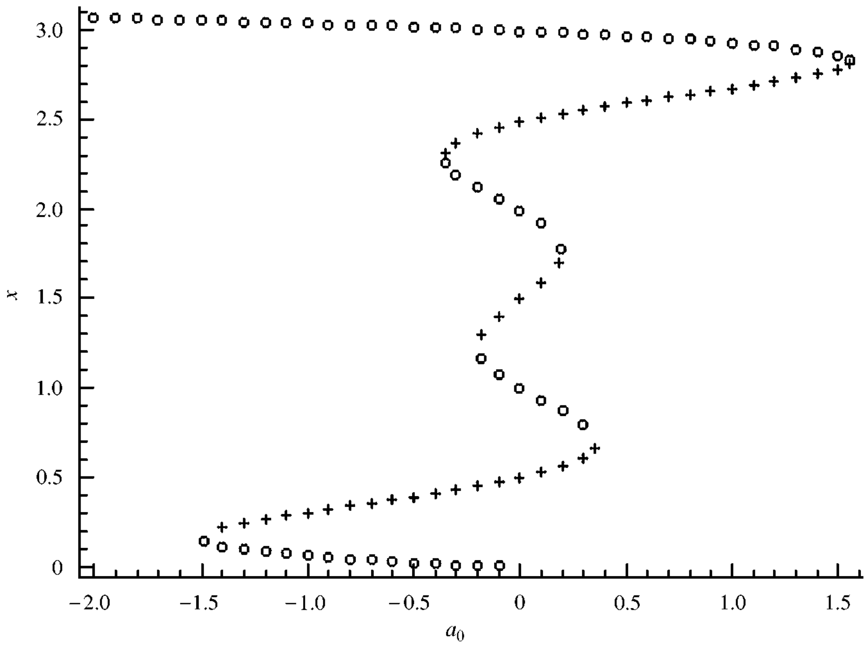

Figure 4. Limit cycle solutions of equation (11): the crosses represent stable equilibria and the circles denote unstable equilibria.

If the damping or restoring coefficients in system (10) are highly non-linear, then more interesting behaviour is possible. The bifurcation diagrams given in Figure 3, for example, show some of the possibilities when the damping coefficient is of degree 14. As a particular example, consider the following differential equation:

$$\ddot{x} + \varepsilon(a_{14}x^{14} + a_{12}x^{12} + a_{10}x^{10} + a_8x^8 + a_6x^6 + a_4x^4 + a_2x^2 - a_0)\dot{x} + x = 0, \tag{11}$$

where $a_{14} = 76{\cdot}38$, $a_{12} = -651{\cdot}638$, $a_{10} = 2133{\cdot}34$, $a_8 = -3359{\cdot}997$, $a_6 = 2598{\cdot}4$, $a_4 = -882$ and $a_2 = 90$. The condition, $0 < \varepsilon \ll 1$, is imposed to make the computations more straightforward. The bifurcation diagram for system (11) is given in Figure 4. As the parameter a_0 is increased and then decreased, a hysteresis loop is formed on the branches of the smallest and largest stable limit cycles. The central stable limit cycle is not involved in the hysteresis.

Wing rock oscillations in aircraft flight dynamics were discussed in reference [27]. It was concluded that the mechanism which caused large-amplitude wing rock in aircraft could not be attributed to the non-linear damping since the polynomial was not of sufficiently high degree. The results presented here for small-amplitude limit cycle bifurcations would tend to suggest that it may be the contributions from the stiffness terms, or other highly non-linear terms in the differential equations which could explain the wing rock phenomenon.

4. CONCLUSION

Two-dimensional differential equations in the plane have been investigated. The question of the maximum number of limit cycles has been addressed to various classes of system which can be used to model mechanical systems with highly non-linear terms. Both small-amplitude and large-amplitude limit cycle bifurcations have been considered for the model of surge oscillations in axial flow compressors, and wing rock oscillations in aircraft.

In summary, this paper demonstrates that non-linear terms in both the damping and restoring coefficients should be considered when bifurcating limit cycles. The analysis presented here can be used on systems containing highly non-linear terms in the differential equations.

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