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PARAMETRIC INSTABILITY OF A COLUMN WITH AN AXIALLY OSCILLATING MASS

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This study investigates the dynamic instability behavior of a column carrying a concentrated mass with oscillating motion along the column axis. The dynamic equation of the column was derived based on the assumed-modes method. The derived dynamic equation, which contains parametrically excited terms associated with modal accelerations, modal velocities, and modal displacements, is a general form of Mathieu's equation. A new analytical method used to determine the instability regions of the column was directly applied to the transition state. This method is different from the traditional perturbation method in which a criterion, involving the determination of the characteristic exponents, is used to yield the transition curves. The principal vibration frequencies, the ratio of principal amplitudes, and the phase difference between the parametrically excited force and the principal frequency response on the transition state were obtained systematically. The parametric instability behavior of a column carrying a periodically moving concentrated mass is different from that of a column subjected to a periodic tangential inertia force. The present case contains the simple resonances and combination resonances of sum type only, while the case with tangential inertia force may contain the combination resonances of the difference type additionally. Four examples are given to demonstrate the instability behavior of various columns carrying concentrated oscillating mass along the column axis at varying positions.

1. INTRODUCTION

The dynamic instability of elastic structural elements, such as rods, beams and columns, induced by parametric excitation has been investigated by many researchers. Extensive bibliographies on this subject were given by Evan-Iwanowski [1] and Nayfeh and Mook [2]. Bolotin [3] provided a general introduction to analyze the dynamic stability problems of various structural elements. Hsu [4–6], Nayfeh and Mook [7], and Yamamoto and Saito [8] used the perturbation method to solve Mathieu–Hill's equation, which governed the behavior of an elastic system under parametric excitation. They established a criterion to yield the transition curves by determining the characteristic exponents in the solution. Several researchers [3, 9] examined simply supported columns subjected to periodic axial loads; they expressed the governing equation of the transverse motion of the column as a set of uncoupled Mathieu–Hill's equation. In

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1972, Iwatsubo *et al.* [10] developed a numerical simulation procedure to investigate the dynamic instability of continuous systems with a periodically time-varying parameter; they applied the procedure to a cantilevered column subjected to axial and tangential periodic loads. Later, Iwatsubo *et al.* [11] used the finite-difference method and experimental approach to examine the simple and combination resonances of clamped–clamped and clamped–simply supported columns in 1973. Iwatsubo *et al.* [12] applied Galerkin's method to investigate the existence of the combination resonances of four typical columns under periodic axial or tangential loads in 1974; they also discussed the effects of damping on combination resonances. Chen and Yeh [13] assessed analytically the instability behavior of a cantilevered column subjected to a periodic load in the direction of a varying tangency coefficient at the free end and physical explanations for the behavior of simple and combination resonances were presented. Yeh and Chen [14] also developed a general formula to determine the regions of simple and combination resonances of an elastic column subjected to a periodic load at any axial position in the direction of a varying tangency coefficient.

Handoo and Sundararajan [15] investigated both analytically and experimentally the parametric instability regions of cantilevered columns carrying concentrated end mass and subjected to periodic axial motion at its fixed end. Elmaraghy and Tabarrok [16] studied the parametric resonance of a beam with encastré ends and subjected to a given periodic axial acceleration. Saito and Koizumi [17] examined the parametrically excited behavior of a simply supported horizontal beam carrying a concentrated mass at one end and subjected to a periodic axial displacement excitation at the other end under the influence of gravity. Buffinton and Kane [18] investigated the dynamic behavior of a uniform beam moving longitudinally at a prescribed rate over two bilateral supports.

Although much research has been carried out on the parametrically excited instability of columns and beams subjected to periodic axial or tangential loads or given periodic axial motion, little information is available on the parametric instability of columns or beams induced by a concentrated mass moving periodically along its axis. This problem may be encountered for columns carrying a Scotch yoke, or a piston, which oscillates along the column axis. The objective of this work is to investigate the parametric instability of an elastic column carrying a concentrated mass that undergoes a periodic motion along the axis of the column. A new analytical method, different from the traditional perturbation method, is developed to determine the transition curves of the general form of Mathieu's equation. In this method, without determining the characteristic exponents as used in the traditional perturbation method, the transition curves between the stability and the instability regions are easily obtained. The study may lead to a better understanding of the dynamic instability behaviour induced by an "axially oscillating mass".

2. DYNAMIC EQUATION OF THE COLUMN

A column carrying a concentrated moving mass m_0 at point P is shown in Figure 1. The column has length L, mass per unit length $\rho(x)$ and bending rigidity

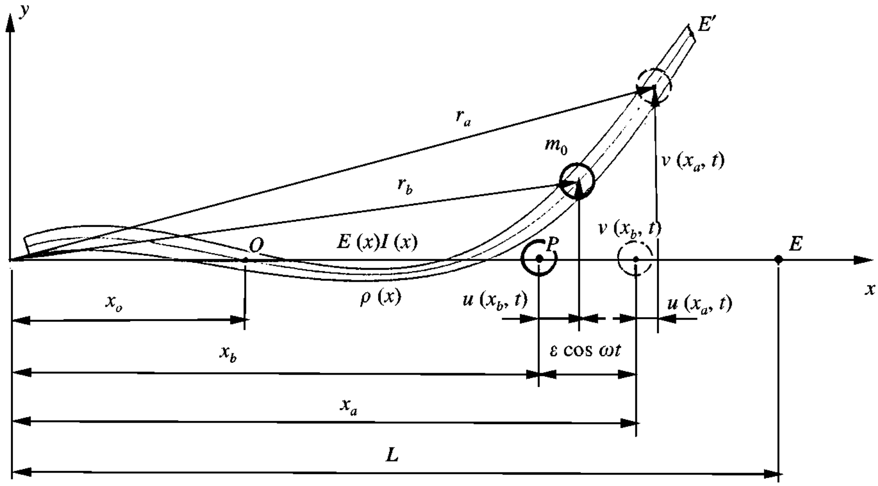

Figure 1. The dynamic system of a general column carrying an axially oscillating mass.

$E(x)I(x)$. The cross sections and material properties of the column, which may vary along the length, are symmetric about the $x-y$ plane. The neutral axis of the column is initially straight. The fixed supporting point of the column is point O, which has no deflection in any direction. The deflections of the column in the x and y directions are denoted as $u(x,t)$ and $v(x,t)$ respectively. The column carries a concentrated mass m_0 which oscillates along the axis of the column. The oscillating mass centered at point P is restricted to have a small sinusoidal motion. In this work the following assumptions were made:

(1) The neutral axis of the column is inextensible
(2) Only small deflections in the x and y directions are considered.
(3) The gravitational effects are neglected.
(4) Only small proportional viscous damping is considered.

As shown in Figure 1, when the concentrated mass is kept at point P on the column, the free transverse vibration equation of the column is

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2 v(x,t)}{\partial x^2}\right) + c(x)\frac{\partial v(x,t)}{\partial t} + [\rho(x) + m_0\delta(x-x_b)]\frac{\partial^2 v(x,t)}{\partial t^2} = 0, \tag{1}$$

where $c(x)$ is the viscous damping coefficient, δ is unit impulse function, and x_b is the x-co-ordinate of point P before the column undergoes deformation. Let the system have undamped natural frequencies ω_n, modal damping coefficient d_n and corresponding mode shape functions $\phi_n(x)$ when boundary conditions are specified. The transverse deflection of the column, shown in Figure 1, can be expressed by mode-shape functions $\phi_n(x)$ and corresponding modal deflection components $V_n(t)$ as

$$v(x,t) = \sum_n \phi_n(x) V_n(t). \tag{2}$$

For convenience, the mode-shape functions are chosen to be a set of orthonormal functions as follows:

$$\int_0^L [\rho(x) + m_0\delta(x - x_b)]\,\phi_n(x)\phi_m(x)\,\mathrm{d}x = \delta_{nm}, \quad n, m = 1, 2, 3, \ldots, \tag{3}$$

where δ_{nm} is Kronecker delta. The governing equation of the free transverse vibration of the system can be expressed with respect to the modal deflection component $V_n(t)$ as

$$\ddot{V}_n(t) + d_n\dot{V}_n(t) + \omega_n^2 V_n(t) = 0, \; n = 1, 2, 3, \ldots . \tag{4}$$

Assuming the inextensibility of the neutral axis and small deflections in the x and y directions, the horizontal displacement of the column, $u(x, t)$, can be approximately expressed by the transverse deflection $v(x, t)$ as

$$\begin{aligned} u(x, t) &= -\frac{1}{2}\int_{x_0}^{x} v'^2(\xi, t)\,\mathrm{d}\xi \\ &= -\frac{1}{2}\sum_n\sum_m \int_{x_0}^{x} \phi_n'(\xi)\phi_m'(\xi)\,\mathrm{d}\xi\, V_n(t)V_m(t), \end{aligned} \tag{5}$$

where x_0 is the x-co-ordinate of the fixed supporting point O.

