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LETTER TO THE EDITOR

HIGHER ORDER ANALYSIS FOR NON-LINEAR VIBRATIONS OF CONTINUOUS SYSTEMS

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1. INTRODUCTION

Many perturbation methods have been developed to construct approximate solutions for non-linear continuous systems. Analytical methods are suitable to study weakly non-linear systems and can be divided into two principal classes, i.e. discretization and direct techniques. In the direct approach a perturbation method is applied directly to the original system and no assumptions are made *a priori* on the form of the solution. The multiple scales method [1, 2] is often used in the direct approach and many continuous systems have been investigated [3–7].

Among the discretization techniques, the Galerkin method is perhaps the best known method. Recently, many papers [3–7] have demonstrated that this technique can lead to incorrect results for response-frequency and response-excitation amplitude curves, because the predicted resulting mode shape exhibits the same self-similarity behaviour as does a linear mode, i.e., at any instant the displacement pattern is a multiple of some basic spatial pattern (the mode shape). However, experimental studies indicate that the vibration modes can deviate from those of the associated linear problem. Thus, this method is not capable of taking into account an important physical feature of non-linear systems. On the contrary, the predicted behaviour of the direct approach methods is not self-similar.

In this paper a reasonably simple direct approach method is presented for the study of non-linear continuous systems. In particular, an asymptotic perturbation method is used, which was developed previously for weakly non-linear ordinary differential equations [8–10], in order to determine the transverse vibrations of a hinged–hinged Euler–Bernoulli beam resting on a non-linear elastic foundation with cubic non-linearities and under the action of a resonant external excitation. The relevant partial differential equation is

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + \varepsilon\, a\, \frac{\partial u}{\partial t} + ku + \varepsilon u^3 = 2\varepsilon F(x)\cos(\Omega t), \tag{1}$$

$$u(0,\, t) = u(1,\, t) = 0 \quad \text{and} \quad \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(1,\, t)}{\partial x^2} = 0, \tag{2}$$

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where $u = u(x, t)$ is the transverse displacement, $x \in [0, 1]$ is the position along the neutral axis, a the damping coefficient, Ω the excitation frequency, and ε is a small non-dimensional parameter. The general solution of arbitrary initial conditions is obtained in the linear, undamped and unforced case ($\varepsilon = 0$) through a mode superposition as

$$u(x, t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + \vartheta_n)\, \Phi_n(x), \tag{3}$$

where A_n and ϑ_n are determined by the initial conditions. The normalized eigenmodes and natural frequencies are

$$\Phi_n(x) = \sqrt{2}\, \sin(n\pi, x), \qquad \omega_n = \sqrt{n^4\pi^4 + k}, \quad n = 1,2, \ldots, \tag{4}$$

In this paper, a method is illustrated to obtain increasingly accurate solutions by increasing the order of approximation in terms of the small parameter ε. This is accomplished by three processes: obtaining the form of solution in terms of harmonic components, introducing a slow time scale, and solving directly for the various harmonic components via harmonic balance. The lowest order results coincide with solutions of the multiple-scale method, while significant differences arise in second order calculations. In view of this, the method is considered as a particular version of the multiple-scale method.

In the following the fundamental resonance is considered, namely $\omega_n \approx \Omega$. A second order approximate solution is calculated, the frequency-response and the excitation amplitude-response curves determined and the results compared with the numerical solution and the multiple-scale method second order solution given in reference [11], where Boyaci and Pakdemirli, following a suggestion coming from Rahman and Burton [12], have developed an improved multiple-scale method.

2. THE APPROXIMATE SOLUTION FOR THE PRIMARY RESONANCE OF THE nth MODE

It is assumed that the system is in primary resonance with the external excitation and in order to express the nearness of the excitation frequency to the natural frequency, detuning parameter σ is defined through the relation

$$\omega_n = \Omega + \varepsilon\sigma, \tag{5}$$

where σ is of O (1). The slow time

$$\tau = \varepsilon t \tag{6}$$

is introduced to include the effects induced by non-linear, damping and excitation terms. These effects are best described in terms of the rescaled variable τ, which accounts for the need to look at larger time scales to obtain a non-negligible

