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A TECHNIQUE FOR THE SYSTEMATIC CHOICE OF ADMISSIBLE FUNCTIONS IN THE RAYLEIGH–RITZ METHOD

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A simple and systematic choice of admissible functions, which are the eigenfunctions of the closest, simple problem extracted from the one considered, is proposed. The extracted problem must be "less-constrained" than the original one; in other words it must be a problem where some constraints or other complications (e.g. added masses) are eliminated. Elastic constraints replace the eliminated rigid ones. The convergence is also analyzed. This approach has practical applications when it is possible to extract a problem with eigenfunctions expressed in closed form. It also allows a very simple calculation of the potential energy of the system. Solutions for several cases involving beams are given in order to show the power of the method. Application of the method to circular plates and shells is also addressed.

1. INTRODUCTION

The Rayleigh–Ritz method [1] assumes deflection shapes in the form of a linear combination of functions which satisfy at least the geometrical boundary conditions of the vibrating structure. Courant [2] addressed the method for solving problems having rigid boundaries by treating such problems as limiting cases of free boundary problems, for which the admissible functions can be simpler. This technique introduces artificial translational and rotational springs at the free boundaries; the stiffness of these springs can be assumed sufficiently high to simulate rigid constraints with the required accuracy. Applications of this technique were made, e.g., by Kao [3], Mizusawa *et al.* [4], Yuan and Dickinson [5] and Cheng and Nicolas [6]. Additional references are given by Laura [7]. Warburton and Edney [8], Gorman [9], Gelos and Laura [10], and Laura and Gutierrez [11] applied the Rayleigh–Ritz method to structures with elastic constraints. In these studies, a large number of different admissible functions were used. In fact, the most critical aspect of the Rayleigh–Ritz method is regarding the choice of appropriate admissible functions. If these functions form a complete set, computed natural frequencies converge to actual ones from above. The nature of natural frequencies obtained by using the Rayleigh–Ritz method and their dependence on the nature of the assumed shape functions was investigated by Bhat [12]. Dickinson and Li [13] introduced a set of admissible functions derived from

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the mode shapes of vibration of plates instead of using the beam functions. They observed that the use of functions for simply supported plates gave better results for a rectangular plate supported in some way, but yielded poor results for plates with some free edges.

The present work proposes a simple and systematic choice of admissible functions which are the eigenfunctions of the closest, simple problem extracted from the one considered. In particular, the problem extracted must be "less-constrained" than the original one; in other words, with less-constrained is indicated a problem where some constraints or other complications (e.g. added masses) are eliminated. The rigid constraints eliminated are replaced by elastic ones. The convergence of the method has been analytically investigated. The approach has practical applications when it is possible to extract a problem having eigenfunctions (mode shapes) that are expressed by analytical expressions in closed form. Moreover, the proposed approach allows a very simple calculation of the potential energy of the system, based on the use of the reference kinetic energy. Several cases with beams are solved in order to show the possibilities of the method. These examples are thought in a way that they can be easily combined to solve more complex cases. Application of the method to circular plates and shells is also addressed.

Another possibility of the proposed approach is in conjunction with the artificial spring method, originally introduced by Yuan and Dickinson [5] and Cheng and Nicolas [6]. In fact, the admissible functions must satisfy all the geometrical boundary conditions or, if the extended Rayleigh–Ritz method (e.g. see reference [14]) is utilized, admissible functions must satisfy geometrical boundary conditions of the unconstrained structure, and the sum of the series of functions must satisfy the additional constraints. When the Rayleigh–Ritz method is applied to a structure obtained by joining several components together, the boundary conditions require the continuity of translational and rotational displacements between all the rigid junctions of the substructures. This continuity condition gives rise to many problems in the choice of the correct admissible functions to be used for each single component.

The use of artificial springs at the junctions allows one to overcome this difficulty. In particular, the joints between the components of the structure are substituted by translational and rotational artificial springs that are distributed along the whole joint length or area. Obviously, each degree of freedom involved in the joint must be simulated by a distributed spring. Then, the spring stiffness is chosen very high with respect to the structure stiffness in order to simulate a rigid junction in the numerical computations. The proposed choice of admissible functions presented in this paper can be repeated for each component from which it is possible to extract the simpler, less-constrained, configuration; consequently, by using the artificial spring method it is possible to study the whole structure.

The Rayleigh–Ritz method and its extension to synthesize substructures can be successfully applied to fluid–structure interaction problems. The Rayleigh–Ritz method applied to fluid–structure interaction was studied by Zhu [15] and Amabili [16]. In this case, it is possible to extract a less-constrained *in vacuo* problem. The wet modes can be described as a sum of *in vacuo* modes. Applications are shown, e.g. by Amabili [17] and Amabili *et al.* [18].

2. ADMISSIBLE FUNCTIONS AND POTENTIAL ENERGY

The equation of motion for free harmonic vibrations of an undamped structure can be written in the following form:

$$\mathbf{N}(\mathbf{u}) = \omega^2 m\mathbf{u}, \tag{1}$$

where $\mathbf{N}$ is a self-adjoint differential operator, $\mathbf{u}$ is the displacement vector of the mean line (for a beam) or surface (for plates and shells) of the structure that gives the mode shape, ω is the corresponding radian frequency and m is the mass per unit length or area. For the system considered, it is possible to write the Rayleigh quotient

$$\omega^2 = \frac{\int_\Omega \mathbf{u}\cdot\mathbf{N}(\mathbf{u})\,\mathrm{d}S}{\int_\Omega m\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}S}, \tag{2}$$

Ω being the domain where $\mathbf{u}$ is defined. The numerator on the right-hand side of equation (2) equals twice the maximum potential energy and the denominator twice the reference kinetic energy of the system. The Rayleigh–Ritz method is used to find natural modes of the system. In particular, $\mathbf{u}$ is expanded by using a sum of admissible functions $\mathbf{x}_i$ (vectorial functions in the case of a structure described by displacements in different directions) and appropriate unknown coefficients a

$$\mathbf{u} = \sum_{i=1}^{\infty} a_i\mathbf{x}_i. \tag{3}$$

The infinite sum in equation (3) is truncated to N terms in the applications. The choice of the admissible functions is very important to simplify the calculations and to guarantee convergence to the actual solution. The choice of admissible functions that are all the eigenfunctions, including eventually rigid-body modes, of the closest, simple "less-constrained" problem extracted from the one considered, is useful in many cases. In particular, it is applicable in all the cases where it is possible to extract a less-constrained problem having mode shapes expressed by analytical expressions in closed form.