The positions of the concentrated mass before and after oscillating motion can be expressed by the position vectors $\mathbf{r}_b$ and $\mathbf{r}_a$ respectively as

$$\mathbf{r}_b = [x_b + u(x_b, t)]\mathbf{i} + v(x_b, t)\mathbf{j}, \tag{6}$$

$$\mathbf{r}_a = [x_a(t) + u(x_a, t)]\mathbf{i} + v(x_a, t)\mathbf{j}, \tag{7}$$

where $x_a(t)$ is the x-co-ordinate of the concentrated mass m_0 after it undergoes an oscillating motion when the column is kept straight. The corresponding kinetic energy of the concentrated mass m_0 before and after the oscillating motion can be expressed respectively as

$$\mathrm{KE}_b = \tfrac{1}{2}m_0\dot{\mathbf{r}}_b^2, \qquad \mathrm{KE}_a = \tfrac{1}{2}m_0\dot{\mathbf{r}}_a^2. \tag{8, 9}$$

Therefore, when the concentrated mass m_0 undergoes an oscillating motion, the dynamic equation of the system can be obtained by adding the terms derived from the energy difference between the oscillating mass and the fixed concentrated mass into equation (4) as follows:

$$\ddot{V}_n(t) + d_n\dot{V}_n(t) + \omega_n^2 V_n(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(\mathrm{KE}_a - \mathrm{KE}_b)}{\partial\dot{V}_n}\right) - \frac{\partial(\mathrm{KE}_a - \mathrm{KE}_b)}{\partial V_n} = 0. \tag{10}$$

Using equations (2) and (5)–(9), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(\mathrm{KE_a}-\mathrm{KE_b})}{\partial \dot{V}_n}\right)-\frac{\partial(\mathrm{KE_a}-\mathrm{KE_b})}{\partial V_n}$$

$$= m_0\sum_m \phi_n(x_a)\phi_m(x_a)\ddot{V}_m(t) + 2m_0\dot{x}_a(t)\sum_m \phi_n(x_a)\phi'_m(x_a)\dot{V}_m(t)$$

$$+ m_0\ddot{x}_a(t)\sum_m\left[\phi_n(x_a)\phi'_m(x_a) - \int_{x_0}^{x_a(t)}\phi'_n(\xi)\phi'_m(\xi)\,\mathrm{d}\xi\right]V_m(t)$$

$$+ m_0\dot{x}_a^2(t)\sum_m \phi_n(x_a)\phi''_m(x_a)V_m(t) - m_0\sum_m \phi_n(x_b)\phi_m(x_b)\ddot{V}_m(t). \tag{11}$$

Assuming that the concentrated mass has a sinusoidal oscillation motion with frequency ω and small amplitude ε

$$x_a(t) = x_b + \varepsilon\cos\omega t. \tag{12}$$

Then $\phi_n(x_a)\phi_m(x_a)$ in equation (11) can be expressed as

$$\phi_n(x_a)\phi_m(x_a) = \phi_n(x_b)\phi_m(x_b) + \varepsilon\cos\omega t[\phi'_n(x_b)\phi_m(x_b) + \phi_n(x_b)\phi'_m(x_b)] + O(\varepsilon^2). \tag{13}$$

Substituting equations (12) and (13) into equation (11) and neglecting the terms of second and higher orders of ε, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(\mathrm{KE_a}-\mathrm{KE_b})}{\partial \dot{V}_n}\right)-\frac{\partial(\mathrm{KE_a}-\mathrm{KE_b})}{\partial V_n}$$

$$\approx \varepsilon m_0\cos\omega t\sum_m[\phi_n(x_b)\phi'_m(x_b) + \phi'_n(x_b)\phi_m(x_b)]\ddot{V}_m(t)$$

$$- 2\varepsilon m_0\omega\sin\omega t\sum_m \phi_n(x_b)\phi'_m(x_b)\dot{V}_m(t)$$

$$- \varepsilon m_0\omega^2\cos\omega t\sum_m\left[\phi_n(x_b)\phi'_m(x_b) - \int_{x_0}^{x_b}\phi'_n(\xi)\phi'_m(\xi)\,\mathrm{d}\xi\right]V_m(t). \tag{14}$$

Assuming small modal damping coefficients d_n, then d_n can be expressed as

$$d_n = 2\varepsilon\mu_n, \tag{15}$$

where μ_n is a damping parameter. After substituting equations (14) and (15) into equation (10), we have

$$\ddot{V}_n(t) + \omega_n^2 V_n(t) + 2\varepsilon\mu_n \dot{V}_n(t) + 2\varepsilon \cos\omega t \sum_m h_{nm} \ddot{V}_m(t) + 2\varepsilon \sin\omega t \sum_m g_{nm} \dot{V}_m(t)$$

$$+ 2\varepsilon \cos\omega t \sum_m f_{nm} V_m(t) = 0, \quad n = 1, 2, 3, \ldots, \tag{16}$$

where

$$h_{nm} = \tfrac{1}{2} m_0 [\phi_n(x_b)\phi'_m(x_b) + \phi'_n(x_b)\phi_m(x_b)],$$

$$g_{nm} = -\, m_0 \omega \phi_n(x_b)\phi'_m(x_b),$$

$$f_{nm} = \tfrac{1}{2} m_0 \omega^2 \left[\int_{x_0}^{x_b} \phi'_n(\xi)\phi'_m(\xi)\,\mathrm{d}\xi - \phi_n(x_b)\phi'_m(x_b) \right]. \tag{17}$$

Equation (16) is the dynamic equation of a column, as shown in Figure 1, carrying a concentrated mass that undergoes a sinusoidal oscillation along the column axis with frequency ω and small amplitude ε. This equation is a general form of Mathieu's equation with parametrically excited terms associated with modal accelerations $\ddot{V}_m(t)$, modal velocities $\dot{V}_m(t)$, and modal displacements $V_m(t)$. The dynamic equation of a column carrying an oscillating concentrated mass, discussed here, is quite different from that of a column subjected to a concentrated periodic loading applied on the axis of the column in the tangential direction or more general in the direction of varying tangency coefficient. In the latter case, the dynamic equation contains parametrically excited terms associated with modal displacement $V_m(t)$ only [14]. The last three terms of equation (16) come out to be the effects of the acceleration of the point on the column which instantaneously coincides with mass m_0, the Coriolis component of acceleration of mass m_0, and the acceleration of mass m_0 relative to the point on the column which instantaneously coincides with mass m_0, respectively.

3. DETERMINATION OF INSTABILITY REGIONS

The instability regions of combination and simple resonances of a general form of Mathieu's equation, equation (16), can be determined as follows:

(a) *Combination resonances of sum type*: From the nth and mth component equations of equation (16), we have

$$\ddot{V}_n + \omega_n^2 V_n + 2\varepsilon\mu_n \dot{V}_n + 2\varepsilon \cos\omega t \sum_k h_{nk} \ddot{V}_k + 2\varepsilon \sin\omega t \sum_k g_{nk} \dot{V}_k + 2\varepsilon \cos\omega t \sum_k f_{nk} V_k = 0, \tag{18}$$

$$\ddot{V}_m + \omega_m^2 V_m + 2\varepsilon\mu_m \dot{V}_m + 2\varepsilon\cos\omega t \sum_k h_{mk}\ddot{V}_k + 2\varepsilon\sin\omega t \sum_k g_{mk}\dot{V}_k + 2\varepsilon\cos\omega t \sum_k f_{mk} V_k = 0. \tag{19}$$

When the excitation frequency ω is near $\omega_n + \omega_m$ and the system is just on the boundary between stable and unstable regions, $V_n(t)$ and $V_m(t)$ have steady oscillations. From equation (18), if one frequency component that consists of $V_n(t)$ and away from ω_n possesses the amplitude of order ε^0, $\ddot{V}_n(t) + \omega_n^2 V_n(t)$ in equation (18) will produce an unbalanced dynamic force of order ε^0. Therefore, only the frequency component that consists of $V_n(t)$ and near ω_n can possess the amplitude of order ε^0. Similarly, from equation (19), only the frequency component which consists of $V_m(t)$ and near ω_m can possess the amplitude of order ε^0 such that when the system is just on the boundary between stable and unstable regions, we get

$$V_n(t) = \bar{V}_n \cos(\omega_n^* t + \theta_n) + \varepsilon(\text{frequencycy components without } \omega_n^*) + \varepsilon^2 \ldots,$$

$$V_m(t) = \bar{V}_m \cos(\omega_m^* t + \theta_m) + \varepsilon(\text{frequency components without } \omega_n^*) + \varepsilon^2 \ldots, \tag{20}$$

where $\omega_n^* \equiv \omega_n + \varepsilon\sigma_n$ and $\omega_m^* \equiv \omega_m + \varepsilon\sigma_m$ are named the principal frequencies of $V_n(t)$ and $V_m(t)$ respectively. The principal frequencies float from the natural frequencies ω_n and ω_m with small magnitudes $\varepsilon\sigma_n$ and $\varepsilon\sigma_m$. $\bar{V}_n$ and $\bar{V}_m$ are the corresponding principal amplitudes of $V_n(t)$ and $V_m(t)$ respectively. Substituting equation (20) into equation (18), we have

