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APPROXIMATE CALCULATION OF THE FUNDAMENTAL FREQUENCY FOR BENDING VIBRATIONS OF CRACKED BEAMS

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A simplified method of evaluating the fundamental frequency for the bending vibrations of cracked Euler–Bernouilli beams is presented. The method is based on the well-known approach of representing the crack in a beam through a hinge and an elastic spring, but here the transverse deflection of the cracked beam is constructed by adding polynomial functions to that of the uncracked beam. With this new admissible function, which satisfies the boundary and the kinematic conditions, and by using the Rayleigh method, the fundamental frequency is obtained. This approach is applied to simply supported beams with a cracked section in any location of the span. For this case, the method provides closed-form expressions for the fundamental frequency. Its validity is confirmed by comparison with numerical simulation results

1. INTRODUCTION

The knowledge of natural frequencies and natural modes of vibration of cracked beams has aroused considerable interest in the last 30 years. Nash [1] investigated the problem of a simply supported beam with an edge crack at mid-span to analyze the dynamic behaviour of a notched beam subjected to impact test. Kishimoto *et al.* [2, 3] derived simple formulas for the dynamic stress intensity factor of elastic three-point and one-point bending specimens in which the fundamental frequency is an important parameter. It has been demonstrated that cracks reduce the natural frequencies of a structure and that this can be used to detect the damage size and its location. This motivates the dynamic analysis of cracked structures.

The natural modes of simply supported beams with symmetric cracks were investigated by Christides and Barr [4] using a two-term Rayleigh–Ritz solution to obtain the variation in the fundamental frequency of beams with a mid-span crack. They considered a crack function which represents the perturbation in the stresses induced by the crack. This function decays exponentially from the crack along the longitudinal axis of the beam. The decay rate has to be found from experimental or numerical analyses. Shen and Pierre [5, 6] used an approximate Galerkin solution to analyze the free bending vibrations of simply

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supported cracked beams. This method is also based on the knowledge of the stress decay rate along the beam.

In other cases, the presence of a crack and the reduction in the stiffness of the beam has been modelled by the introduction of a linear spring [7]. Ju *et al.* [8]. theoretically related the magnitude of the equivalent linear spring constants to the length of the crack in the beam, based on Fracture Mechanics. Haisty and Springer [9] developed a beam element to be used in finite-element codes. The crack was simulated as a linear spring for axial vibrations and as a torsional spring for bending vibrations. These form the base for models that have been applied to simply supported [10, 11], cantilever [12] and free–free beams [13].

To the knowledge of the authors, closed-form expressions for the fundamental frequency have not been found.

This work presents a simplified analytical method of obtaining the fundamental frequency of a cracked Euler–Bernouilli beam. For the simply supported beam a closed form for the fundamental frequency has been achieved. All the results obtained by using the proposed method are compared with numerical calculations.

2. PROPOSED METHOD

Consider the bending vibrations of a uniform Euler–Bernouilli beam in the x–y plane (see Figure 1) which is assumed to be a plane of symmetry for any cross-section. The length and height of the beam are, respectively, L and W, and it has a crack, of depth a, at a distance b from the left support. The crack is assumed to be always open. The presence of the crack introduces a discontinuity in the slope of the beam, which is proportional to the bending moment transmitted through the cracked section. Thus, if $\Theta(x)$ represents the slope, the above-mentioned discontinuity $\Delta\Theta$ can be written as

$$\Delta\Theta = C_m M, \tag{1}$$

where M is the bending moment transmitted by the cracked section and C_m is a flexibility constant which from dimensional considerations can be expressed as

$$C_m = \frac{W}{EI} m(a/W, \text{cross-section geometry}), \tag{2}$$

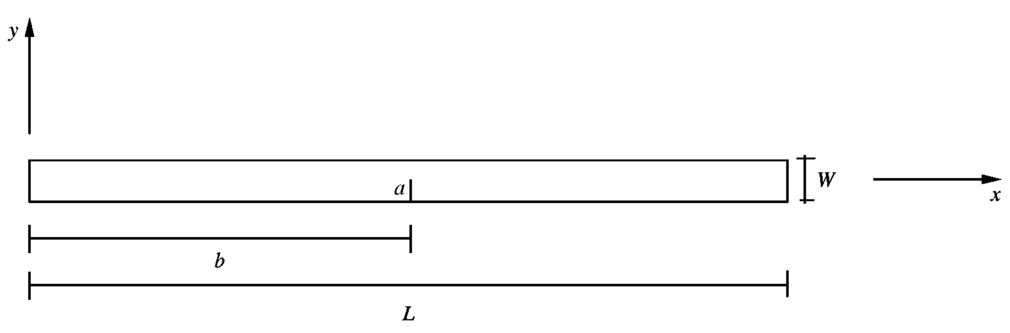

Figure 1. Schematic model of the cracked beam.