In fact, the expansion theorem states that any function (or vectorial function) $\mathbf{u}$, defined in the structure, satisfying the homogeneous boundary conditions of the system and for which $\mathbf{N}(\mathbf{u})$ is continuous can be represented by an absolutely and convergent series of eigenfunctions of the system [19]. This theorem can be applied to the present case where $\mathbf{u}$ is a mode of the considered system and the eigenfunctions $\mathbf{x}_i$ are those of the less-constrained system. In fact, $\mathbf{u}$ satisfies the same boundary conditions of the less-constrained system, if all the additional constraints are replaced by translational and rotational springs. The nature of the convergence of the method and its rate in the case of a beam are discussed in Section 4.

The use of admissible functions, which are the natural modes of a less-constrained problem, allows an interesting simplification. In fact, the maximum potential energy of the system can be obtained as the multiplication of the reference kinetic energy of natural modes in the less-constrained problem by the corresponding eigenvalue ω_i^2 (the squared radian frequency) of the same problem and by the coefficient a_i, and then adding all the products [16–18]. For each term of the expansion, it is possible to write

$$\int_{\Omega} \mathbf{x}_i \cdot \mathbf{N}(\mathbf{x}_i)\, \mathrm{d}S = \omega_i^2 \int_{\Omega} m\mathbf{x}_i \cdot \mathbf{x}_i\, \mathrm{d}S. \tag{4}$$

Equation (4) is obtained by equalizing the maximum potential and the maximum kinetic energies of each natural mode of the less-constrained problem. Therefore, twice the maximum potential energy of the system can be written as

$$\int_{\Omega} \mathbf{u} \cdot \mathbf{N}(\mathbf{u})\, \mathrm{d}S = \sum_{i=1}^{N} a_i^2 \omega_i^2 \int_{\Omega} m\mathbf{x}_i \cdot \mathbf{x}_i\, \mathrm{d}S\,. \tag{5}$$

In equation (5), the orthogonality of the eigenfunctions of the less-constrained problem has been used.

3. APPLICATIONS

In order to illustrate the proposed approach, six different cases, relative to beams, are considered. They are: (a) simply supported beam with intermediate elastic or rigid support and concentrated mass; (b) clamped beam; (c) simply supported beam on an intermediate elastic foundation; (d) simply supported beam of varying cross-section; (e) free-edge beam with two intermediate supports; and (f) simply supported beam. A simple combination of the cases presented can cover many of the practical applications of beams; beams with intermediate hinges and stepped beams can be studied by modelling the structure with substructures and using the artificial spring method to obtain the solution. For simplicity, shear deformation and rotary inertia are neglected in the following examples. At the end of this section, applications to circular plates and shells with uniform and non-uniform constraints are addressed without solving specific problems.

3.1. CASE (a): SIMPLY SUPPORTED BEAM WITH INTERMEDIATE ELASTIC SUPPORT AND CONCENTRATED MASS. FROM ELASTIC TO RIGID SUPPORT.

Figure 1 shows the considered problem. It concerns a simply supported beam of uniform circular cross-section having an elastic support at distance x_1 from the left-side support and carrying a concentrated mass M at distance x_2 from the left-side support, where L is the length of the beam. The problem is solved by using the eigenfunctions of the "analogue" simply supported beam as admissible functions. The simply supported beam is surely the closest, simple problem that can

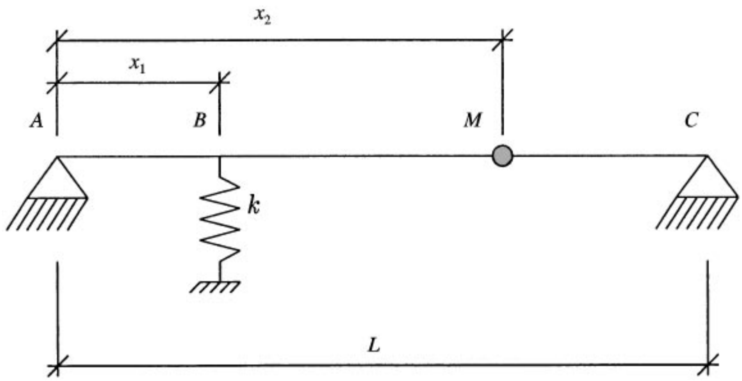

Figure 1. Simply supported beam with intermediate elastic support and concentrated mass.

be extracted from the one considered. In particular, the transverse displacement w can be written as

$$w(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L), \tag{6}$$

where n is the number of longitudinal half-waves, a_n are the appropriate coefficients and x is the longitudinal co-ordinate (see Figure 1). The natural radian frequencies ω_n of the simply supported beam are given by

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}}, \tag{7}$$

where I is the cross-sectional area moment of inertia, E is the Young's modulus, A is the cross-sectional area and $m = \rho A$ is the mass per unit length. The reference kinetic energy of the beam is

$$T_B^* = \frac{1}{2} m \int_0^L w^2\,\mathrm{d}x = \frac{1}{2} m \frac{L}{2} \sum_{n=1}^{\infty} a_n^2. \tag{8}$$

The maximum potential energy of the beam can be computed by using equation (5), so that it is given by

$$V_B = \frac{1}{2} m \frac{L}{2} \sum_{n=1}^{\infty} a_n^2 \omega_n^2. \tag{9}$$

The maximum potential energy stored by the elastic intermediate support is

$$V_k = \frac{1}{2} k w^2(x_1) = \frac{1}{2} k \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_n a_i \sin(n\pi x_1/L)\sin(i\pi x_1/L). \tag{10}$$

The reference kinetic energy due to the concentrated mass M is

$$T_M^* = \frac{1}{2} M w^2(x_2) = \frac{1}{2} M \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_n a_i \sin(n\pi x_2/L) \sin(i\pi x_2/L). \tag{11}$$