$$-2\varepsilon\sigma_n\omega_n\bar{V}_n\cos(\omega_n^* t + \theta_n) - 2\varepsilon\mu_n\omega_n\bar{V}_n\sin(\omega_n^*\mathrm{t} + \theta_n) + 2\varepsilon\cos\omega t \sum_k h_{nk}\ddot{V}_k$$

$$+ 2\varepsilon\sin\omega t \sum_k g_{nk}\dot{V}_k + 2\varepsilon\cos\omega t \sum_k f_{nk} V_k = \varepsilon(\text{frequency components}$$

$$\text{without } \omega_n^*) + \varepsilon^2 \ldots \,. \tag{21}$$

The first two terms on the left-hand side of the above equation are always unbalanced and so the last three terms on the left-hand side of the above equation have to contain the principal frequency ω_n^* with the amplitude of order ε to maintain self-equilibrium for the component of the principal frequency ω_n^*. To induce the terms with the principal frequency ω_n^* and the amplitude of order ε in the last three terms on the left-hand side of the above equation, it is necessary that the excitation frequency ω, near $\omega_n + \omega_m$, must be equal to $\omega_n^* + \omega_m^*$. That is

$$2\varepsilon\cos\omega t \sum_k h_{nk}\ddot{V}_k$$

$$= -\varepsilon\omega_m^2 h_{nm}\bar{V}_m\cos(\omega_n^* t - \theta_m) + \varepsilon(\text{frequency components without } \omega_n^*) + \varepsilon^2 \ldots,$$

$$2\varepsilon \sin \omega t \sum_k g_{nk} \dot{V}_k$$

$$= -\varepsilon\omega_n g_{nm} \bar{V}_m \cos(\omega_n^* t - \theta_m) + \varepsilon(\text{frequency components without } \omega_n^*) + \varepsilon^2 \dots ,$$

$$2\varepsilon \cos\omega t \sum_k f_{nk} V_k$$

$$= -\varepsilon f_{nm} \bar{V}_m \cos(\omega_n^* t - \theta_m) + \varepsilon(\text{frequency components without } \omega_n^*) + \varepsilon^2 \dots . \tag{22}$$

Substituting equation (22) into equation (21), we get the dynamic equilibrium of the principal frequency ω_n^* as

$$-2\varepsilon\sigma_n\omega_n \bar{V}_n \cos(\omega_n^* t + \theta_n) - 2\varepsilon\mu_n\omega_n \bar{V}_n \sin(\omega_n^* t + \theta_n) - \varepsilon\omega_m^2 h_{nm} \bar{V}_m \cos(\omega_n^* t - \theta_m)$$

$$-\varepsilon\omega_m g_{nm} \bar{V}_m \cos(\omega_n^* t - \theta_m) + \varepsilon f_{nm} \bar{V}_m \cos(\omega_n^* t - \theta_m) = 0. \tag{23}$$

The dynamic equilibrium diagram of the above equation can be plotted as shown in Figure 2(a). Following similar procedure, the dynamic equilibrium diagram of the principal frequency ω_m^* can be plotted as shown in Figure 2b. The dynamic forces in the dynamic equilibrium diagrams constitute two similar closed-loop right triangles. From Figures 2(a, b), we found that

(i) The phase difference, $\theta = \theta_n + \theta_m$, between the parametrically excited force, $(f_{nm} - \omega_m g_{nm} - \omega_m^2 h_{nm})\bar{V}_m \cos(\omega_n^* t - \theta_m)$ or $(f_{mn} - \omega_n g_{mn} - \omega_n^2 h_{mn})\bar{V}_n \cos(\omega_m^* t - \theta_n)$ and the principal frequency response, $\bar{V}_n \cos(\omega_n^* t - \theta_n)$ or $\bar{V}_m \cos(\omega_m^* t + \theta_m)$, satisfies the following relationship:

$$\tan\theta = \tan(\theta_n + \theta_m) = \frac{\mu_n}{\sigma_n} = \frac{\mu_m}{\sigma_m}. \tag{24}$$

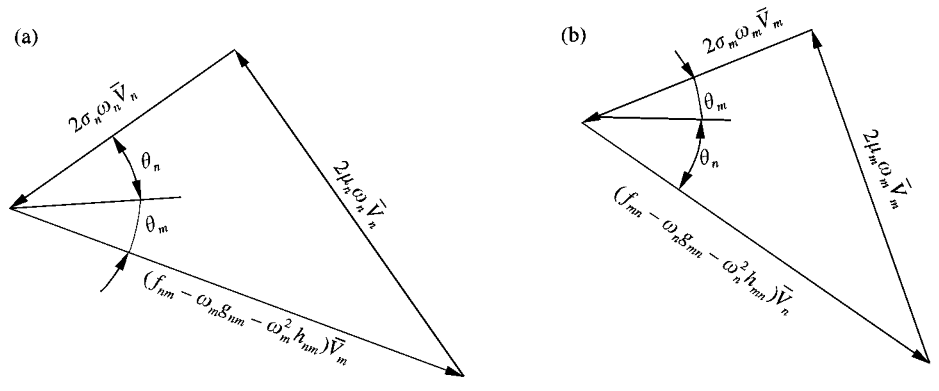

Figure 2. The dynamic equilibrium diagram of the principal frequencies ω_n^* and ω_m^*.

(ii) The principal amplitude ratio between $\bar{V}_n$ and $\bar{V}_m$ is

$$\frac{\bar{V}_n}{\bar{V}_m} = \left\{ \left(\frac{\mu_m \omega_m}{\mu_n \omega_n} \right) \left(\frac{f_{nm} - \omega_m g_{nm} - \omega_m^2 h_{nm}}{f_{mn} - \omega_n g_{mn} - \omega_n^2 h_{mn}} \right) \right\}^{1/2}. \tag{25}$$

(iii) The frequency floating parameters σ_n and σ_m satisfy the following relationship:

$$4(\sigma_n \sigma_m + \mu_n \mu_m) \omega_n \omega_m = (f_{nm} - \omega_m g_{nm} - \omega_m^2 h_{nm})(f_{mn} - \omega_n g_{mn} - \omega_n^2 h_{mn}). \tag{26}$$

Substituting equation (24) into equation (26), we obtain

$$\sigma_n = \pm \frac{\mu_n}{\mu_n + \mu_m} G_{nm}^{1/2},$$

$$\sigma_m = \pm \frac{\mu_m}{\mu_n + \mu_m} G_{nm}^{1/2},$$

$$\sigma = \sigma_n + \sigma_m = \pm G_{nm}^{1/2}, \tag{27}$$

where the instability bandwidth parameter G_{nm} is

$$G_{nm} = \frac{(\mu_n + \mu_m)^2}{\mu_n \mu_m} \left[\frac{(f_{nm} - \omega_m g_{nm} - \omega_m^2 h_{nm})(f_{mn} - \omega_n g_{mn} - \omega_n^2 h_{mn})}{4\omega_n \omega_m} - \mu_n \mu_m \right]. \tag{28}$$

Equations (24)–(27) are available when G_{nm} is positive. The transition curves separating stable and unstable regions are

$$\omega = \omega_n^* + \omega_m^* = \omega_n + \omega_m \pm \varepsilon G_{nm}^{1/2} \quad \text{when } G_{nm} > 0. \tag{29}$$

(b) *Simple resonances*: When the excitation frequency ω is near $2\omega_n$, letting $n = m$ in equation (28), we have

$$G_{nn} = \left(\frac{f_{nn} - \omega_n g_{nn} - \omega_n^2 h_{nn}}{\omega_n} \right)^2 - 4\mu_n^2. \tag{30}$$

The transition curves separating stable and unstable regions are

$$\omega = 2\omega_n \pm \varepsilon G_{nn}^{1/2} \quad \text{when } G_{nn} > 0. \tag{31}$$

(c) *Combination resonances of difference type*: When the excitation frequency ω is near $\omega_n - \omega_m$ ($\omega_n > \omega_m$), the instability bandwidth parameter G_{nm} can be obtained by changing the sign of ω_m in equation (28) as

$$G_{nm} = \frac{(\mu_n + \mu_m)^2}{\mu_n \mu_m} \left[\frac{(f_{nm} + \omega_m g_{nm} - \omega_m^2 h_{nm})(f_{mn} - \omega_n g_{mn} - \omega_n^2 h_{mn})}{-4\omega_n \omega_m} - \mu_n \mu_m \right], \tag{32}$$

when $G_{nm} > 0$ the transition curves are

$$\omega = \omega_n - \omega_m \pm \varepsilon G_{nm}^{1/2}. \tag{33}$$

The frequency floating parameters are

$$\sigma_n = \pm \frac{\mu_n}{\mu_n + \mu_m} G_{nm}^{1/2},$$

$$\sigma_m = \pm \frac{\mu_m}{\mu_n + \mu_m} G_{nm}^{1/2},$$

$$\sigma = \sigma_n + \sigma_m = \pm G_{nm}^{1/2}. \tag{34}$$

The principal amplitude ratio is

$$\frac{\bar{V}_n}{\bar{V}_m} = \left\{ \left(-\frac{\mu_m \omega_m}{\mu_n \omega_n} \right) \left(\frac{f_{nm} + \omega_m g_{nm} - \omega_m^2 h_{nm}}{f_{mn} - \omega_n g_{mn} - \omega_n^2 h_{mn}} \right) \right\}^{1/2} \tag{35}$$

and the phase difference θ is the same as equation (24).