where E is Young's modulus, I is the geometric moment of inertia, and $m(a/W)$ is a function depending on the crack ratio, a/W, and on the section geometry. This function can be evaluated by Fracture Mechanics, and for the case of a rectangular section, it takes the form [14]

$$m\left(\frac{a}{W}\right) = 2\left(\frac{a/W}{1 - a/W}\right)^2\left[5{\cdot}93 - 19{\cdot}69\left(\frac{a}{W}\right) + 37{\cdot}14\left(\frac{a}{W}\right)^2 - 35{\cdot}84\left(\frac{a}{W}\right)^3 + 13{\cdot}12\left(\frac{a}{W}\right)^4\right]. \tag{3}$$

The vertical displacement, $y(x, t)$, of a section located at a distance x from the left end of the specimen at the time t, is assumed to be

$$y(x, t) = u(x)\cos\omega t, \tag{4}$$

where ω is the frequency of the harmonic vibration and $u(x)$ is the transverse deflection of the cracked beam.

In comparison to other approaches, the beam is not divided into two sub-beams at the crack, but the transverse deflection of the cracked beam, $u(x)$, is assumed to be that of the uncracked beam, $u_{nc}(x)$, plus a polynomial function of x:

$$u(x) = \begin{cases} u_1(x) = u_{nc}(x) + B_0 + B_1 x + B_2 x^2 + B_3 x^3, & 0 \leqslant x \leqslant b, \\ u_2(x) = u_{nc}(x) + C_0 + C_1 x + C_2 x^2 + C_3 x^3, & b \leqslant x \leqslant L. \end{cases} \tag{5}$$

The eight constants B_i and C_i $(i = 0, 1, \ldots, 3)$ that appear in equation (5) have to be evaluated to satisfy the boundary conditions (four boundary conditions, two for each end of the beam) and the kinematic conditions, as follows: equality for the transverse deflection, bending moment and shear force at the cracked section $(x = b)$ (three conditions), i.e.,

$$u_1(x) = u_2(x), \frac{\mathrm{d}^2 u_1(x)}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 u_2(x)}{\mathrm{d}x^2}, \qquad \frac{\mathrm{d}^3 u_1(x)}{\mathrm{d}x^3} = \frac{\mathrm{d}^3 u_2(x)}{\mathrm{d}x^3}, \tag{6}$$

slope discontinuity at the cracked section $(x = b)$ (one condition),

$$\Delta\Theta = \frac{\mathrm{d}u_2(x)}{\mathrm{d}x} - \frac{\mathrm{d}u_1(x)}{\mathrm{d}x} = Wm\left(\frac{a}{W}\right)\frac{\mathrm{d}^2 u_2(x)}{\mathrm{d}x^2}. \tag{7}$$

To obtain the fundamental frequency, the Rayleigh method [15] is used, so the maximum values of potential energy, $U_{\max}$, and kinetic energy, $T_{\max}$, must be calculated. Taking into account the flexural rigidity of the beam and the slope discontinuity at the cracked section, one finds that the maximum potential energy of the beam is

$$U_{\max} = \frac{EI}{2}\int_0^b \left(\frac{\mathrm{d}^2 u_1(x)}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \frac{EI}{2}\int_0^L \left(\frac{\mathrm{d}^2 u_2(x)}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \frac{1}{2}\Delta\Theta M. \tag{8}$$

The bending moment transmitted by the cracked section can be written as

$$M = EI\,\mathrm{d}^2 u_1(x)/\mathrm{d}x^2|_{x=b}.$$

The maximum kinetic energy is

$$T_{\max} = \frac{\rho A\omega_1^2}{2}\int_0^b (u_1(x))^2\,\mathrm{d}x + \int_b^L (u_2(x))^2\,\mathrm{d}x, \tag{9}$$

where ρ is the mass density of the material and A is the cross-sectional area of the beam.

Finally, by equating expressions (8) and (9), the fundamental frequency, ω_1, can be obtained.

3. APPLICATION TO SIMPLY-SUPPORTED BEAMS

To check its applicability, the method was applied to a simply supported Euler–Bernouilli beam.

The equation for the transverse deflection of the uncracked beam can be written as

$$u_{\mathrm{nc}}(x) = D\sin(\pi x/L), \tag{10}$$

where D is an arbitrary constant with dimensions of length.