contribution. The solution $u(x, t)$ of equations (1) and (2) can be expressed by means of a power series in the expansion parameter ε,

$$u(x, t) = \sum_{m[odd]=-\infty}^{+\infty} \varepsilon^{\gamma_m} \psi_m(x, \tau, \varepsilon) \exp(-\mathrm{i}m\Omega t), \tag{7}$$

where $\gamma_m = (|m| - 1)/2$ and $\psi_m(x, \tau, \varepsilon) = \psi^*_{-m}(x, \tau, \varepsilon)$, because $u(x, t)$ is real. Only a mode in the expansion is assumed, because the system is excited near the natural frequency of a specific linear mode and that mode is not involved in an internal resonance with any other mode. For a damped system, the modes that are not directly excited by an external source or indirectly by an internal resonance will decay with time.

Equation (7) can be written more explicitly as

$$u(x, t) = (\psi(x, \tau; \varepsilon) \exp(-\mathrm{i}\Omega t) + \varepsilon\psi_3(x, \tau; \varepsilon) \exp(-3\mathrm{i}\Omega t) + \text{c.c.}) + o(\varepsilon^2), \tag{8}$$

where c.c. stands for complex conjugate of the preceeding terms. The functions $\psi_m(x, \tau, \varepsilon)$'s depend on the parameter ε and it is supposed that the limit of the ψ_m's for $\varepsilon \to 0$ exists and is finite and, moreover, can be expanded in power series of ε, i.e.,

$$\psi_m(x, \tau; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \psi_m^{(i)}(x, \tau). \tag{9}$$

In the following for simplicity the abbreviations $\psi_m^{(0)} = \psi_m$ for $m \neq 1$ and $\psi_1^{(0)} = \psi$ for $m = 1$ are used. Note that the variable change (6) implies that

$$\frac{\partial}{\partial t}(\psi_m \exp(-\mathrm{i}m\Omega t)) = \left(-\mathrm{i}m\Omega\psi_m + \varepsilon \frac{\partial \psi_m}{\partial \tau}\right) \exp(-\mathrm{i}m\Omega t). \tag{10}$$

The expansion of solution (8) is used for the elimination of the predominant linear part of equations (1) and (2) and it allows the possible interactions among the different harmonics, created by the non-linear terms to be calculated. After inserting equation (8) in complete equations (1) and (2), a number of equations are obtained for every harmonic and for a fixed order of approximation.

For $n = 1$ (order ε^0) the linear equation becomes

$$\frac{\partial^4 \psi}{\partial x^4} + k\psi - \omega_n^2 \psi = 0, \qquad \psi(0, t) = \psi(1, t) = \frac{\partial^2 \psi}{\partial x^2}(0, t) = \frac{\partial^2 \psi}{\partial x^2}(1, t) = 0. \tag{11}$$

The solution is

$$\psi(x, \tau) = \Phi_n(x) \Psi(\tau), \tag{12}$$

where $\Phi_n(x)$ and ω_n are furnished by equation (4).

Considering equations (1) and (2) for $n = 3$ (order ε) yields

$$\frac{\partial^4 \psi_3}{\partial x^4} + k\psi_3 - 9\omega_n^2 \psi_3 = -\Phi_n^3 \Psi^3, \qquad \psi_3(0, t) = \psi_3(1, t) = \frac{\partial^2 \psi_3}{\partial x^2}(0, t) = \frac{\partial^2 \psi_3}{\partial x^2}(1, t) = 0. \tag{13}$$

The solution of equation (13) is

$$\psi_3(x, \tau) = g_3(x)\ \Psi^3(\tau), \qquad g_3(x) = \frac{3\sqrt{2}}{16\omega_n^2} \sin(n\pi x) + \frac{\sqrt{2}}{16(9\omega_n^2 - 10k)} \sin(3n\pi x), \tag{14a}$$

with the boundary conditions

$$g_3(0) = g_3(1) = \frac{\mathrm{d}^2 g_3}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2 g_3}{\mathrm{d}x^2}(1) = 0. \tag{14b}$$