In the absence of damping, i.e. $\mu_n = 0$, $n = 1, 2, \ldots,$ the instability regions of simple and combination resonances can be obtained as follows:

(a) *Combination resonances of sum type*: When the excitation frequency ω is near $\omega_n + \omega_m$ and the system oscillates steadily, from the dynamic equilibrium diagrams of Figures 2(a, b), we have

$$-2\sigma_n \omega_n \bar{V}_n + (f_{nm} - \omega_m g_{nm} - \omega_m^2 h_{nm}) \bar{V}_m = 0, \tag{36}$$

$$(f_{mn} - \omega_n g_{mn} - \omega_n^2 h_{mn}) \bar{V}_n - 2\sigma_m \omega_m \bar{V}_m = 0. \tag{37}$$

Combining the above two equations, we obtain

$$\sigma^2 = (\sigma_n + \sigma_m)^2 \geqslant 4\sigma_n \sigma_m = G_{nm}, \tag{38}$$

where the instability bandwidth parameter G_{nm} is

$$G_{nm} = \left[\frac{(f_{nm} - \omega_m g_{nm} - \omega_m^2 h_{nm})(f_{mn} - \omega_n g_{mn} - \omega_n^2 h_{mn})}{\omega_n \omega_m} \right]. \tag{39}$$

The total frequency floating parameter σ in equation (38) is solvable when G_{nm} is positive, and the transition curves separating neutrally stable and unstable regions are

$$\omega = \omega_n + \omega_m \pm \varepsilon G_{nm}^{1/2}. \tag{40}$$

(b) *Simple resonances*: When the excitation frequency ω is near $2\omega_n$, the transition curves are

$$\omega = 2\omega_n \pm \varepsilon G_{nn}^{1/2}, \tag{41}$$

where the instability bandwidth parameter G_{nn} is

$$G_{nn} = \left(\frac{f_{nn} - \omega_n g_{nn} - \omega_n^2 h_{nn}}{\omega_n}\right)^2. \tag{42}$$

(c) *Combination resonances of difference type*: When the excitation frequency ω is near $\omega_n - \omega_m$ ($\omega_n > \omega_m$), the instability bandwidth parameter G_{nm} can be obtained by changing the sign of ω_m in equation (39) as

$$G_{nm} = \left[\frac{(f_{nm} + \omega_m g_{nm} - \omega_m^2 h_{nm})\,(f_{mn} - \omega_n g_{mn} - \omega_n^2 h_{mn})}{-\,\omega_n\omega_m}\right]. \tag{43}$$

Then, when $G_{nm} > 0$ the transition curves are

$$\omega = \omega_n - \omega_m \pm \varepsilon G_{nm}^{1/2}. \tag{44}$$

It is noted that

(i) When $h_{nm} = g_{nm} = 0$, n, m $= 1, 2, \ldots,$ the transition curves obtained in equations (28)–(33) are the same as those obtained by Nayfeh and Mook [2].
(ii) When $h_{nm} = g_{nm} = 0, n, m = 1, 2, \ldots,$ and $\mu_n = 0, n = 1, 2, \ldots,$ the transition curves obtained in equations (39)–(44) are the same as those obtained by Hsu [5] and Nayfeh and Mook [7].

Consider the problem shown in Figure 1. When the excitation frequency ω is near $\omega_n - \omega_m(\omega_n > \omega_m)$, from equation (17) and substituting ω by $(\omega_n - \omega_m)$, we have

$$f_{nm} + \omega_m g_{nm} - \omega_m^2 h_{nm} = f_{mn} - \omega_n g_{mn} - \omega_n^2 h_{mn}$$
$$= \tfrac{1}{2} m_0 (\omega_n - \omega_m)^2 \int_{x_0}^{x_b} \phi_n'(\xi)\phi_m'(\xi)\,d\xi - \tfrac{1}{2} m_0 [\omega_n^2 \phi_n(x_b)\phi_m'(x_b) + \omega_m^2 \phi_n'(x_b)\phi_m(x_b)]. \tag{45}$$

Substituting equation (45) into equations (32) and (43), G_{nm} is found to be always less or equal to zero, which implies that no combination resonances of difference type may occur in the column system, as shown in Figure 1. This problem is quite different from that of a column subjected to a periodic loading in the tangential direction or in the direction of a varying tangency coefficient, in which the combination resonances of difference type may occur in addition to simple resonances and combination resonances of sum type.

The transition curves of simple resonances and combination resonances of sum type can be summarized as follows:

(a) The transition curves of simple resonance

$$\omega = 2\omega_n \pm \varepsilon G_{nn}^{1/2} \quad \text{when } G_{nn} > 0, \tag{46}$$

where

$$G_{nn} = \begin{cases} \left[m_0\omega_n \left(2\int_{x_0}^{x_b} \phi_n'^2(\xi)\,\mathrm{d}\xi - \phi_n(x_b)\phi_n'(x_b) \right) \right]^2 - 4\mu_n^2, & \mu_n \neq 0, \\ \left[m_0\omega_n \left(2\int_{x_0}^{x_b} \phi_n'^2(\xi)\,\mathrm{d}\xi - \phi_n(x_b)\phi_n'(x_b) \right) \right]^2, & \mu_n = 0. \end{cases} \tag{47}$$

(b) The transition curves of combination resonance of sum type

$$\omega = \omega_n + \omega_m \pm \varepsilon G_{nm}^{1/2} \quad \text{when } G_{nm} > 0, \tag{48}$$

where

$$G_{nm} = \frac{(\mu_n + \mu_m)^2}{\mu_n\mu_m} \times \left\{ \frac{m_0^2\left[(\omega_n + \omega_m)^2 \int_{x_0}^{x_b} \phi_n'(\xi)\phi_m'(\xi)\,\mathrm{d}\xi - \omega_n^2\phi_n(x_b)\phi_m'(x_b) - \omega_m^2\phi_n'(x_b)\phi_m(x_b) \right]^2}{16\omega_n\omega_m} - \mu_n\mu_m \right\} \tag{49}$$

for the case with damping; and

$$G_{nm} = \frac{m_0^2\left[(\omega_n + \omega_m)^2 \int_{x_0}^{x_b} \phi_n'(\xi)\phi_m'(\xi)\,\mathrm{d}\xi - \omega_n^2\phi_n(x_b)\phi_m'(x_b) - \omega_m^2\phi_n'(x_b)\phi_m(x_b) \right]^2}{4\omega_n\omega_m} \tag{50}$$

for the case without damping.

4. EXAMPLES AND DISCUSSION

From equations (46)–(50) the transition curves of simple and combination resonances of a system, as shown in Figure 1, can be completely determined by the following parameters:

ε the excitation amplitude of the concentrated mass
ω_n the modal natural frequencies of the system when the concentrated mass is kept fixed on the column
d_n modal damping coefficients of the system when the concentrated mass is kept fixed on the column, $d_n = 2\varepsilon\mu_n$
ϕ_n normalized mode shape functions of the system when the concentrated mass is kept fixed on the column
m_0 the concentrated mass
x_0 x-co-ordinate of the fixed supporting point
x_b x-co-ordinate of the center point of the moving concentrated mass before the column undergoes deformation

The natural frequencies ω_n, the corresponding modal damping coefficients d_n and normalized mode shape functions $\phi_n(x)$ can be obtained by using an experimental, numerical, or analytic method. Once ω_n, d_n and $\phi_n(x)$ are known, the transition curves of simple and combination resonances can be obtained.

