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THE CONSERVATION LAWS AND THE EXACT SOLUTIONS FOR THE SINGULAR NON-LINEAR OSCILLATOR

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1. INTRODUCTION

The problem of non-linear singular oscillator systems is considered in references [1–5]. In reference [1] systems with "soft" and "hard" singularity are analyzed. For a special type of non-linear equation with soft singularity the exact solution which can be written in closed form is shown. The non-linear equation

$$\ddot{x} + x = -\lambda x(\dot{x}^2 + x^2)/(1 - x^2), \tag{1}$$

which contains a hard singularity is also discussed. It is concluded that the exact solution of this equation cannot be expressed in simple, closed form. In references [1–5] the approximate solutions of (1) for small non-linearity when $\lambda \ll 1$ are obtained by applying the slowly varying amplitude and phase method and the harmonic balance method. The case of strong non-linearity is not considered.

In this paper the strong non-linear system (1) with $\lambda \geqslant 1$ is analyzed. It is a pure non-conservative system. In this paper the conservation laws for such non-conservative systems are obtained. Noether's theorem adopted for non-conservative systems is applied. Besides using the first integrals of the equation (1) the exact solutions for the strong non-linear system with hard singularity of the type (1) are determined.

2. CONSERVATION LAW

The dynamical system being considered (1) is characterized by the Lagrangian function

$$L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2\dot{x}^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 - (\lambda/4)x^4, \tag{2}$$

and a generalized force

$$Q = -(1 + \lambda)x\dot{x}^2, \tag{3}$$

where x is a generalized co-ordinate and $\dot{x}$ is a generalized velocity. The system is a purely non-conservative holonomic dynamical system. In reference [6] it is shown that in non-conservative systems conservation laws may exist. Noether's

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theory used for obtaining conservation laws for conservative systems [7] (to every infinitesimal transformation of the dynamical variables that leaves the action integral invariant there corresponds a conserved quantity) is generalized for non-conservative systems. The theory is based on the variational principle of Hamilton's type for purely non-conservative systems [8]. The main idea of the principle is that the variations of the generalized velocity $\dot{x}$ are not completely determined by the variations of the generalized co-ordinates x but by these quantities and by the generalized dissipative force Q. Following this idea it is assumed that the variations of time and generalized co-ordinates are independent and that these transformations together with the dissipative force determine the infinitesimal transformation of the generalized velocity. It means that for a continuous one-parameter transformation of time, the generalized co-ordinate and generalized velocity

$$\bar{t} \approx t + \varepsilon f(t, x, \dot{x}), \quad \bar{x} \approx x + \varepsilon F(t, x, \dot{x}), \quad \dot{\bar{x}} \approx \dot{x} + \varepsilon(\dot{F} - \dot{x}\dot{f} + \Phi), \tag{4}$$

there exists an infinitesimal transformation of the form

$$\Delta t \approx \varepsilon f, \quad \Delta x \approx \varepsilon F, \quad \Delta \dot{x} \approx \varepsilon(\dot{F} - \dot{x}\dot{f} + \Phi), \tag{5}$$

where ε is a small parameter of the transformation and f, F and Φ are functions of time, generalized co-ordinates and generalized velocities. Further, by assuming that the infinitesimal transformation (5) induces a Lagrangian function L that is gauge invariant, i.e., is invariant up to an exact differential in the sense

$$L(\bar{x}, \dot{\bar{x}}, \bar{t})\,\mathrm{d}\bar{t} - L(x, \dot{x}, t)\,\mathrm{d}t = \varepsilon\,\mathrm{d}P(x, \dot{x}, t), \tag{6}$$

then combining (2), (4) and (6) developing the term $L(\bar{x}, \dot{\bar{x}}, \bar{t})$ in series and retaining only members linear in the small parameter ε becomes

$$\begin{aligned} &\varepsilon\Big[F(-x\dot{x}^2 - x + x^3 - \lambda x^3) + \dot{x}(1 - x^2)(\dot{F} - \dot{x}\dot{f} + \Phi) \\ &\qquad + \dot{f}\left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2\dot{x}^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 - \frac{\lambda}{4}x^4\right)\Big]\,\mathrm{d}t = \varepsilon\,\mathrm{d}P(x, \dot{x}, t), \end{aligned} \tag{7}$$

where P is a known function of x, $\dot{x}$ and t.