From equations (1) and (2) for $n = 1$ and at the order ε

$$2\mathrm{i}\Omega\Phi_n(x) \frac{\mathrm{d}\Psi}{\mathrm{d}\tau} + \mathrm{i}a\Omega\Phi_n\ (x)\ \Psi - 2\Omega\sigma\Psi\Phi_n(x) - 3\,|\,\Psi\,|^2\,\Psi\Phi_n^3\ (x) + F(x) = 0. \tag{15}$$

Making use of the normalization of the eigenmodes and with the definition

$$f_n = \int_0^1 F(x)\ \Phi_n(x)\ \mathrm{d}x \tag{16}$$

the ordinary differential equation becomes

$$2\mathrm{i}\Omega \frac{\mathrm{d}\Psi}{\mathrm{d}\tau} + \mathrm{i}a\Omega\Psi - \frac{9}{2}\,|\,\Psi\,|^2\ \Psi + f_n = 0, \tag{17}$$

where $\int_0^1 \Phi_n^4(x)\,\mathrm{d}x = \frac{3}{2}$. Expressing the function Ψ in real and imaginary parts,

$$\Psi(\tau) = \rho(\tau)\ \exp(\mathrm{i}\vartheta(\tau)), \tag{18}$$

the model equations become

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = -\frac{a}{2}\,\rho + \frac{f_n}{2\Omega} \sin\vartheta, \qquad \rho\,\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} = -\frac{9}{4\Omega}\,\rho^3 + \frac{f_n}{2\Omega} \cos\vartheta - \sigma\rho\,. \tag{19, 20}$$

Equations (19, 20) are invariant under the transformation $f_n \to -f_n$, $\vartheta \to \vartheta - \pi$, and hence possess the corresponding symmetry. Thus, if there is an equilibrium point at (f_n^0, ϑ_0), then there is also one at $(-f_n^0, \vartheta_0 - \pi)$. In order to simplify the following analysis, only half of the system is considered. If it is assumed that the system

contains an equilibrium point, then it actually contains two equilibria, the other one being located at the symmetrical position under the above-mentioned transformation. From equations (8), (12) and (18) we can express the deflection to the lowest order of approximation as

$$u(x, t) = 2\Phi_n(x)\ \rho\ \cos\ (\Omega t - \vartheta) + o(\varepsilon), \tag{21}$$

where ρ and ϑ are given by equations (19) and (20). In order to improve the validity of the analysis, a second order calculation must be performed. Equations (1) and (2) yields

$$\varepsilon(-2i\Omega\frac{\partial\psi}{\partial\tau} - ia\Omega\psi + 2\Omega\sigma\psi + 3|\psi|^2\ \psi - F)$$

$$+\varepsilon^2\left(\frac{\partial^2\psi}{\partial\tau^2} - \sigma^2\psi + a\frac{\partial\psi}{\partial\tau} + 3\psi_3\bar{\psi}^2\right) = 0. \tag{22}$$

The term $\partial^2\psi/\partial\tau^2$ in equation (22) can be eliminated, because equation (17) can be differentiated with respect to τ

$$\frac{\partial^2\psi}{\partial\tau^2} = \left(\frac{a^2}{4} + ia\sigma - \sigma^2\right)\psi + \frac{3}{\Omega}(ia - \sigma)|\psi|^2\ \psi - \left(\frac{9}{4\Omega^2}\right)|\psi|^4\psi$$

$$+\frac{F}{4\Omega}(2\sigma - ia) - \frac{3}{4\Omega^2}F\psi^2 + \frac{3}{2\Omega^2}F|\psi|^2. \tag{23}$$

Equation (23) can be now substituted into equation (22). Subsequently, making use of the normalization of the eigenmodes and with the definitions

$$\beta_1 = \int_0^1 g_3(x)\ \Phi_n^3\ (x)\ dx = \frac{15}{16}\frac{(8\omega_n^2 - 9k)}{\omega_n^2\ (9\omega_n^2 + 26k)}, \qquad \beta_2 = \int_0^1 F(x)\,\Phi_n^3\ (x)\ dx. \tag{24}$$

then

$$\varepsilon(-2i\Omega\frac{d\Psi}{d\tau} - ia\Omega\Psi + 2\Omega\sigma\Psi + \frac{9}{2}|\Psi|^2\ \Psi - f_n)$$