In this case the boundary conditions which must be satisfied are

$$\text{for } x = 0, \quad u_1 = 0 \quad \text{and} \quad \mathrm{d}^2u_1/\mathrm{d}x^2 = 0, \tag{11}$$

$$\text{for } x = L, \quad u_2 = 0 \quad \text{and} \quad \mathrm{d}^2u_2/\mathrm{d}x^2 = 0. \tag{12}$$

The expressions for the transverse deflection of the cracked beam, which satisfy the above boundary conditions and the kinematic conditions expressed by equations (6) and (7), are

$$u_1(x) = D\left[\sin\frac{\pi x}{L} - \frac{bW}{L^3}\left(1 - \frac{L}{b}\right)m\left(\frac{a}{W}\right)\pi^2 \sin\frac{\pi b}{L}\right], \quad 0 \leqslant x \leqslant b, \tag{13}$$

$$u_2(x) = D\left[\sin\frac{\pi x}{L} + \frac{bW}{L^2}\,m\left(\frac{a}{W}\right)\pi^2 \sin\frac{\pi b}{L}\left(1 - \frac{x}{L}\right)\right], \quad b \leqslant x \leqslant L, \tag{14}$$

The bending moment transmitted by the cracked section and the slope discontinuity are, respectively,

$$M = -\,D\,EI\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi b}{L}, \qquad \varDelta\Theta = -\,DWm\left(\frac{a}{W}\right)\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi b}{L}. \qquad (15, 16)$$

Therefore, the expressions for $u_1(x)$ and $u_2(x)$ (equations (13) and (14)) and the values of M and $\varDelta\Theta$ (equations (15) and (16)) are substituted in equations (8) and (9), and the values of U_{max} and T_{max} are evaluated as

$$U_{max} = D^2(EI\pi^4/4L^3)(1+\eta\gamma) \qquad (17)$$

and

$$T_{max} = \omega_1^2 D^2 \frac{\rho AL}{4}\left[1 + 2\eta\gamma + \frac{\pi^4}{3}\eta^2\left[\left(\frac{b}{L}\right)^4\gamma - 2\left(\frac{b}{L}\right)^3\gamma + \left(\frac{b}{L}\right)^2\gamma\right]\right], \qquad (18)$$

where η and γ respectively, taken the values,

$$\eta = (W/L)\,m(a/W) \quad \text{and} \quad \gamma = 1 - \cos(2\pi b/L).$$

By equating equations (17) and (18), the following closed-form solution is obtained for the fundamental frequency of the cracked beam:

$$\omega_1 = \omega_0\left[\frac{1+\eta\cdot\gamma}{1 + 2\eta\gamma + (\pi^4/3)\eta^2[(b/L)^4\gamma - 2(b/L)^3\gamma + (b/L)^2\gamma]}\right]^{1/2}. \qquad (19)$$

Here ω_0 is the fundamental frequency for the simply supported uncracked beam, i.e.,

$$\omega_0 = (\pi/L)^2\sqrt{EI/\rho A}.$$

When the crack is located at mid-span of the beam ($b/L = 1/2$), the above expression reduced to

$$\omega_1 = \omega_0\left[\frac{1+2\eta}{1 + 4\eta + (\pi^4/24)\eta^2}\right]^{1/2}. \qquad (20)$$

The same methodology can be applied to cracked beams with other end conditions, but closed-form solutions have not been obtained as yet.

4. NUMERICAL COMPARISON

To validate the proposed method, the results were compared with those obtained from numerical simulation of cracked beams with different boundary conditions.

The fundamental frequencies of the beams were calculated by using the ABAQUS finite-element code [16].

The cases analyzed were (a) a simply supported beam with a crack at mid-span ($b/L = 1/2$), and (b) a simply supported beam with a crack at 0·75 of the total length ($b/L = 0{\cdot}75$). For all cases the beam geometry was $L = 200$ mm and $W = 10$ mm. The beam material had a mass density $\rho = 7850\ \mathrm{kg/m^3}$ and Young's modulus $E = 200$ GPa.