After simple manipulation, one has

$$\begin{aligned} \varepsilon\Big\{ &\frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{x}(1 - x^2)(F - \dot{x}f) + f\left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2\dot{x}^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 - \frac{\lambda}{4}x^4\right)\right] \\ &+ \Phi\dot{x}(1 - x^2) + (1 + \lambda)x\dot{x}^2(F - \dot{x}f) \\ &- [\ddot{x}(1 - x^2) + x(1 - x^2) + \lambda x(\dot{x}^2 + x^2)](F - \dot{x}f)\Big\}\,\mathrm{d}t = \varepsilon\,\mathrm{d}P. \end{aligned} \tag{8}$$

Assuming that the function f, F and Φ satisfy the algebraic equation

$$\Phi\dot{x}(1 - x^2) = -(1 + \lambda)x\dot{x}^2(F - \dot{x}f), \tag{9}$$

and that the dynamical system moves in agreement with the Euler–Lagrange equation (1) one may deduce the following theorem:

Theorem 1 (*Noether's theorem*): *If under the continuous infinitesimal one-parameter transformation* (5) *which satisfies the algebraic equation* (9) *the Lagrangian is gauge invariant in the sense of the equation* (6) *then the quantity*

$$D(x,\dot{x},t)=\dot{x}(1-x^2)(F-\dot{x}f)+f\left(\tfrac{1}{2}\dot{x}^2-\tfrac{1}{2}x^2\dot{x}^2-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4-(\lambda/4)x^4\right)-P, \quad (10)$$

is constant for the non-conservative system (1).

2.1. *Killing equations*

The functions f, F and P have to satisfy the relation (7) with (9) which is called Noether's identity

$$F(-x\dot{x}^2-x+x^3-\lambda x^3)+\dot{x}(1-x^2)(\dot{F}-\dot{x}\dot{f})-(1+\lambda)x\dot{x}^2(F-\dot{x}f)$$
$$+\dot{f}\left(\tfrac{1}{2}\dot{x}^2-\tfrac{1}{2}x^2\dot{x}^2-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4-(\lambda/4)x^4\right)=\dot{P}. \quad (11)$$

Writing the identity (11) explicitly it can be decomposed into a system of linear partial differential equations of the first order with respect to the generators f and F and gauge function P by equating to zero the terms of corresponding degrees in $\ddot{x}$

$$\dot{x}(1-x^2)\frac{\partial F}{\partial \dot{x}}+\left(-\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}x^2\dot{x}^2-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4-\frac{\lambda}{4}x^4\right)\frac{\partial f}{\partial \dot{x}}-\frac{\partial P}{\partial \dot{x}}=0, \quad (12)$$

$$F(-x\dot{x}^2-x+x^3-\lambda x^3)+\dot{x}(1-x^2)\left(\frac{\partial F}{\partial t}+\frac{\partial F}{\partial x}\dot{x}-\frac{\partial f}{\partial t}\dot{x}-\frac{\partial f}{\partial x}\dot{x}^2\right)$$
$$+\left(\tfrac{1}{2}\dot{x}^2-\tfrac{1}{2}x^2\dot{x}^2-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4-\frac{\lambda}{4}x^4\right)(\partial f/\partial t+(\partial f/\partial x)\dot{x})$$
$$-(1+\lambda)x\dot{x}^2(F-f\dot{x})-\partial P/\partial t-(\partial P/\partial x)\dot{x}=0. \quad (13)$$

These equations are called the generalized Killing equations.