In the following, the method developed above is demonstrated to obtain the instability bandwidth parameters G_{nm} of an undamped system of cantilevered, simply supported, clamped–simple supported and clamped–clamped columns carrying a concentrated mass which undergoes a sinusoidal motion along the axis of the column with frequency ω and small vibrating amplitude ε. Once the instability bandwidth parameters G_{nm} are known, the transition curves separating stable and unstable regions can be obtained by substituting the natural frequency ω_n and the vibrating amplitude ε into equations (46) and (48). For simplicity, the column in each case was chosen to have uniform cross-section with length L, mass per unit length ρ and bending rigidity EI. The configuration of each system is shown in Figures 3–6 respectively. The natural frequencies ω_n and corresponding normalized mode-shape functions $\phi_n(x)$ of the free transverse vibration of each column with concentrated mass m_0 fixed at an arbitrary point on the column was

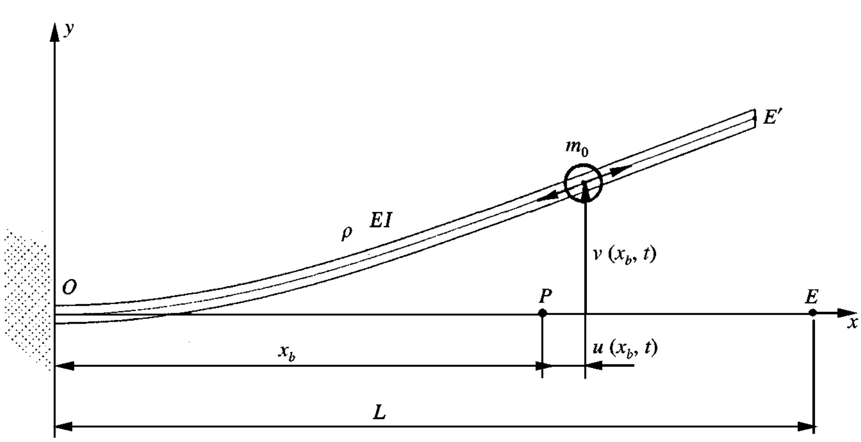

Figure 3. The dynamic system of a cantilevered column carrying an axially oscillating mass.

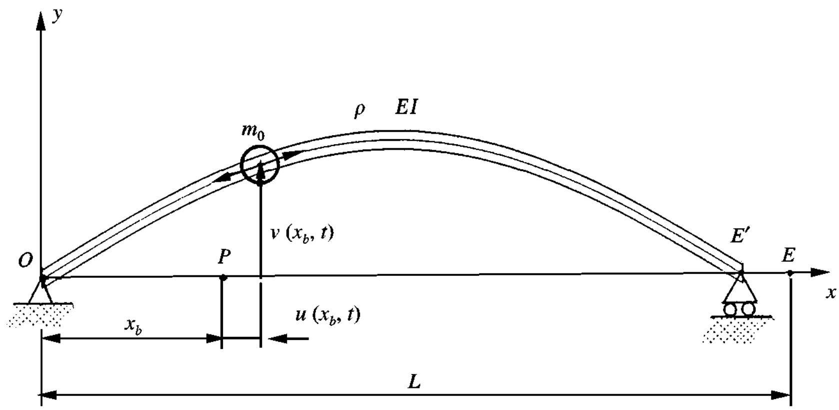

Figure 4. The dynamic system of a simply-supported column carrying an axially oscillating mass.

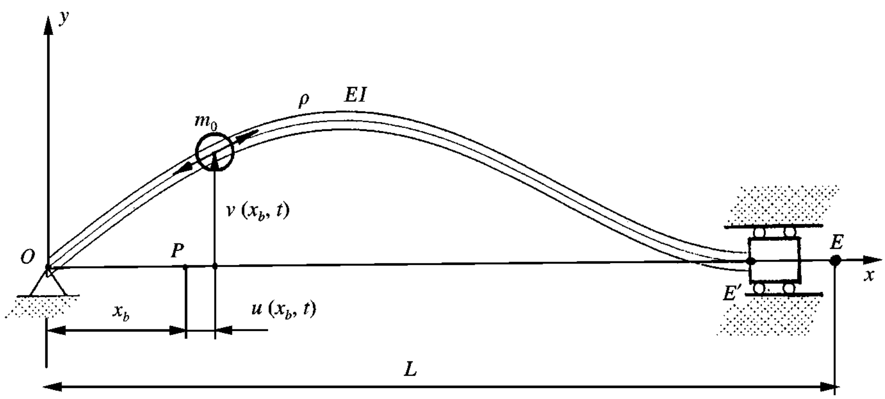

Figure 5. The dynamic system of a clamped-simply supported column carrying an axially oscillating mass.

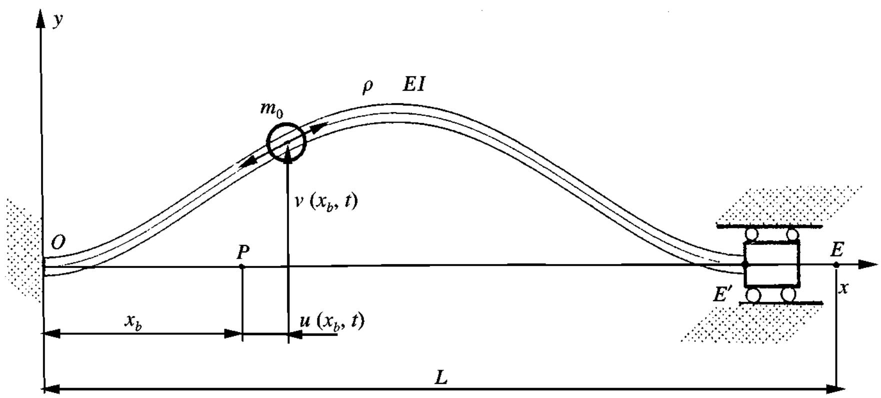

Figure 6. The dynamic system of a clamped-clamped column carrying an Axially oscillating mass.

calculated by the assumed-modes method. The first four natural frequencies ω_n of each column with various concentrated mass fixed at varying positions on the column were shown in Figures 7–10 respectively. The abscissa of each plot represents the center position of the oscillating concentrated mass and the ordinate of each plot represents the natural frequency ω_n. The instability bandwidth parameters G_{nm}, n, $m = 1, 2, 3, 4$, of each column carrying various concentrated masses which undergo sinusoidal motion along the axis of the column and centered at varying positions, calculated from equations (47) and (50), are shown in Figures 11–14 respectively. The abscissa of each represents the center position of the oscillating concentrated mass and the ordinate of each plot represents the instability bandwidth parameters.

5. CONCLUSIONS

The parametrically excited instability behavior of a general column carrying a concentrated moving mass that undergoes a small sinusoidal motion along the axis of the column has been investigated analytically. The following conclusions can be drawn.

(1) The governing equation of the system is a Mathieu's equation with multiple degrees of freedom that contains parametrically excited terms associated with modal accelerations, modal velocities, and modal displacements. The governing

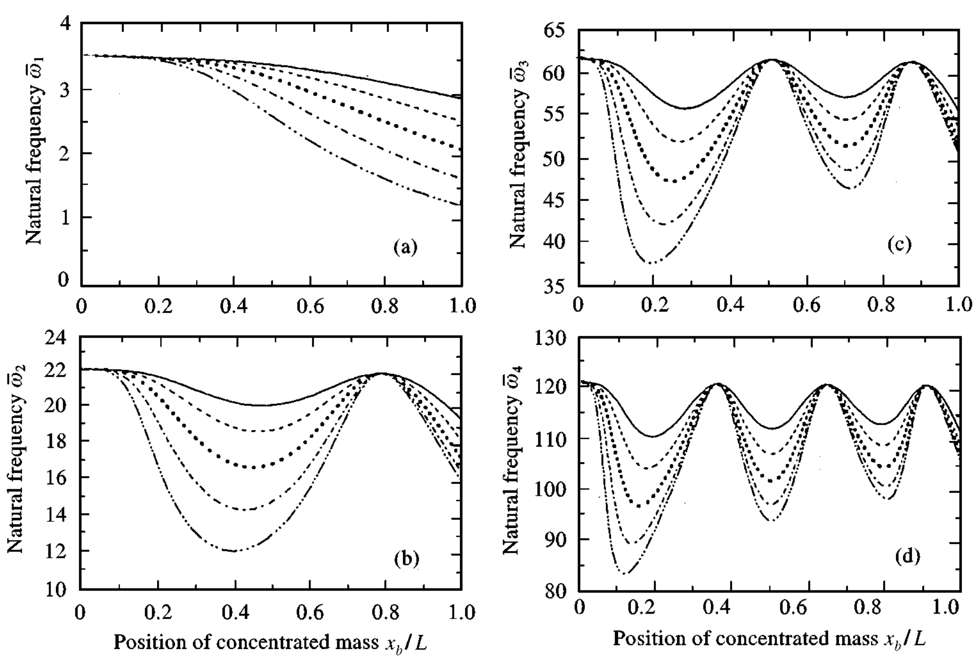

Figure 7. The first four natural frequencies of cantilevered column carrying various mass fixed on the column. ($\bar{\omega}_n = \omega_n L^2 \rho^{1/2} (\mathrm{EI})^{-1/2}$; $\bar{m}_0 = m_0(\rho L)^{-1}$). —$\bar{m}_0 = 0.1$,- - - - -$\bar{m}_0 = 0.2$, ... $\bar{m}_0 = 0.4$, -·-·-· $\bar{m}_0 = 0.8$, —· · —$\bar{m}_0 = 1.6$