By introducing a vectorial notation and truncating all the infinite sums to N terms, it is possible to write

$$T_B^* = \frac{1}{2} m \frac{L}{2} \mathbf{q}^{\mathrm{T}} \mathbf{M}_{\mathrm{B}} \mathbf{q}. \tag{12}$$

where $\mathbf{q}^{\mathrm{T}} = \{a_1, \ldots, a_n\}$, $\mathbf{M}_{\mathrm{B}}$ is the $N \times N$ matrix of elements $(\mathbf{M}_{\mathrm{B}})_{n,i} = \delta_{n,i}$ and $\delta_{n,i}$ is the Kronecker delta. Similarly, it is useful to write

$$V_B = \frac{1}{2} m \frac{L}{2} \mathbf{q}^{\mathrm{T}} \mathbf{K}_{\mathrm{B}} \mathbf{q}, \tag{13}$$

where

$$(\mathbf{K}_{\mathrm{B}})_{n,i} = \delta_{n,i} \omega_n^2, \quad n, i = 1, \ldots, N, \tag{14}$$

and

$$V_k = \frac{1}{2} k \mathbf{q}^{\mathrm{T}} \mathbf{K}_{\mathrm{K}} \mathbf{q}, \tag{15}$$

with

$$(\mathbf{K}_{\mathrm{K}})_{n,i} = \sin(n\pi x_1/L) \sin(i\pi x_1/L), \quad n, i = 1, \ldots, N. \tag{16}$$

Then, for the concentrated mass, it gives

$$T_M^* = \frac{1}{2} m \mathbf{q}^{\mathrm{T}} \mathbf{M}_{\mathrm{M}} \mathbf{q}, \tag{17}$$

where

$$(\mathbf{M}_{\mathrm{M}})_{n,i} = \sin(n\pi x_2/L) \sin(i\pi x_2/L), \qquad n, i = 1, \ldots, N. \tag{18}$$

The natural frequencies and mode shapes are obtained from the solution of the eigenvalue problem associated with the following Galerkin equation:

$$[m(L/2)\, \mathbf{K}_{\mathrm{B}} + k\mathbf{K}_{\mathrm{K}}]\mathbf{q} - \Lambda^2[m(L/2)\, \mathbf{M}_{\mathrm{B}} + M\mathbf{M}_{\mathrm{M}}]\mathbf{q} = 0, \tag{19}$$

where Λ are the natural radian frequencies of the studied beam.

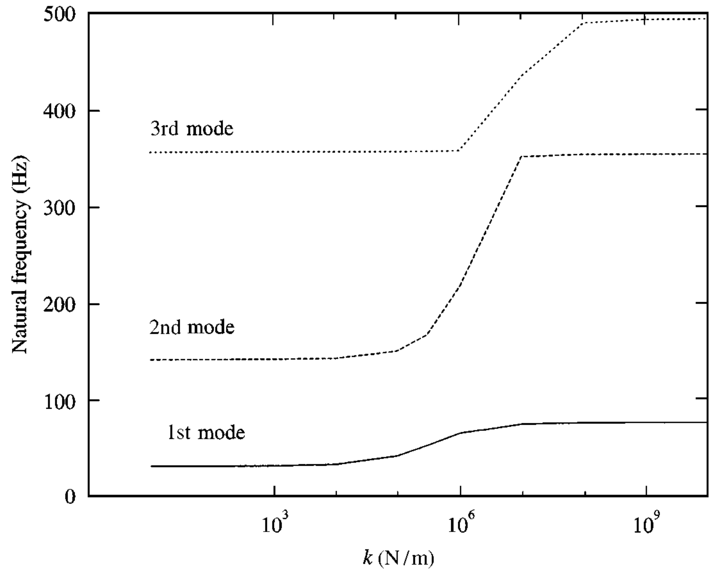

Figure 2. First three natural frequencies of the beam with intermediate elastic support and concentrated mass, Case (a), versus the stiffness of the intermediate support k; $N = 30$.

The geometrical and material properties of the system are: length $L = 1\,\text{m}$, $x_1 = L/\pi$, $x_2 = 2L/\pi$, radius of cross-section $r = 10\,\text{mm}$, concentrated mass $M = 1\,\text{kg}$, material density $\rho = 7850\,\text{kg/m}^3$ and Young's modulus $E = 206\,\text{GPa}$. All the numerical results are obtained by using the software *Mathematica* [20].

In Figure 2, the three lower natural frequencies, obtained by using $N = 30$ terms in the expansion of w, are given versus the stiffness of the intermediate support k. It is shown that, for large k values, the curves in Figure 2 reach horizontal asymptotes. In particular, a rigid intermediate support is well simulated with $L^3k/(EI) = 6.18 \times 10^6$ ($k = 10^{10}\,\text{N/m}$), at least for the three modes considered in Figure 2. Figure 3 shows the first three mode shapes for $k = 10^{10}$ N/m and $N = 100$; it is clearly shown that, in this case, there is no movement of the beam at the intermediate support. Table 1 shows the rate of convergence of the natural frequencies versus the number of terms used in the expansion of w. It is clear that 10 terms give a very good evaluation of the first three natural frequencies and that there is no significant difference in the results obtained using 30 and 100 terms.

3.2. CASE (b): CLAMPED BEAM

The clamped beam presents a simple and well-known solution. However, it is possible to obtain the solution using the admissible functions of a simply supported beam, which can be seen as a simple problem extracted from the clamped beam. This case is interesting for comparison of the method with an exact solution and because the eigenfunctions used seem to be quite far from the clamped ones, thus making it an interesting test. All the calculations made for the previous case can be