Now assume that the unknown functions are

$$F=0, \quad f\equiv f(x), \quad P\equiv P(x). \quad (14)$$

For (14) the equation (12) is identically satisfied and equation (13) transforms into two equations obtained by separating the terms of corresponding degrees in $\dot{x}$

$$\tfrac{1}{2}(1-x^2)\partial f/\partial x+xf(1+\lambda)=0, \quad \left(-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4-(\lambda/4)x^4\right)\partial f/\partial x=\partial P/\partial x. \quad (15, 16)$$

Integrating equations (15) and (16) gives

$$f=1/(1-x^2)^{\lambda+1}, \quad P=[(1+\lambda)/4]x^4/(1-\dot{x}^2)^{\lambda+1}. \quad (17, 18)$$

Substituting (17) and (18) into (10) the conservation law is

$$-(x^2+\dot{x}^2)/2(1-x^2)^{\lambda}=\text{const.} \tag{19}$$

The same conservation law is obtained using two other sets of functions:

$$f=0, \quad F=\dot{x}/(1-x^2)^{1+\lambda}, \quad P=(\dot{x}^2-x^2)/2(1-x^2)^{\lambda}, \tag{20}$$

and

$$f=1/(1-x^2)^{1+\lambda}-2\Psi/\dot{x}^2, \quad F=-2\Psi/\dot{x},$$

$$P=-\frac{2}{\dot{x}^2}\Psi\left(-\frac{1}{2}x^2+\frac{1}{4}x^4-\frac{\lambda}{4}x^4\right), \tag{21}$$

where

$$\Psi=-\frac{1}{x^2-1}\frac{1+\lambda}{2}\left(-\frac{1}{2}\frac{1}{(1-x^2)^{1+\lambda}}+\frac{1}{(1-x^2)^{\lambda}}-\frac{1}{2}\frac{1}{(1-x^2)^{\lambda-1}}\right).$$

The conservation law is a specific functional relation between physical and geometrical parameters that is satisfied due to the differential equations of motion (1). The conservation law in some specific way reflects the physical mechanism acting in the dynamical system. This conservation law can considerably simplify the integration of the differential equation of motion.

Now consider some specific cases for various values of parameter λ.

3. SOLUTION OF EQUATION (1) FOR $\lambda = 1$

For $\lambda = 1$ the conservation law (10) is

$$(x^2+\dot{x}^2)/(1-x^2)=K, \tag{22}$$

where for the initial conditions $x(0) = x_0 \neq \pm 1$ and $\dot{x}(0)=\dot{x}_0$ it is

$$K=(x_0^2+\dot{x}_0^2)/(1-x_0^2). \tag{23}$$

The trajectory in phase plane is for $K>0$ an ellipse

$$\dot{x}^2/K+x^2/[K/(K+1)]=1, \tag{24}$$

with characteristic points

$$\dot{x}=0, \quad x=\pm\sqrt{K/(K+1)},$$

and

$$x=0, \quad \dot{x}=\pm\sqrt{K}.$$

Based on equation (24) the phase plane diagrams for various values of parameter K are plotted (see Figure 1).

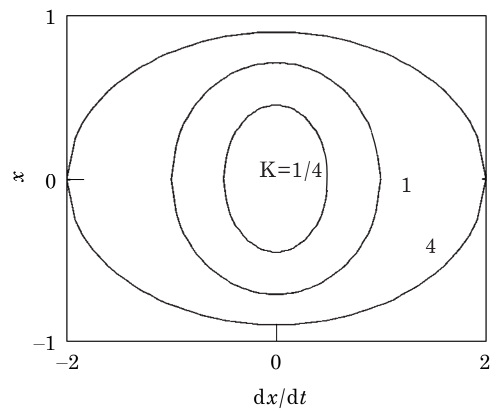

Figure 1. The phase plane curves for $\lambda = 1$ and K = 1/4; 1; 4.

The solution of (24) is a harmonic function

$$x = A\cos\omega t, \tag{25}$$

where the amplitude of vibration is

$$A = \pm\sqrt{K/(1+K)}, \tag{26}$$

and the frequency of vibration is

$$\omega = \sqrt{1+K}. \tag{27}$$

For $K < -1$ the trajectory in phase plane is a hyperbola

$$x^2/[|K|/(|K|-1)] - \dot{x}^2/|K| = 1, \tag{28}$$

with characteristic points

$$\dot{x} = 0, \quad x = \pm\sqrt{|K|/(|K|-1)}.$$

In Figure 2 the curves in the phase plane described with equation (28) for $K = -2$ and $K = -4$ are plotted.