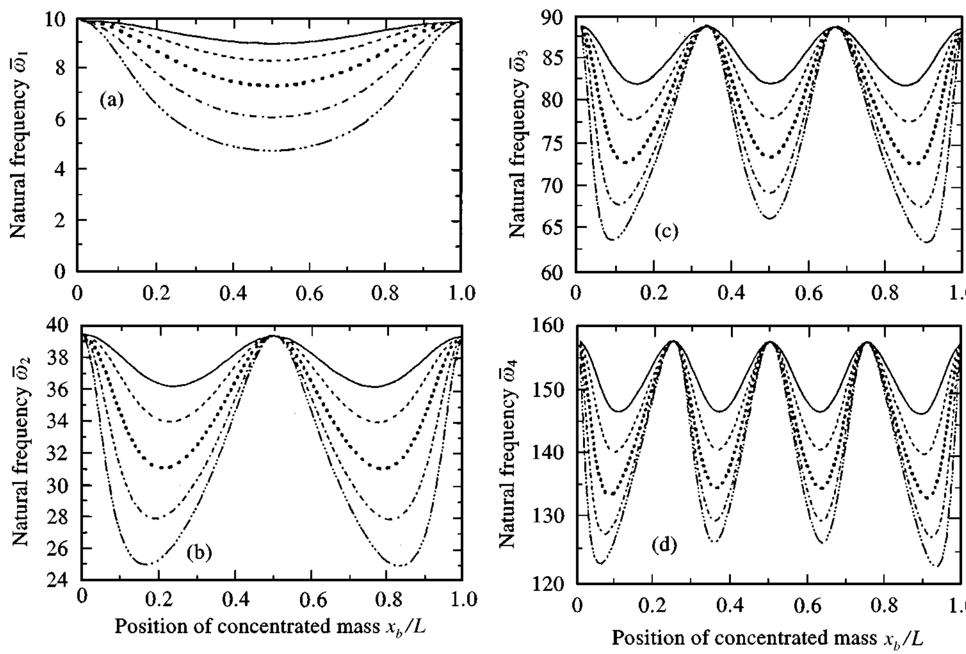

Figure 8. The first four natural frequencies of simply-supported column carrying various mass fixed on the Column. ($\bar{\omega}_n = \omega_n L^2 \rho^{1/2}(\mathrm{EI})^{-1/2}$; $\bar{m}_0 = m_0(\rho L)^{-1}$). —$\bar{m}_0 = 0.1$,- - - - -$\bar{m}_0 = 0.2$, ... $\bar{m}_0 = 0.4$, -·-·-· $\bar{m}_0 = 0.8$, —··—$\bar{m}_0 = 1.6$

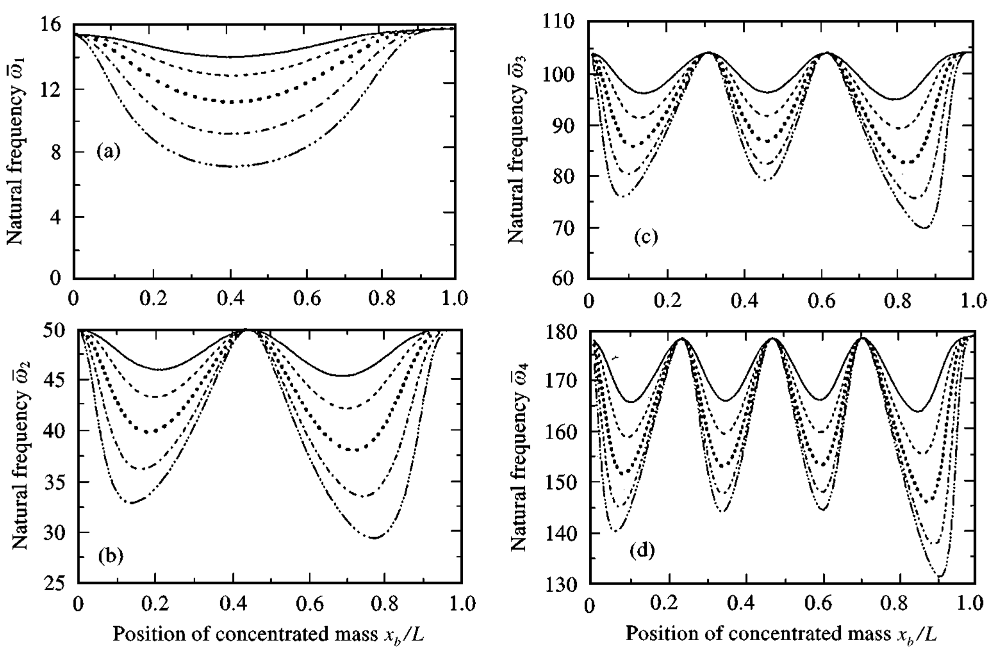

Figure 9. The first four natural frequencies of clamped-simply supported column carrying various mass fixed on the Column. ($\bar{\omega}_n = \omega_n L^2 \rho^{1/2}(\mathrm{EI})^{-1/2}$; $\bar{m}_0 = m_0(\rho L)^{-1}$). —$\bar{m}_0 = 0.1$,- - - - -$\bar{m}_0 = 0.2$, ... $\bar{m}_0 = 0.4$, -·-·-· $\bar{m}_0 = 0.8$, —··—$\bar{m}_0 = 1.6$

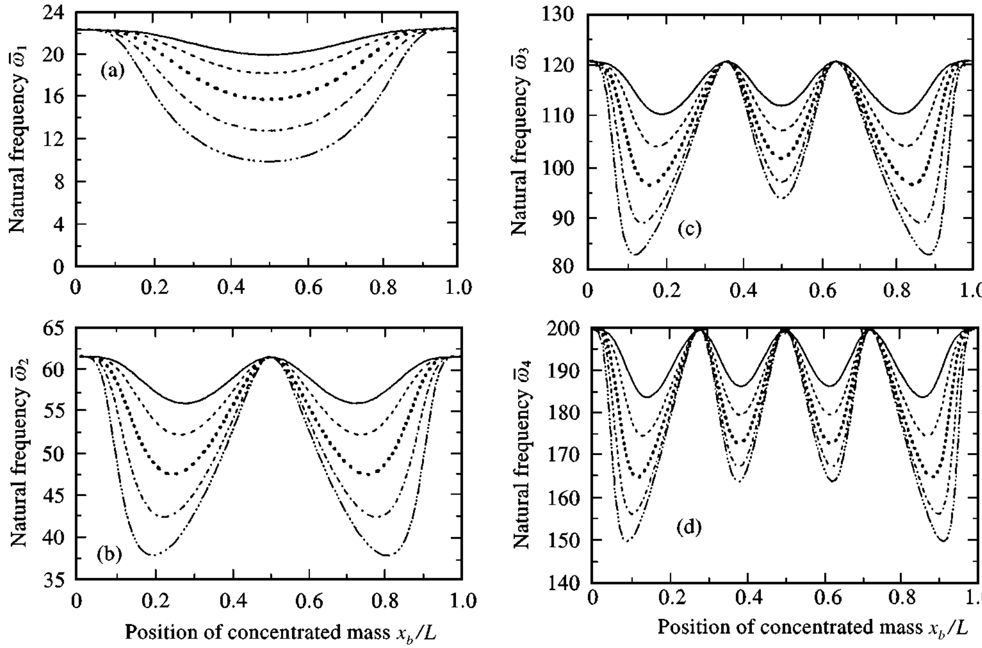

Figure 10. The first four natural frequencies of clamped-clamped column carrying various mass fixed on the Column. ($\bar{\omega}_n = \omega_n L^2 \rho^{1/2}(\mathrm{EI})^{-1/2}$; $\bar{m}_0 = m_0(\rho L)^{-1}$). —$\bar{m}_0 = 0.1$,- - - -$\bar{m}_0 = 0.2$, ... $\bar{m}_0 = 0.4$, ------ $\bar{m}_0 = 0.8$, —··—$\bar{m}_0 = 1.6$

equation is different from that of a column subjected to a concentrated periodic loading applied on the column in the direction of the tangency coefficient. In the latter case, the governing equation contains parametrically excited terms associated with modal displacements only.

(2) The approach developed in this study determines the instability regions of a general form of the Mathieu's equation systematically. In the evaluation of the instability regions, only the oscillating amplitude, the center position and the value of the concentrated mass, the position of the fixed supporting point, and the modal natural frequencies, the modal damping, and the normalized mode-shape functions of the system with the concentrated mass fixed on the column are necessary.

(3) The method can solve the principal vibration frequencies, the ratio of principal amplitudes and phase difference between the parametrically excited forces and principal frequency responses of a damping system on the transition state as well as the transition curves.

(4) When the parametrically excited terms associated with modal accelerations and modal velocities of the general Mathieu's equation vanish, the instability regions obtained by our approach are the same as those obtained by Hsu [5] and Nayfeh and Mook [2,7].