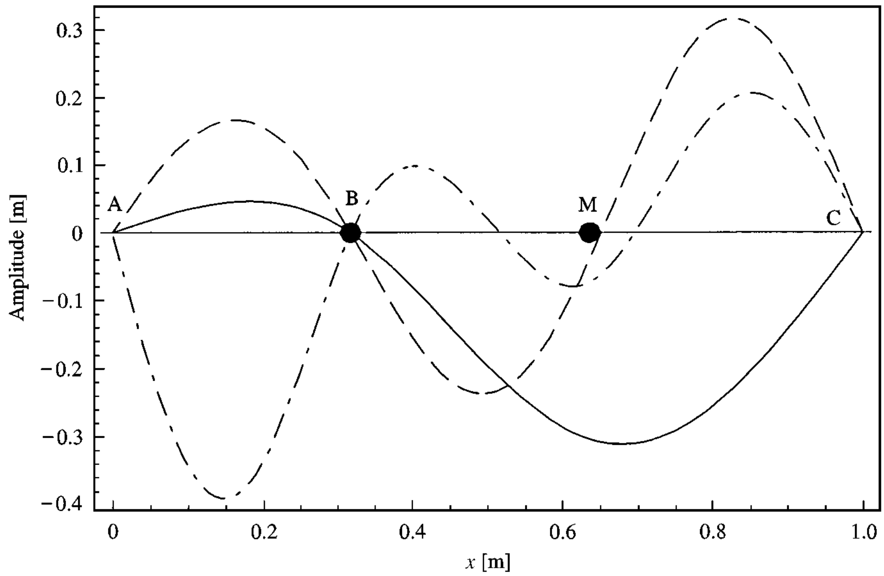

Figure 3. First three mode shapes of the beam with intermediate elastic support and concentrated mass, Case (a), for $k = 10^{10}$ N/m and $N = 100$. — First mode; – – – second mode; — · — · — third mode.

TABLE 1

Natural frequencies (Hz) of the first three modes of the beam shown in Figure 1 [Case (a)] with $k = 10^{10}$ N/m versus the number N of terms used in the expansion

N	1st mode	2nd mode	3rd mode
4	76·15	354·2	501·3
10	75·93	354·0	495·1
30	75·90	353·9	494·2
100	75·90	353·9	494·2

retained, setting $k = 0$ and $M = 0$. It is necessary to add at the edges two artificial rotational springs whose stiffnesses must be very large.

The maximum potential energy stored by the rotational spring at the left edge is

$$V_R = \frac{1}{2} c \left[\frac{\mathrm{d}w(0)}{\mathrm{d}x} \right]^2 = \frac{1}{2} c \frac{\pi^2}{L^2} \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_n a_i ni, \tag{20}$$

where c is the stiffness of the rotational spring. Similarly, for the rotational spring at the right edge

$$V_R = \frac{1}{2} c \left[\frac{\mathrm{d}w(L)}{\mathrm{d}x} \right]^2 = \frac{1}{2} c \frac{\pi^2}{L^2} \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_n a_i ni(-1)^n(-1)^i. \tag{21}$$

TABLE 2

Natural frequencies (Hz) of the first three modes of the clamped beam [Case (b)] with $c = 10^9$ Nm versus the number N of terms used in the expansion

N	1st mode	2nd mode	3rd mode
5	108·0	322·9	615·5
10	100·2	275·3	549·5
30	94·10	259·3	509·2
100	92·25	254·3	498·6

The first three natural frequencies obtained by using the Rayleigh–Ritz method are given in Table 2 for different N up to 100 and are computed for $Lc/(EI) = 6{\cdot}18 \times 10^5$ ($c = 10^9$ Nm). The exact solutions are: 91·5, 252·3, 494·5 Hz, that are close to those obtained with 100 terms (less than 0·9% difference). It is clear that the rate of convergence in this case is slower than in Case (a). In general, rotational constraints require more terms than translational constraints (see Section 4 for the explanation). In any case, the advantage of the method is that all the energy terms are automatically generated and the computational effort is very limited, so that a computation of a quite large number of terms in the expansion of mode shapes does not present a problem.

Note that the present case can be immediately combined with the previous one to give more complex cases where several translational and rotational springs (and eventually concentrated and distributed masses) can be inserted.

3.3. CASE (c): SIMPLY SUPPORTED BEAM ON AN INTERMEDIATE ELASTIC FOUNDATION

Figure 4 shows the considered problem. It is a simply supported beam of uniform circular cross-section having an elastic foundation from x_3 to x_4, where L is the length of the beam. The closest, simple problem extracted from the one considered is the simply supported beam. All the expressions obtained for Case (a) are retained, setting $k = 0$ and $M = 0$. The additional maximum potential energy stored by the elastic foundation is given by

$$V_F = \frac{1}{2} k_F \int_{x_3}^{x_4} w^2 \, \mathrm{d}x = \frac{1}{2} k_F \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_n a_i \int_{x_3}^{x_4} \sin(n\pi x/L) \sin(i\pi x/L) \, \mathrm{d}x, \tag{22}$$

where the integral on the right-hand side is easily expressed in a closed form that is not reported here for brevity and k_F is the stiffness of the elastic foundation. However, the closed-form expression was inserted in the computer program to speed up the computation. Numerical results are given in Table 3 for $N = 100$ terms and for different stiffnesses k_F; all the values in Table 3 correspond to very rigid foundations, that represent the more critical cases for convergence of the

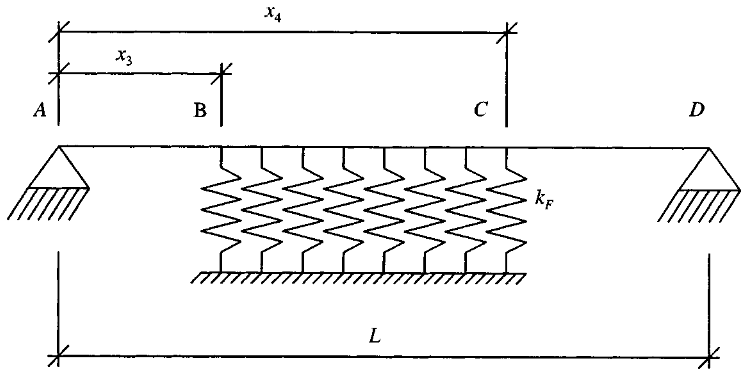

Figure 4. Simply supported beam on an intermediate elastic foundation.

TABLE 3

Natural frequencies (Hz) of the first three modes of the beam shown in Figure 4 [Case (c)] with N = 100 versus the stiffness k_F of the foundation

k_F (N/m^2)	1st mode	2nd mode	3rd mode
10^{10}	225·7	480·8	730·0
10^{12}	243·4	538·1	788·8
10^{14}	249·9	559·9	810·0
10^{16}	254·6	575·5	824·9

method. In particular, $L^4 k_F/(EI) = 6{\cdot}18 \times 10^{12}$ ($k_F = 10^{16}$ N/m^2) approximates very well a beam completely fixed between x_3 to x_4 (beam on rigid foundation). The last case corresponds to two clamped-simply supported beams; one between A and B and the second between C and D. The exact natural frequencies are: 252·3 Hz (first mode of the right-hand beam), 567·6 Hz (first mode of the left-hand beam) and 824·9 Hz (second mode of the right-hand beam); they are close to the ones computed for $k_F = 10^{16}$ (less than 1·4% difference). The relative mode shapes are given in Figure 5. This case represents a very severe test for the accuracy of the method.