$$-\varepsilon^2\left(\left(2\sigma^2 + \frac{a^2}{4}\right)\Psi + \frac{9}{2\Omega}\left(\frac{ia}{2} - \sigma\right)\left|\Psi\right|^2\ \Psi - \left(3\beta_1 - \frac{45}{8\Omega^2}\right)\left|\Psi\right|^4\ \Psi\right. \tag{25}$$

$$\left. -\frac{f_n}{4\Omega}(2\sigma + ia) - \frac{3}{2\Omega^2}\beta_2|\Psi|^2 + \frac{3}{4\Omega^2}\beta_2\ \Psi^2\right) = 0,$$

with $\int_0^1 \Phi_n^6(x)\,\mathrm{d}x = \frac{5}{2}$. The second order approximate solution can be written as

$$u(x, t) = 2\Phi_n(x)\,\rho\cos(\Omega t - \vartheta) + 2\varepsilon\rho^3 g_3(x)\cos[3(\Omega t - \vartheta)] + o(\varepsilon^2). \tag{26}$$

Equation (25) can be separated into real and imaginary parts in the usual way and the refined model equations

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = -\frac{a}{2}\rho + \frac{f_n}{2\Omega}\sin\vartheta + \varepsilon\left(-\frac{9a}{8\Omega^2}\rho^3 + \frac{f_n}{8\Omega^2}(-2\sigma\sin\vartheta + a\cos\vartheta) - \frac{9\beta_2}{8\Omega^3}\sin\vartheta\right), \tag{27}$$

$$\rho\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} = -\frac{9}{4\Omega}\rho^3 + \frac{f_n}{2\Omega}\cos\vartheta - \sigma\rho + \varepsilon\left(\frac{8\sigma^2 + a^2}{8\Omega}\rho - \frac{9\sigma}{4\Omega^2}\rho^3 + \frac{3}{2\Omega}\left(\frac{15}{8\Omega^2} - \beta_1\right)\rho^5 + \frac{f_n}{8\Omega^3}(-2\sigma\cos\vartheta + a\sin\vartheta) - \frac{3}{8\Omega^3}\beta_2\cos\vartheta\right) \tag{28}$$

can be obtained.

3. FREQUENCY-RESPONSE AND EXCITATION AMPLITUDE-RESPONSE CURVES

Equilibrium points ($\mathrm{d}\rho/\mathrm{d}\tau = \mathrm{d}\vartheta/\mathrm{d}\tau = 0$) of equations (19, 20) or equations (27) and (28) correspond to periodic solutions of the original equations (1) and (2).

Using the trigonometric identity $\sin^2\vartheta + \cos^2\vartheta = 1$, the frequency-response curve can be obtained from equations (19, 20):

$$\sigma = -\frac{9}{4\Omega}\rho^2 \pm \sqrt{\left(\frac{f}{2\Omega\rho}\right)^2 - \frac{a^2}{4}}. \tag{29}$$

However, a more accurate expression can be derived from second order equations (27) and (28):

$$\sigma = -\frac{9}{4\Omega}\rho^2 \pm \sqrt{\left(\frac{f}{2\Omega\rho}\right)^2 - \frac{a^2}{4}} + \varepsilon\left(\frac{a^2 - 4\sigma^2}{8\Omega} + \frac{\sigma}{2\Omega^2}\left(\frac{3\beta_2}{f} - \frac{27}{2}\right)\rho^2 + \frac{1}{2\Omega}\left(\frac{27\beta_2}{4\Omega^2 f} + \frac{45}{8\Omega^2} - 3\beta_1\right)\rho^4\right), \tag{30}$$

where β_1, β_2 are given by equation (24). Note that the second order correction to the frequency-response curve of the multiple-scale method (see equation (51) of reference [11]), consists of only a single term, which is essentially the last term in equation (30), i.e. $(3\beta_1/2\Omega)\,\rho^4$, and is independent of the excitation amplitude f_n,