(5) Only the simple resonances and the combination resonances of sum type occur in the general system, as shown in Figure 1. This is regardless of the

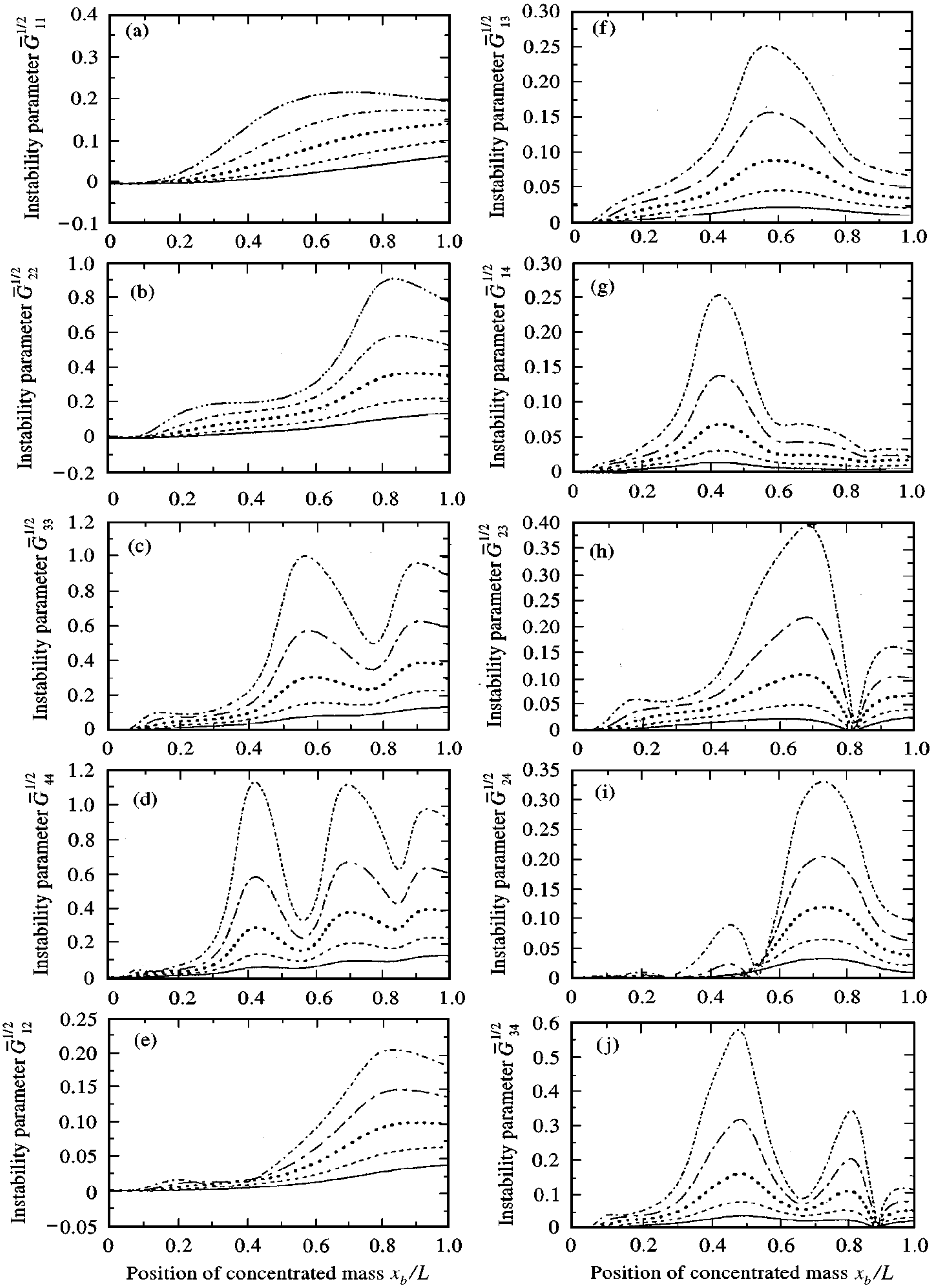

Figure 11. The instability bandwidth parameters G_{nm} of cantilevered column carrying various mass fixed on the column. ($\bar{G}_{nm}^{1/2} = G_{nm}^{1/2} L\omega_1(\omega_n + \omega_m)^{-2}$; $\bar{m}_0 = m_0(\rho L)^{-1}$). —$\bar{m}_0 = 0.1$,- - - - -$\bar{m}_0 = 0.2$, ... $\bar{m}_0 = 0.4$, -·-·-· $\bar{m}_0 = 0.8$, —· · —$\bar{m}_0 = 1.6$

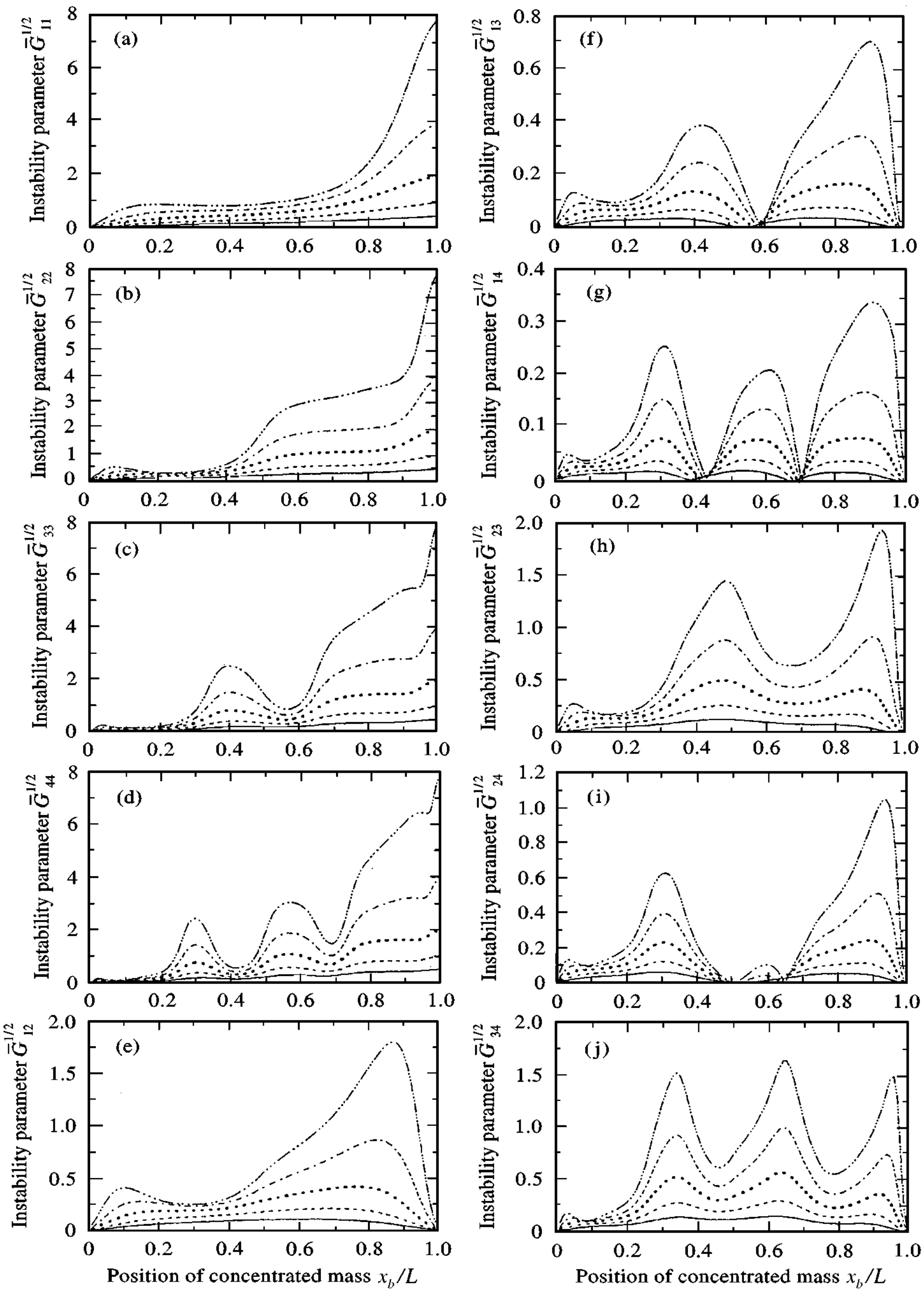

Figure 12. The instability bandwidth parameters G_{nm} of simply-supported column carrying various mass fixed on the column. ($\bar{G}_{nm}^{1/2} = G_{nm}^{1/2} L\omega_1(\omega_n + \omega_m)^{-2}$; $\bar{m}_0 = m_0(\rho L)^{-1}$). —$\bar{m}_0 = 0.1$,- - - - -$\bar{m}_0 = 0.2$, ... $\bar{m}_0 = 0.4$, -·-·- $\bar{m}_0 = 0.8$, —· · —$\bar{m}_0 = 1.6$