A finite-element mesh of 1000 eight-node plane stress elements was used as shown in Figure 2. The beam was divided into four zones, each part at both sides of the crack having 300 elements. The height of each element was $h/L = 1/200$ and the width varied from $d_1/L = 1/200$ near the crack to $d_2/L = 1/80$ far away. The other

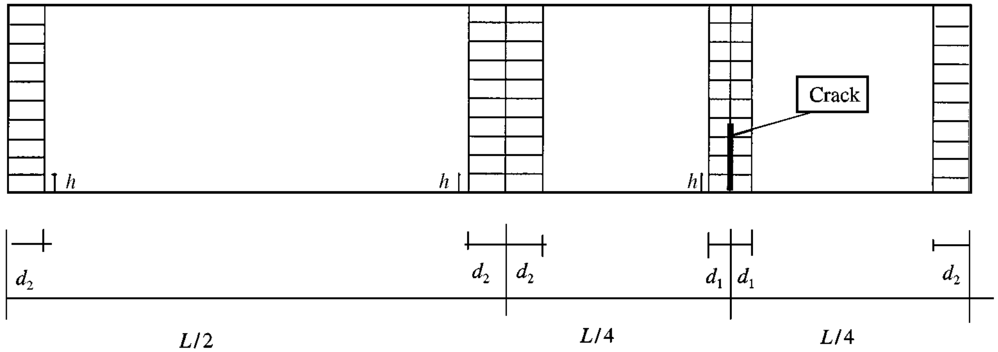

Figure 2. Mesh used in the numerical analysis.

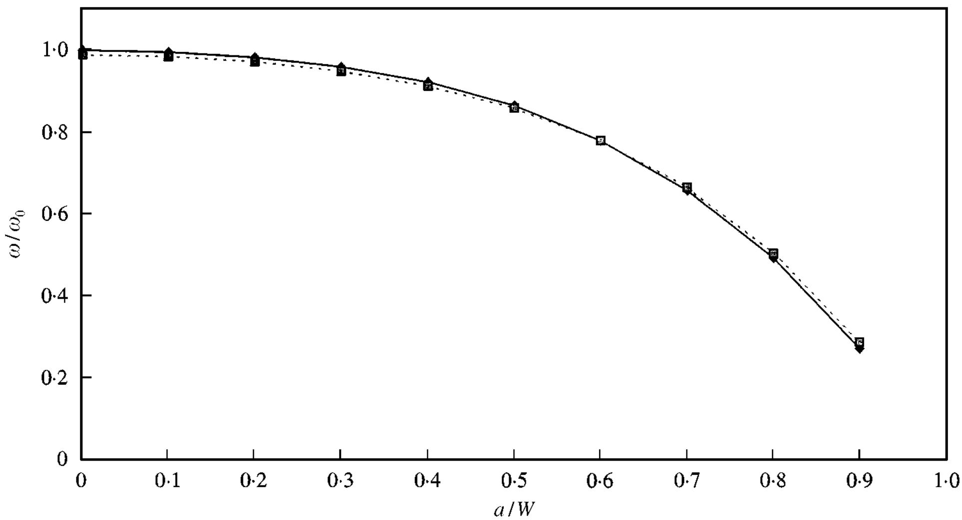

Figure 3. Variation of fundamental frequency with crack ratio of a simply supported beam ($b/L = 0{\cdot}5$). —◆—, Proposed method; – – □ – – , numerical results.

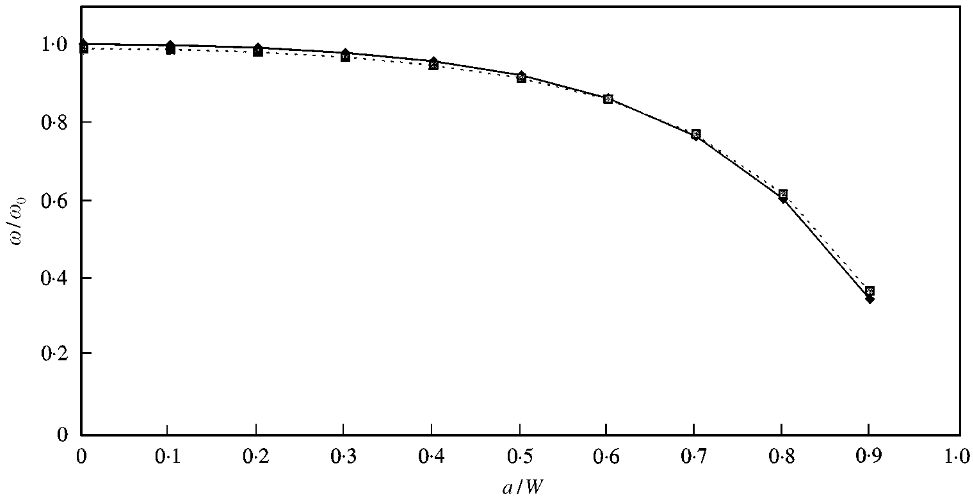

Figure 4. Variation of fundamental frequency with crack ratio of a simply supported beam. ($b/L = 0{\cdot}75$). —◆—, Proposed method; – – □ – – , numerical results.

two parts are of 200 elements each and their dimensions were $d_2/L = 1/80$ and $h/L = 1/200$. For each case, 10 different analyses were made for values of the crack ratio, a/W, ranging from 0 to 0·9.