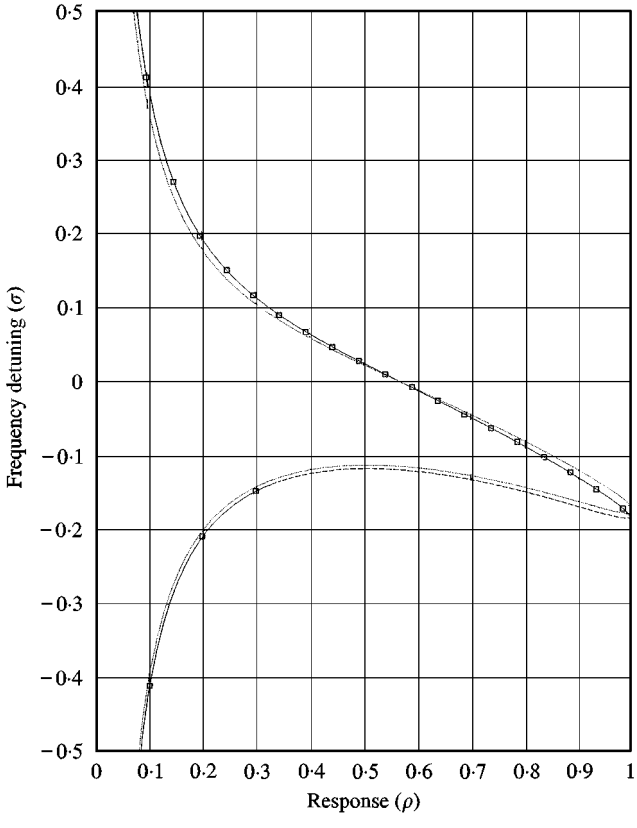

Figure 1. The frequency-response curve (30) for the case of primary resonance of the first mode, i.e., the detuning (σ) as function of the response (ρ). Solid lines stand for the stable solutions and dashed lines for the unstable solutions. ($a = 0{\cdot}1$, $\varepsilon = 0{\cdot}03$, $f_1 = 1$). Dot lines represent the solution given in reference [11] and boxes the numerical results.

whilst, the second order correction of the method described here takes into account both the external excitation and the damping coefficient. In order to establish the stability of steady state solutions, small perturbations are superimposed in the amplitudes and the phases on the steady state solutions and the resulting equations are then linearized. Subsequently, the eigenvalues of the corresponding system of first order differential equations with constant coefficients (the Jacobian matrix can be considered). A positive real root indicates an unstable solution, whereas if the real part of the eigenvalues are all negative then the steady state solution is stable. Dot lines represent the solution given in reference [11] and boxes the numerical results.

Results of numerical stability analysis are shown in Figure 2 in the plane $(\mu = a/\sigma, f_n) =$ (damping coefficient-to-detuning ratio, excitation amplitude). However, if calculations are limited to the lowest order calculations, the following analytical results are obtained:

(i) if $3\mu^2 > 4$ there is only one equilibrium point:
(ii) if $3\mu^2 < 4$, $\sigma < 0$, $f_n > f_n^0$, where

$$f_n^0 = \frac{\sigma\Omega}{9}\,[2\Omega\sigma(16 - 40\,\mu^2 + (6\mu - 8)(4 - 3\mu^2)^{1/2})]^{1/2} \tag{31}$$