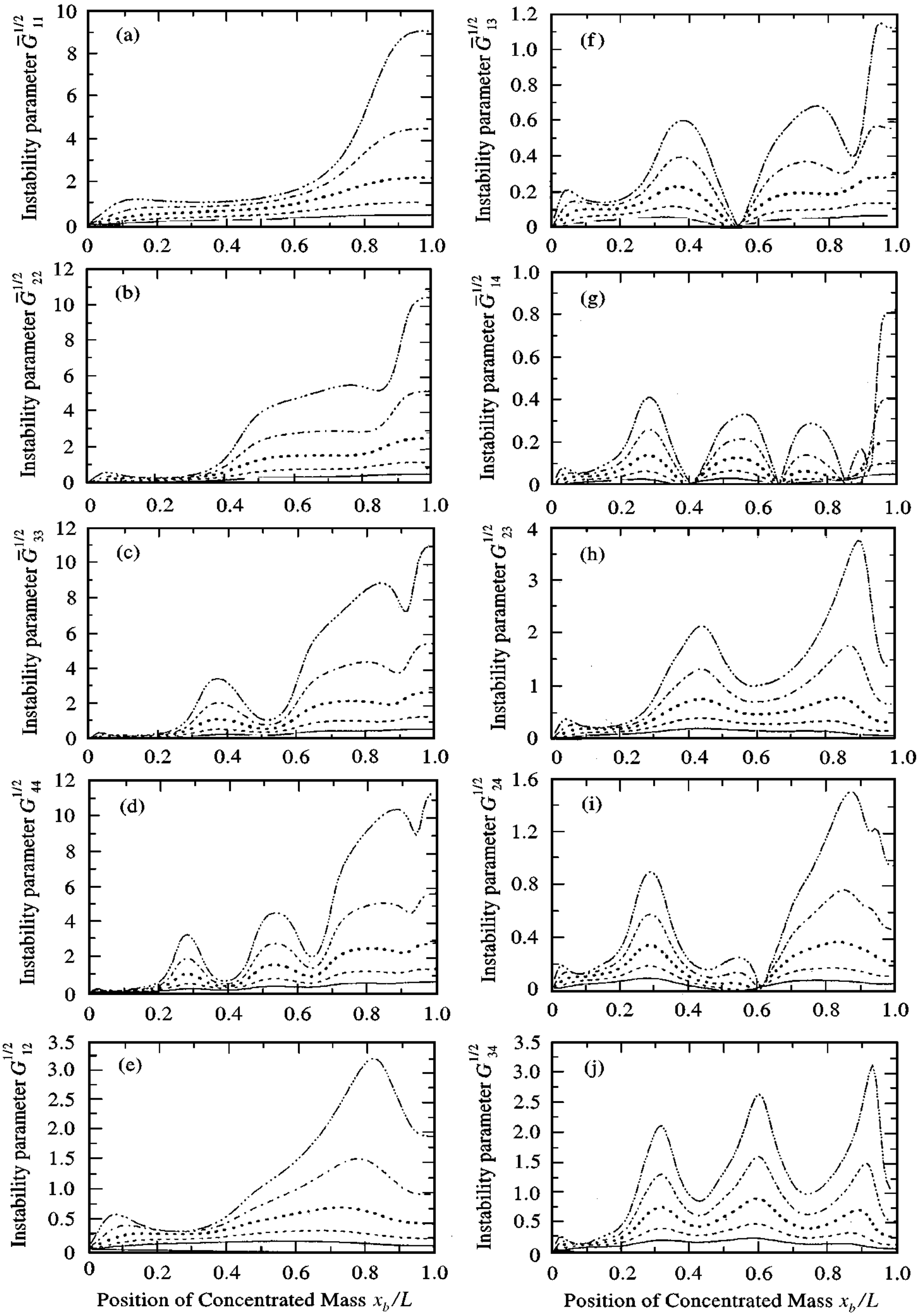

Figure 13. The instability bandwidth parameters G_{nm} of clamped-simply supported column carrying various mass fixed on the column. ($\bar{G}_{nm}^{1/2} = G_{nm}^{1/2} L\omega_1(\omega_n + \omega_m)^{-2}$; $\bar{m}_0 = m_0(\rho L)^{-1}$). —$\bar{m}_0 = 0.1$,- - - - $\bar{m}_0 = 0.2$, ... $\bar{m}_0 = 0.4$, -·-·- $\bar{m}_0 = 0.8$, —··—$\bar{m}_0 = 1.6$

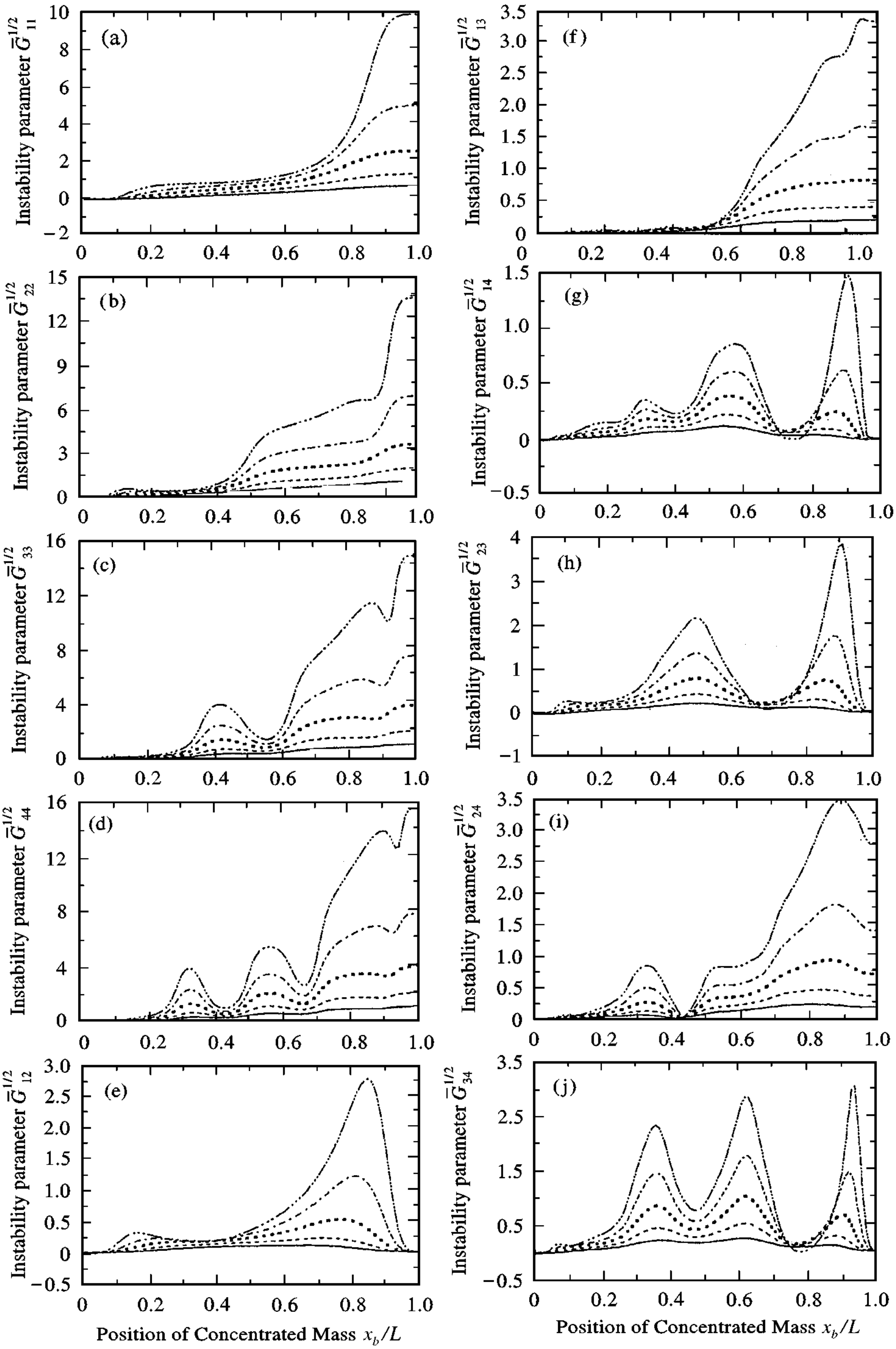

Figure 14. The instability bandwidth parameters G_{nm} of clamped-clamped column carrying various mass fixed on the column. ($\bar{G}_{nm}^{1/2} = G_{nm}^{1/2} L\omega_1(\omega_n + \omega_m)^{-2}$; $\bar{m}_0 = m_0(\rho L)^{-1}$). —$\bar{m}_0 = 0.1$,----- $\bar{m}_0 = 0.2$, ... $\bar{m}_0 = 0.4$, -·-·- $\bar{m}_0 = 0.8$, —··—$\bar{m}_0 = 1.6$

distribution of mass and the bending rigidity of the column, the supporting condition of the column, and the value and central position of the oscillating concentrated mass.

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