The exact solution of (28) is a hyperbolic function

$$x = B\cosh\Omega t, \tag{29}$$

where

$$B = \sqrt{|K|}, \quad \Omega = \sqrt{|K|-1}. \tag{30}$$

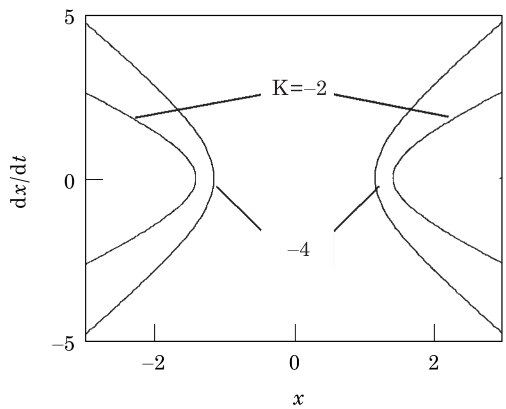

Figure 2. The phase plane curves for $\lambda = 1$ and K = −2; −4.

4. SOLUTION OF EQUATION (1) FOR $\lambda = 2$

The conservation law for $\lambda = 2$ is

$$(x^2 + \dot{x}^2)/(1 - \dot{x}^2)^2 = K, \tag{31}$$

where for the initial conditions $x(0) = x_0 \neq \pm 1$ and $\dot{x}(0) = \dot{x}_0$ it is

$$K = (x_0^2 + \dot{x}_0^2)/(1 - x_0^2)^2. \tag{32}$$

The mathematical description of the phase plane portrait for $K > 0$ is

$$\dot{x}^2/K = (1 - x^2)^2 - (1/K)x^2, \tag{33}$$

i.e.,

$$\dot{x}^2/K = (x + 1/2\sqrt{K} - \sqrt{1 + [1/4K]})(x + 1/2\sqrt{K} + \sqrt{1 + [1/4K]})$$
$$\times (x - 1/2\sqrt{K} - \sqrt{1 + [1/4K]})(x - 1/2\sqrt{K} + \sqrt{1 + [1/4K]}). \tag{34}$$

The solution of (1) exists for $K > 0$ and

$$x \geqslant (1 + \sqrt{1 + 2K})/2\sqrt{K}, \quad x \leqslant (1 + \sqrt{1 + 2K})/2\sqrt{K}, \tag{35, 36}$$

and

$$-(\sqrt{1 + 4K} - 1)/2\sqrt{K} \leqslant x \leqslant (\sqrt{1 + 4K} - 1)/2\sqrt{K}. \tag{37}$$

In Figure 3 the phase plane curves for $K = 2$ are plotted.

For $\lambda = 2$ and (37) assume the solution of (1) in the form

$$x = A\ \mathrm{cn}(\omega t + \theta, k^2), \tag{38}$$

where cn is the elliptic Jacobi function (see references [9–11]), ω is the frequency, k is the modulus of Jacobi function and θ and A are arbitrary constants. Substituting (38) and its time derivatives into (1) gives

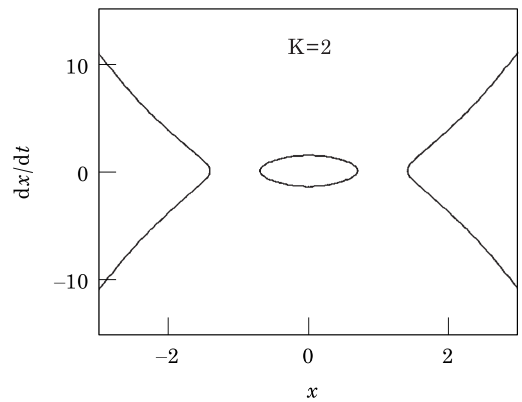

Figure 3. The phase plane curves for $\lambda = 2$.