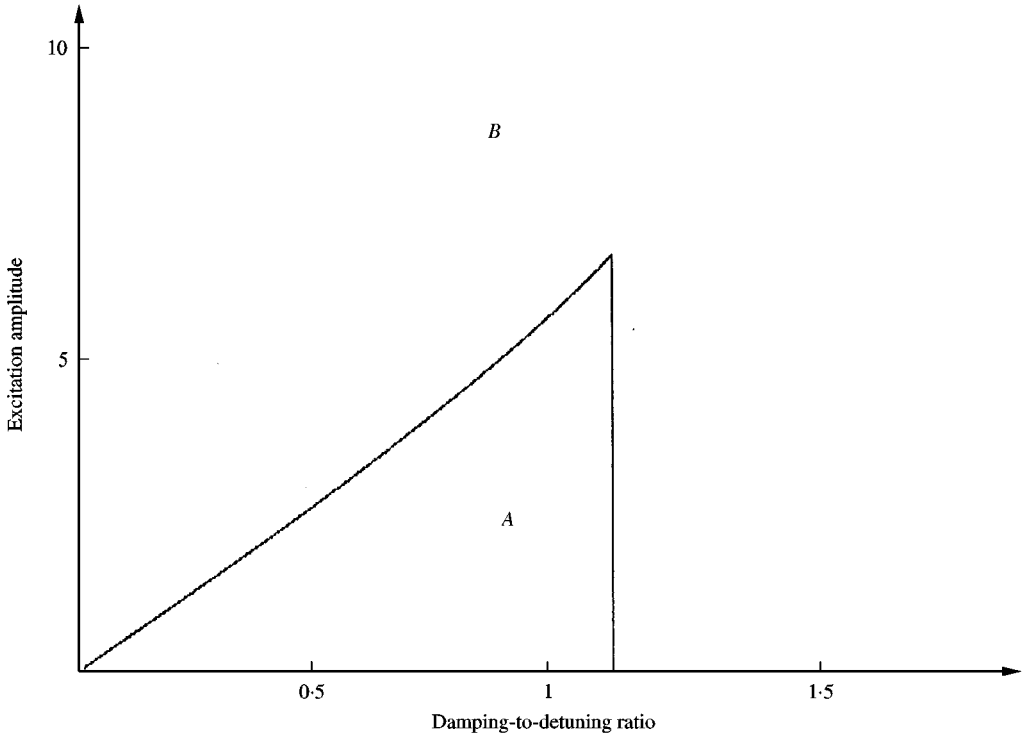

Figure 2. Stability analysis in the (μ, f_1) = (damping coefficient-to-detuning ratio, excitation amplitude) plane for $\sigma < 0$ in the case of fundamental resonance of the first mode. In region A there is only an equilibrium point, while in region B there are three equilibrium points.

there is only one equilibrium point;

(iii) if $3\mu^2 < 4$, $\sigma < 0$, $f_n < f_n^0$ there are three equilibrium points;
(iv) if $3\mu^2 < 4$, $\sigma > 0$ there is only one equilibrium point.

Boundary curves in Figure 2 are approximated for small values of ε by analytical expressions furnished in cases (i)–(iv).

Figure 3 shows a typical excitation amplitude-response curve. In particular, note a saddle-node fold (or cyclic fold) bifurcation when the solution jumps up to a larger stable orbit as the amplitude of the external force is increased. A fold bifurcation corresponds to a vertical tangency in the external force-response space, where the derivative of the response with respect to the control parameter is infinite. Dotted lines represent the solution given in reference [11] and boxes the numerical results.

Figure 4 shows the spatial distribution of $u(x, t)$ for the first mode as obtained from equation (26). In order to check the formulation, solutions of equations (1)–(2) were also obtained numerically and results from the multiple-scale method given in reference [11] are shown. Results given by the proposed method and the multiple-scale technique are very similar at the lowest order, but there are remarkable

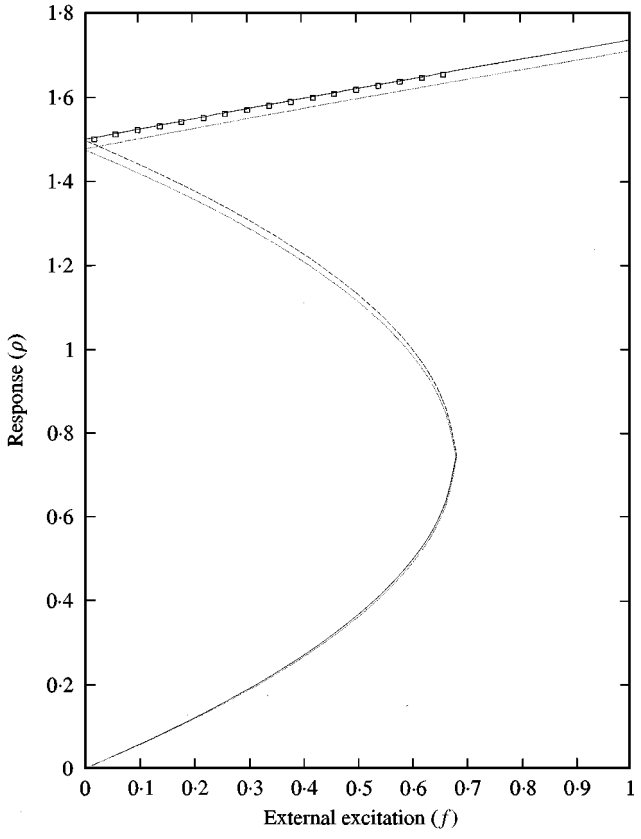

Figure 3. Amplitude of the response (ρ) as function of the excitation (f_1) for the case of fundamental resonance of the first mode. Solid lines stand for stable and dashed lines for unstable solutions. Dotted lines represent the solution given in reference [11] and boxes the numerical results.