3.4. CASE (d): SIMPLY SUPPORTED BEAM OF VARYING CROSS-SECTION

A simply supported beam of rectangular cross-section of constant height h and width $b(x)$ is considered. In the example, $h = 10$ mm, $b(x) = \tilde{b}\,(f + x/L)$, $f = 0.1$ and $\tilde{b} = 10$ mm. The mass per unit length is $m(x) = \rho h \tilde{b}\,(f + x/L)$ and the cross-sectional area moment of inertia is $I(x) = (1/12)\, h^3 \tilde{b}\,(f + x/L)$. The closest, simple problem extracted is the simply supported beam; therefore the assumed deflection is still given by equation (6).

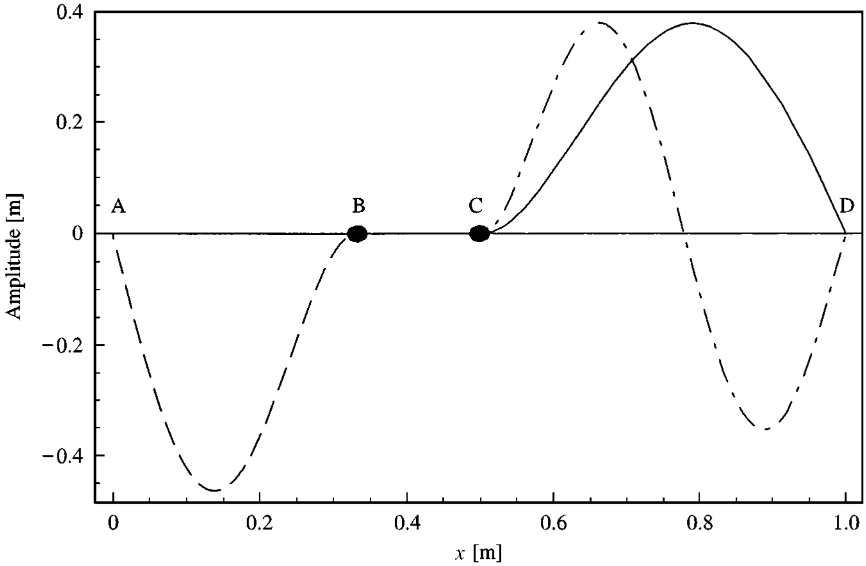

Figure 5. First three mode shapes of the beam on elastic foundation, Case (c), for $k = 10^{16}$ N/m^2 and $N = 100$. — First mode; – – – second mode; — · — · — third mode.

The reference kinetic energy of the beam is

$$T_B^* = \frac{1}{2}\int_0^L m(x) w^2 \,\mathrm{d}x = \frac{1}{2}\rho h \tilde{b}\left[f \frac{L}{2}\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} a_n a_i \Psi_{n,i}\right], \tag{23}$$

where

$$\Psi_{n,i} = \begin{cases} L/4 & \text{if } n = i, \\ \dfrac{L[-4in + (i+n)^2(-1)^{i-n} - (i-n)^2(-1)^{i+n}]}{2(i-n)^2(i+n)^2\pi^2} & \text{if } n \neq i. \end{cases}$$

In this case, the maximum potential energy of the beam cannot be obtained by using equation (5) anymore. It is given by

$$V_B = \frac{1}{2}E\int_0^L I(x)\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = \frac{1}{2}E\frac{h^3\tilde{b}}{12}\frac{\pi^4}{L^4}\left[f\frac{L}{2}\sum_{n=1}^{\infty} a_n^2 n^2 i^2 + \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} a_n a_i n^2 i^2 \Psi_{n,i}\right]. \tag{24}$$

The natural frequencies of the system are given in Table 4 versus the number of terms used in the expansion. The convergence rate is high. Note that natural frequencies of the present case are quite close to the ones of any simply supported beam of rectangular cross-section having the same height h. In contrast, mode

TABLE 4

Natural frequencies (Hz) of the first three modes of the beam with varying cross-section [Case (d)] versus the number N of terms used in the expansion

N	1st mode	2nd mode	3rd mode
3	22·56	93·95	214·9
10	22·50	93·39	210·5
100	22·50	93·38	210·4

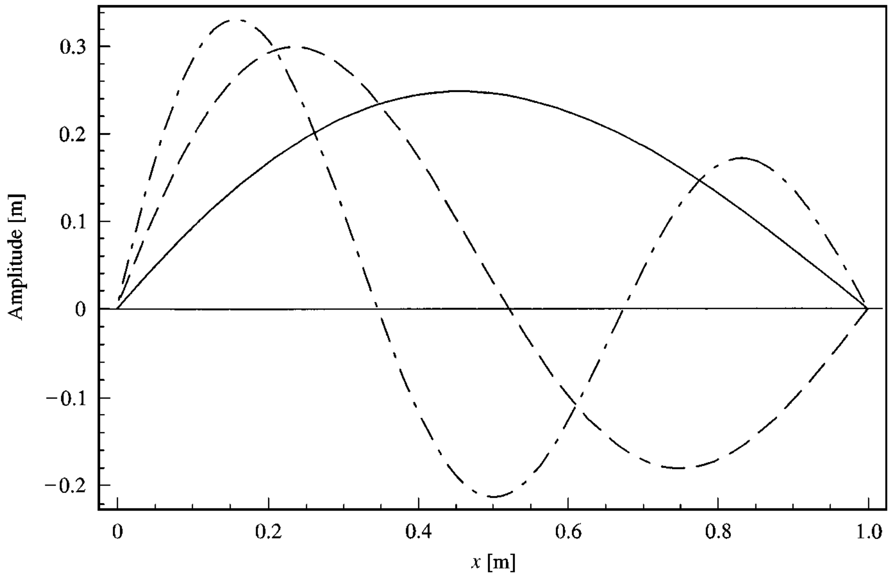

Figure 6. First three mode shapes of the beam of varying cross-section, Case (d), for $N = 100$. — First mode; – – – second mode; — · — · — third mode.

shapes are sensibly different, as shown in Figure 6. The narrow part (left side) of the beam presents larger movement than the wide part (right side).