$$\omega^2 = (1 + A^2)/(1 - A^2), \quad k^2 = A^4/(A^4 - 1). \tag{39, 40}$$

From equations (39) and (40) it is evident that for

$$A < 1, \tag{41}$$

then

$$\omega^2 > 0, \quad k^2 < 0, \tag{42}$$

and for

$$A > 1, \tag{43}$$

then

$$\omega^2 < 0, \quad k^2 > 0. \tag{44}$$

This means that the special strong non-linear differential equation with hard singularity

$$\ddot{x} + x = -2x(\dot{x}^2 + x^2)/(1 - x^2), \tag{45}$$

has an exact solution in the closed form for $A < 1$

$$x = A\ \mathrm{cn}(t\sqrt{(1 + A^2)/(1 - A^2)} + \theta, \quad -A^4/[1 - A^4]), \tag{46}$$

and for $A > 1$

$$x = A\ \mathrm{cn}(\mathrm{i}t\sqrt{(1 + A^2)/(1 - A^2)} + \theta, \quad A^4/[A^4 - 1]), \tag{47}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, A and θ are arbitrary constants. The solutions for (46) and (47) are now analyzed.

Solution for $A < 1$. Using the relations between elliptic functions with positive and elliptic functions with negative modulus [11] the solution (46) can be transformed to

$$x = A\,\mathrm{cd}(t/(1 - A^2) + \theta_1,\, A^4), \tag{48}$$

where $\theta_1 = \theta/\sqrt{1 - A^4}$ is an arbitrary constant and cd is the Jacobi elliptic function [9].

The constants A and θ_1 are obtained according to the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \tag{49}$$

from the equations

$$x_0 = A\,\mathrm{cd}(\theta_1,\, A^4), \quad \dot{x}_0 = -A(1 + A^2)\,\mathrm{sd}(\theta_1,\, A^4)\,\mathrm{nd}(\theta_1,\, A^4), \tag{50, 51}$$

where sd and nd are Jacobi elliptic functions [11].

Two special types of initial conditions will be considered:

(a) For

$$x(0) = x_0, \quad \dot{x}(0) = 0, \tag{52}$$

it is

$$A = x_0, \quad \theta_1 = 2n\mathrm{K}, \tag{53}$$

where $n = 0, 1, 2, \ldots$ and $\mathrm{K} \equiv \mathrm{K}(k) \equiv \mathrm{K}(x_0^2)$ is the complete elliptic integral of the first kind [10]. The solution of equation (46) is

$$x = x_0\,\mathrm{cd}(t/(1 - x_0^2) + 2n\mathrm{K}(x_0^2),\, x_0^4), \tag{54}$$

for

$$x_0 < 1. \tag{55}$$

(b) For

$$x(0) = 0, \quad \dot{x}(0) = \dot{x}_0, \tag{56}$$

it is

$$\theta_1 = (2n + 1)\mathrm{K}, \tag{57}$$

and

$$\dot{x}_0 = -A\omega = -A/(1 - A^2), \tag{58}$$

i.e.,

$$A_{1,2} = (1 \pm \sqrt{4\dot{x}_0^2 + 1})/2\dot{x}_0, \tag{59}$$

where $n = 0, 1, 2, \ldots,$ and $\mathrm{K} = \mathrm{K}(A^2)$. Finally, the solution (46) is transformed to

$$x = A\,\mathrm{cd}(t/(1 - A^2) + (2n + 1)\mathrm{K}(A^2),\, A^4), \tag{60}$$

where A has the form (59).

Solution for $A > 1$. For the initial conditions $x(0) = x_0 > 1$ and $\dot{x}(0) = \dot{x}_0$ the solution of (47) is

$$x = A\ \mathrm{cn}(\mathrm{i}t\sqrt{(1 + A^2)/(1 - A^2)}, \quad A^4/(A^4 - 1)), \tag{61}$$

where $A = x_0$. The elliptic function is with imaginary argument. Using the relation between the elliptic function with imaginary argument and elliptic function with real argument (see reference [9]) the solution (61) is modified to

$$x = A\ \mathrm{nc}(\sqrt{[(1 + A^2)/(A^2 - 1)]}t, \quad -1/(A^4 - 1)), \tag{62}$$

where nc is a Jacobi elliptic function. The modulus of the function is negative. Transforming it into the elliptic function with positive argument it becomes

$$x = A\ \mathrm{dc}(A^2 t/(A^2 - 1), 1/A^4). \tag{63}$$

The relation (63) represents the exact solution of (47) for $A = x_0 > 1$.

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