differences in the approximate solutions at the second order. For the chosen values of parameters, the proposed method furnishes a slightly better solution. For example, in Figure 4, numerical results show that the solution (mean difference with respect to the numerical solution 0·013) gives an accuracy greater than that of multiple-scale method (mean difference 0·046).

4. CONCLUDING REMARKS

A method has been presented which is essentially based on the harmonic balance and the multiple- scale methods. It has been used for the theory of approximate analytic solutions to non-linear oscillations of continuous systems with cubic non-linearities in the case of primary resonance. No *a priori* assumption on the spatial dependence of the motion has been employed.

A system of non-linear model equations has been derived describing the modulation of the amplitude and of the phase of the oscillation. Frequency-response and excitation amplitude-response curves can be easily deduced. It has been demonstrated that for the lowest order analysis approximate solutions given by this

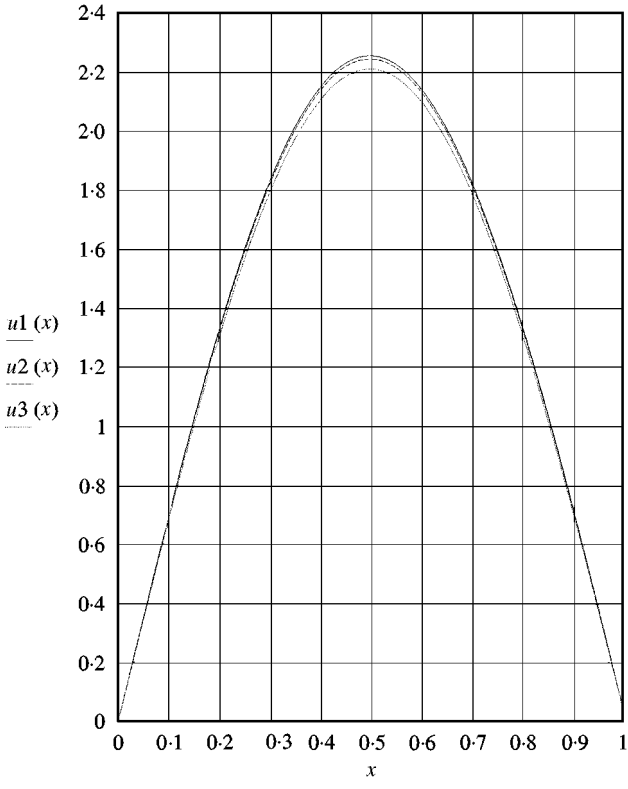

Figure 4. Spatial distribution of the displacement in the case of primary resonance of the first mode after a period ($T = 2\pi/\Omega$) of the external excitation. (——) numerical solution ($u(x, T) = u1(x)$), (– – –) AP method ($u(x, T) = u2(x)$), (. . .) multiple-scale method ($u(x, T) = u3(x)$) ($a = 0{\cdot}1$, $\varepsilon = 0{\cdot}03$, $f_1 = 1$).

method and the multiple-scale technique are essentially identical, whilst, the second order calculations lead to different results. For the chosen values of parameters, this method furnishes more accurate approximate solutions, but this may not be true for other non-linear systems.

A possible drawback of the method is that time dependence is assumed at the beginning. In this respect the usual multiple-scale method is more straightforward since no assumptions are made on the temporal dependence of solutions, but in this way spurious solutions can be introduced [12]. On the other hand, the assumed temporal dependence of the solution allows more rapid calculations and furnishes different results for higher order analysis.

Numerical results demonstrate the validity of this particular version of the multiple-scale method for higher order calculations; further study of other non-linear continuous systems are suggested.

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