3.5. CASE (e): FREE-EDGE BEAM WITH TWO INTERMEDIATE SUPPORTS

Figure 7 shows the problem considered. It concerns a free-edge beam of uniform circular cross-section having two intermediate supports at x_1 and x_2. The same dimensions x_1, x_2 and cross-section of Case (a) are assumed. The closest, simple problem extracted from the one considered is the free-edge beam. The transverse displacement w can be written as

$$w(x) = \sum_{n=1}^{\infty} a_n \alpha_n W_n(x), \tag{25}$$

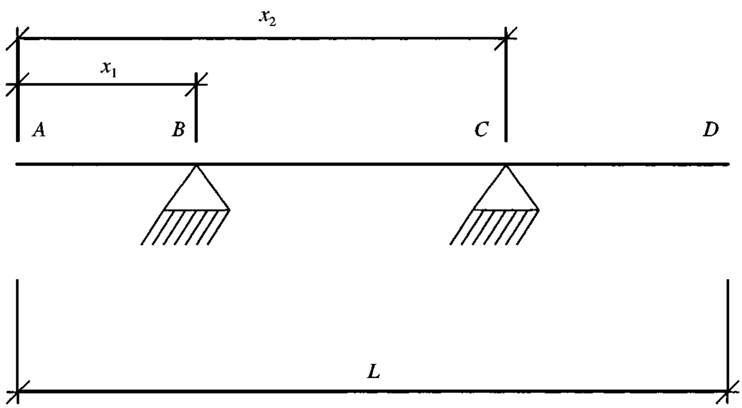

Figure 7. Free-edge beam with two intermediate supports.

where the eigenfunctions of the free-edge beam are

$$W_n(x) = \beta_n[\cos(\lambda_n L) - \cosh(\lambda_n L)][\sin(\lambda_n x) + \sinh(\lambda_n x)]$$
$$- [\sin(\lambda_n L) - \sinh(\lambda_n L)][\cos(\lambda_n x) + \cosh(\lambda x)], \tag{26}$$

and λ_n are the roots of the following equation:

$$\cos(\lambda_n L)\cosh(\lambda_n L) = 1, \quad n = 1, 2, \ldots, \tag{27}$$

including the first two zero roots associated with the two rigid-body modes, and

$$\beta_n = [\cos(\lambda_n L) - \cosh(\lambda_n L)]/[\sin(\lambda_n L) - \sinh(\lambda_n L)]. \tag{28}$$

In equation (25), a_n are appropriate unknown coefficients, as usual, and α_n is a normalization coefficient introduced in order to have $a_n^2 \int_0^L W_n^2 \,\mathrm{d}x = 1$. Equations (27) and (28) must be evaluated with a very good accuracy. The natural frequencies of the free-edge beam are given by $\omega_n = \lambda_n^2 \sqrt{EI/m}$; the first two frequencies, associated to rigid-body modes, are zero.

The reference kinetic energy of the beam is

$$T_B^* = \frac{1}{2} m \int_0^L w^2 \,\mathrm{d}x = \frac{1}{2} m \sum_{n=1}^{\infty} a_n^2. \tag{29}$$

The maximum potential energy of the beam can be evaluated by using equation (5), so that it is given by

$$V_B = \frac{1}{2} m \sum_{n=3}^{\infty} a_n^2 \,\omega_n^2. \tag{30}$$

In equation (30), the sum starts at 3 because the first two frequencies are zero.

TABLE 5

Natural frequencies (Hz) of the first three modes of the beam shown in Figure 7 [Case (e)] with $k = 10^{10}$ N/m versus the number N of terms used in the expansion

N	1st mode	2nd mode	3rd mode
5	69·74	103·6	477·5
10	69·49	101·2	474·0
30	69·47	101·0	473·7
50	69·47	101·0	473·7

The maximum potential energy stored by the artificial translational spring of stiffness k placed at B to replace the support is given by

$$V_{KB} = \frac{1}{2} k w^2(x_1) = \frac{1}{2} k \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_n a_i \alpha_n \alpha_i W_n(x_1) W_i(x_1). \tag{31}$$

Similarly, for the artificial translational spring of stiffness k placed at C

$$V_{KC} = \frac{1}{2} k w^2(x_2) = \frac{1}{2} k \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_n a_i \alpha_n \alpha_i W_n(x_2) W_i(x_2). \tag{32}$$

The solution is obtained via an eigenvalue problem, similar to Case (a). The natural frequencies of the system are given in Table 5 versus the number of terms used in the expansion for $k = 10^{10}$ N/m. The convergence rate is very high. The first three mode shapes are shown in Figure 8.

3.6. CASE (f): SIMPLY SUPPORTED BEAM

The simply supported beam also presents a simple and well-known solution. However, it is possible to obtain the solution using the admissible functions of a free-edge beam, which can be seen as a simple problem extracted from the simply supported beam. This case, similar to Case (b), is interesting for comparison of the method with an exact solution and because the eigenfunctions used are quite far from the simply supported case. All the calculations made for the previous case can be retained, setting $k = 0$. It is necessary to add at the edges two artificial translational springs whose stiffnesses must be very high.

The maximum potential energy stored by the artificial translational spring of stiffness k_A placed at the left edge to replace the support is given by

$$V_{KA} = \frac{1}{2} k_A w^2(0) = \frac{1}{2} k_A \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_n a_i \alpha_n \alpha_i W_n(0) W_i(0). \tag{33}$$