The comparison of the results of the numerical simulation with those of the proposed method is given in Figure 3 (crack at mid-span) and Figure 4 (crack at 0·75 of total length), which show the variation of the fundamental frequency (dimensionless, corresponding to the uncracked beam, ω_0). For the analyzed cases the numerical value of ω_0 is 3595·3 rad/s.

In all the cases studied, the estimated values of the fundamental frequency of the cracked beam are very close to those obtained from finite-element calculations.

For other bean support conditions, closed-form solutions for the fundamental frequency are not available yet. For cantilever, fixed–simply support and fixed–fixed cracked beams however, the proposed method gives values of the fundamental frequency which differ only by about 2% from those obtained from fully numerical simulation of the problem.

5. CONCLUSIONS

This work presents a simplified method of evaluating the fundamental frequency for the bending vibrations of a simply supported cracked Euler–Bernouilli beam.

The transverse deflection of the cracked beam is constructed by adding polynomial functions to that of the uncracked beam. With this new admissible function, which satisfies the boundary and the kinematic conditions, and using the Rayleigh method, closed-form expressions for the fundamental frequency are obtained. In all the cases considered in this paper, the results are very close to those obtained numerically by the finite-element method.

REFERENCES

1. G. E. Nash 1969 *International Journal of Fracture Mechanics* **5**, 269–286. An analysis of the forces and bending moments generated during the notched beam impact test.
2. K. Kishimoto, S. Aoki and M. Sakata 1980 *Engineering Fracture Mechanics* **13**, 501–508. Simple formula for dynamic stress intensity factor of pre-cracked Charpy specimen.
3. K. Kishimoto, Y. Fujino, S. Aoki and M. Sakata 1990 *JSME International Journal Series I* **33**, 51–56. A simple formula for the dynamic stress intensity factor of an impacted freely-supported bend specimen.
4. S. Christides, and A. S. D. Barr 1984 *International Journal of Mechanical Science* **26**, 639–648. One-dimensional theory of cracked Bernouilli–Euler beams.
5. M.-H. H. Shen, and C. Pierre 1990 *Journal of Sound and Vibration* **138**, 115–134. Natural modes of Bernouilli–Euler beams with symmetric cracks.
6. M.-H. H. Shen and C. Pierre 1994 *Journal of Sound and Vibration* **170**, 237–259. Free vibrations of beams with a single-edge crack.
7. R. D. Adams, P. Cawley, C. J. Pye, and B. J. Stone 1978 *Journal of Mechanical Engineering Science* **20**, 93–100. A vibration technique for non-destructive assessing the integrity of structures.
8. F. D. Ju, M. Akgun, T. L. Paez, and E. T. Wong 1982 *Bureau of Engineering Research Report No. CE*-62(82) *AFOSR*-993-1 *University of New Mexico*, *Alburquerque*, *NM*. Diagnosis of fracture damage in simple structures.
9. B. S . Haisty and W. T. Springer 1984 *Journal of Vibrations*, *Acoustic*, *Stress and Reliability in Design* **110**, 389–394. A general beam element for use in damage assesment of complex structures.
10. Y. Narkis 1994 *Journal of Sound and Vibration* **172**, 549–558. Identification of crack location vibrating simply supported beams.
11. H. P. Lee and T. Y. Ng 1994 *Applied Acoustic* **43**, 151–163. Natural frequencies and modes for the flexural vibration of a cracked beam.
12. G. Bamnios and A. Trochides 1995 *Applied Acoustic* **45**, 97–112. Dynamic Behaviour of a cracked cantilever beam.
13. M. Boltezar, B. Strancar and A. Kuhelj 1998 *Journal of Sound and Vibration* **211**, 729–734. Identification of transverse crack location in flexural vibrations of free-free beams.
14. H. Tada, P. Paris, and G. Irwin 1985 *The Stress Analysis of Cracks Handbook*. St Louis, Missouri: Research Corporation, second edition.
15. S. S. Rao 1990. *Mechanical Vibrations*. New York: Addison-Wesley, second edition.
16. *ABAQUS User's Manual Version* 5·7, 1997. Hibbcitt, Karlsson & Sorensen, Inc.