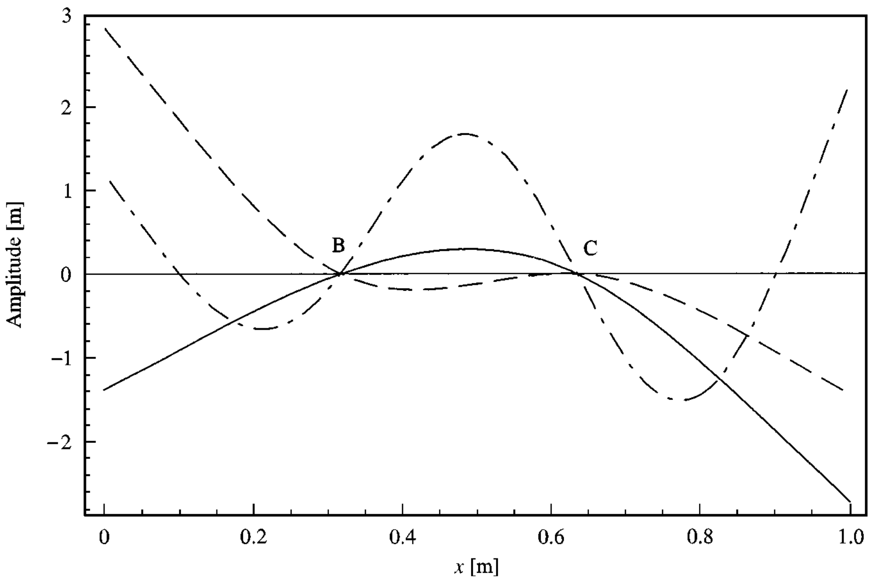

Figure 8. First three mode shapes of the free-edge beam with two intermediate supports, Case (e), for $k = 10^{10}$ N/m and $N = 50$. — First mode; – – – second mode; — · — · — third mode.

TABLE 6

Natural frequencies (Hz) of the first three modes of the simply supported beam [Case (f)] with $k_A = 10^{10}$ N/m versus the number N of terms used in the expansion

N	1st mode	2nd mode	3rd mode
5	40·47	165·1	372·6
10	40·38	161·6	364·7
30	40·36	161·5	363·3
50	40·36	161·4	363·3

Similarly, for the artificial translational spring of stiffness k_A placed at the right edge

$$V_{KD} = \frac{1}{2} k_A w^2(L) = \frac{1}{2} k_A \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_n a_i \alpha_n \alpha_i W_n(L)\ W_i(L). \tag{34}$$

The natural frequencies of the first three modes are given in Table 6 versus the number of terms used in the expansion for $k_A = 10^{10}$ N/m; this stiffness gives an accurate approximation of the rigid support. The effect of the spring stiffness on the natural frequencies of the first three modes is shown in Table 7. The accuracy of the solution is excellent, as can be verified comparing it with the exact solutions (40·36, 161·4 and 363·3 Hz). The rate of convergence of natural frequencies in this case is very high.

TABLE 7

Natural frequencies (Hz) of the first three modes of the simply supported beam [Case (f)] with N = 30 versus the stiffness k_A of the artificial spring

k_A (N/m)	1st mode	2nd mode	3rd mode
10^8	40·35	161·3	362·3
10^{10}	40·36	161·5	363·3
10^{12}	40·36	161·5	363·3

3.7. CIRCULAR PLATES AND SHELLS

Free-edge, simply supported and clamped circular plates and simply supported shells present eigenfunctions given by simple analytical expressions. This allows the use of the proposed method to solve many problems where these elements are employed. An interesting case is given by a circular plate having non-homogenous boundary conditions. This problem was initially addressed by Leissa *et al.* [21] by employing a polynomial expansion in the Rayleigh–Ritz method. Amabili *et al.* [22] solved the same problem by employing an expansion based on the eigenfunctions of the free-edge plate; this expansion corresponds to the extraction of the free-edge plate as a simple, less-constrained problem. This simple choice allows one to use many terms and to study more practical applications, e.g. bolted plates, with a sufficient accuracy. It is interesting to note that for an axisymmetric structure, e.g. a circular plate, there exist two families of eigenfunctions rotated by π/n, where n is the number of circumferential waves. Therefore, a double series expansion must be used when the axial symmetry of the constraints is lost. In particular, in order to study a plate with non-uniform elastic translational and rotational constraints it is possible to employ the following mode expansion [22]:

$$
\begin{aligned}
w(r,\theta) = \sqrt{2}a_{00} + \sum_{n=1}^{\infty} a_{0n}W_{0n}(r) + 2\frac{r}{a}(a_{10}\cos\theta + b_{10}\sin\theta) \\
+ \sum_{n=1}^{\infty} W_{1n}(r)\,(a_{1n}\cos\theta + b_{1n}\sin\theta) \\
+ \sum_{m=2}^{\infty}\sum_{n=0}^{\infty} W_{mn}(r)\,[a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)],
\end{aligned}
\tag{35}
$$

where

$$
W_{mn}(r) = [A_{mn}\mathrm{J}_m(\lambda_{mn}r/a) + C_{mn}\mathrm{I}_m(\lambda_{mn}r/a)], \tag{36}
$$

m and n are the number of nodal diameters and circles in the mode shape of a free-edge plate, a is the plate radius, J_m and I_m the Bessel and the modified Bessel functions of the first kind, respectively, A_{mn} and C_{mn} are the mode shape constants and λ_{mn} are the frequency parameters of a free-edge plate. In equation (35) the three rigid-body modes are included.

It must be remarked that for circular plates elastically restrained at the edges different approaches have been used in the past; e.g., the use of polynomial co-ordinate functions has been proved to be a convenient choice by Laura *et al.* [23].

The present approach can be applied to study simply supported shells with additional translational and rotational constraints, elastic bed and mass distributions, even if it is not axisymmetric [24]. It is also useful to solve fluid–shell interaction problems [25].

4. ON THE NATURE OF THE CONVERGENCE

For the sake of simplicity, the case analyzed is that of a beam with an intermediate support simulated by an artificial translational spring, although this extimation can be extended in a general form. Consider a beam of length L and assume harmonic, undamped free vibrations. The mode shapes of the system are indicated with w, that is assumed to be a continuous function with continuous derivative in all the interval $(0, L)$ where w is defined; moreover, it is assumed that w and its derivative respect the Dirichlet condition. The function w is expanded in a series of admissible functions ϕ_n that are the eigenfunctions of the less-constrained problem extracted from the one considered (e.g. the simply supported beam). It has to be noted that, as a consequence of the former choice, ϕ_n constitute a complete set of orthogonal, admissible functions. Therefore, it is possible to write

$$w(x) = \sum_{n=1}^{\infty} a_n \phi_n(nx), \tag{37}$$

where n is the wave number. For the expansion theorem [19], the series on the right-hand side of equation (37) is absolutely and uniformly convergent to w. Moreover, the derivative of the right-hand side of equation (37) is absolutely and uniformly convergent to w', where the prime indicates the derivative with respect to x. The coefficients a_n can be evaluated by using the following relation (if w is known):

$$a_n = \int_0^L w(x)\phi_n(nx)\,\mathrm{d}x, \tag{38}$$

where the following normalization criterion has been used $\int_0^L \phi_n^2(nx)\,\mathrm{d}x = 1$. Equation (38) has been obtained by using the orthogonality of ϕ_n.

As a consequence of the hypotheses made on w and w', the possible discontinuities on w'' are jumps and therefore w'' respects the Dirichlet condition; moreover, a finite number of discontinuities of infinite type but giving finite integral can be present on w''' and therefore this function also respects the Dirichlet condition. These properties of w and its derivatives allows one to write

$$w''(x) = \sum_{n=1}^{\infty} a_n n^2 \phi_n''(nx), \tag{39}$$

where the series on the right-hand side of equation (39) is absolutely convergent to w'' [26]. Moreover, the coefficients $a_n n^2$ have asymptotic behavior of the order $1/n^2$ [26] and therefore a_n has an absolute value lower than μ/n^4, where μ is a positive, finite number independent of n.

The reference kinetic energy of the beam is given by

$$T_B^* = \frac{1}{2} m \sum_{n=1}^{\infty} a_n^2, \tag{40}$$

where m is the mass per unit length. The maximum potential energy of the beam, by using equation (5), is

$$V_B = \frac{1}{2} m \sum_{n=1}^{\infty} a_n^2 \omega_n^2, \tag{41}$$

where ω_n is the radian frequency of the nth mode of the less-constrained problem. The maximum potential energy of the artificial, translational spring is

$$V_k = \frac{1}{2} k \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_n a_i \, \phi_n(n\bar{x}) \, \phi_i(i\bar{x}), \tag{42}$$

where k is the stiffness of the spring located at $x = \bar{x}$.

In order to guarantee the convergence of the method, it is necessary to verify that the difference between the squared radian frequency Ω_N^2 computed with N terms in the expansion and the squared radian frequency Ω_∞^2 computed with infinity terms is as small as desired when a sufficiently large number N of terms is employed. Moreover, for the inclusion principle [19] $\Omega_N^2 > \Omega_\infty^2$. Therefore, it is possible to write

$$\Omega_N^2 - \Omega_\infty^2 = \frac{(V_B)_N + (V_S)_N}{(T_B^*)_N} - \frac{(V_B)_\infty + (V_S)_\infty}{(T_B^*)_\infty}, \tag{43}$$

where the subscripts N and ∞ indicate the number of terms in the series. It can be manipulated into

$$\Omega_N^2 - \Omega_\infty^2 < \frac{[(V_B)_N + (V_S)_N] - [(V_B)_\infty + (V_S)_\infty]}{(T_B^*)_\infty}. \tag{44}$$

Equation (44) can be easily transformed into

$$\Omega_N^2 - \Omega_\infty^2 < \frac{m\sum_{n=N+1}^{\infty} a_n^2\,\omega_n^2 + 2k\sum_{n=1}^{\infty}\sum_{i=N+1}^{\infty} a_n\,a_i\,\phi_n(n\bar{x})\phi_i(i\,\bar{x})}{(T_B^*)_\infty}. \tag{45}$$

Then, it is possible to write

$$\omega_n = \beta n^2, \quad (T_B^*)_\infty = \delta, \quad \mathrm{Max}[\phi_n(n\bar{x})\,\phi_i(i\,\bar{x})] = \gamma \qquad \forall n, i, \tag{46}$$

where β, δ, and γ are finite numbers. The first of equations (46) is obvious for a beam, the second one is due to the asymptotic behavior of a_n and the last one is based on the fact that the eigenfunctions are limited in $(0, L)$ and respect the normalization criterion. Substituting these equations into equation (45), gives

$$\Omega_N^2 - \Omega_\infty^2 < \frac{m\beta^2\,\mu\sum_{n=N+1}^{\infty}\mu/n^4 + 2k\gamma\sum_{n=1}^{\infty}\mu/n^4\sum_{i=N+1}^{\infty}\mu/i^4}{\delta} \tag{47}$$

If $\sum_{n=1}^{\infty}\mu/n^4 = \eta$, where η is a finite number for the asymptotic behavior of a_n, equation (47) is transformed into

$$\Omega_N^2 - \Omega_\infty^2 < \frac{(m\beta^2\mu + 2k\gamma\,\eta)\ \sum_{n=N+1}^{\infty}\mu/n^4}{\delta}. \tag{48}$$

Equation (48) shows that the difference $(\Omega_N^2 - \Omega_\infty^2)$ can be made as small as desired if a sufficiently large number N of terms is chosen, even if k is assumed to be very large in order to simulate a rigid support. In particular, the right-hand side of equation (48) has an asymptotic order k/N^3.

This analysis can be extended to rotational springs, where $\mathrm{Max}\{[\mathrm{d}\phi_n(n\bar{x})/\mathrm{d}x][\mathrm{d}\phi_i(i\bar{x})/\mathrm{d}x]\} = ni\gamma$ replaces the third expression in equation (46). In this case, equation (48) has an asymptotic order k/N^2. This last expression explains the reason for a slower convergence in the case of a high stiffness rotational spring with respect to a high stiffness translational spring. In fact, it is to be noted that if rotational springs are not present in the system, in general the Dirichlet condition is satisfied by w'''' and therefore the asymptotic order of equation (48) becomes k/N^4.

5. DISCUSSION AND CONCLUSIONS

The present study shows that the use of eigenfunctions (including eventually rigid-body modes) of a less-constrained problem extracted from the one considered is a simple (and often smart) choice of admissible functions in the Rayleigh–Ritz method. This choice has practical applications when it is possible to extract a less-constrained problem having eigenfunctions expressed by analytical expressions in closed form; the extracted problem should be the closest, simple problem in order to simplify calculations. The rigid constraints eliminated are replaced by elastic ones. The rate of convergence is always sufficiently high; in particular, it is higher for additional translational constraints added to the less-constrained problem with respect to rotational constraints. This fact has been explained and the convergence of the method has been analytically investigated. Specific examples show the possibilities and potential of the method.

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