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NON-LINEAR VIBRATION OF CABLE–DAMPER SYSTEMS PART I: FORMULATION

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This paper presents a formulation for determining the non-linear dynamic response of sag cables equipped with discrete oil dampers and subject to harmonic loading. The state-space method is first employed to convert the second order non-linear partial differential equations of motion of the system to first order non-linear partial differential equations. Then, in terms of the complex modes of vibration and their orthogonality properties achieved by a hybrid method, the generalized modal superposition method is used to reduce the first order non-linear partial differential equations to first order non-linear ordinary differential equations with respect to time functions only. Finally, the harmonic balance method is applied to obtain the non-linear algebraic equations, from which the real solutions for the time functions and non-linear dynamic responses of the cable–damper system are found. The application and verification of the suggested approach are presented in Part II of this paper.

1. INTRODUCTION

Cable structures can span a long distance or cover a large area with little material weight. Thus, they have found many applications, such as in long-span cable-supported bridges, guyed masts, and ocean mooring systems. However, due to their overall flexibility and low-energy dissipation capacity, cables are susceptible to large-amplitude vibration that may eventually degrade their performance [1]. To overcome cable vibration problems, oil dampers have been installed to some long cables to increase their energy dissipation capacity [2].

To evaluate the effectiveness of oil dampers, some theoretical studies have been carried out [3–6], but the oscillation of cable–damper systems was assumed to be small and the linear cable vibration theory was applied. While this treatment makes the problem manageable, the cable–oil damper system with a relatively large-amplitude vibration may still be possible due to either improper design of oil dampers or for long cables with large sag. This motivates the writers to study non-linear vibration of a sag cable with oil dampers.

Dynamic non-linearity occurring in large-amplitude vibration of a sag cable mainly arises from the quadratic and cubic non-linear terms in its equations of motion. These non-linear terms come up due to the stretching of the cable associated with the large-amplitude vibration [7]. The existence of the quadratic and cubic non-linear terms makes the in-plane cable motion couple with the out-of-plane cable motion and induces modal interaction [8]. To determine non-linear dynamic response of a sag cable, the Galerkin method is often used to convert non-linear partial differential equations of motion to non-linear ordinary differential equations with respect to time functions only. Then, either the perturbation method [9,10] or the harmonic balance method [7] is applied to find the solution for the time functions.

When a sag cable is equipped with oil dampers and exhibits non-linear vibration, the cable–damper system becomes a non-classically damped non-linear system. Very little information is available on how to determine the dynamic response and behavior of such a system. This paper thus aims to present a general approach for studying non-linear vibration of a sag cable with oil dampers. The second order non-linear partial differential equations of motion of the cable–damper system are first converted to first order partial non-linear differential equations in terms of the state-space method. The complex modes of vibration of the linear cable–damper system and their orthogonality properties are then derived, by which the generalized modal superposition method is applied to obtain the first order non-linear ordinary differential vibrations with respect to time functions only. The harmonic balance method is finally applied to seek the real solutions for the time functions and non-linear dynamic responses of the cable–damper system. The application and verification of the suggested approach are presented in Part II of this paper.

2. BASIC EQUATIONS

Three-dimensional equations of motion of an inclined sag cable with one pair of dampers being symmetrically installed (see Figure 1) can be expressed as follows [5]:

$$\frac{1}{\sqrt{1+y_x^2}}\frac{\partial}{\partial x}\left[(H+h)\left(1+\frac{\partial u}{\partial x}\right)\right]+f_x\delta(x-x_c)+F_x=m\frac{\partial^2 u}{\partial t^2}+c_1\frac{\partial u}{\partial t}, \tag{1}$$

$$\frac{1}{\sqrt{1+y_x^2}}\frac{\partial}{\partial x}\left[(H+h)\frac{\partial w}{\partial x}+hy_x\right]+f_y\delta(x-x_c)+F_y=m\frac{\partial^2 w}{\partial t^2}+c_1\frac{\partial w}{\partial t}, \tag{2}$$

$$\frac{1}{\sqrt{1+y_x^2}}\frac{\partial}{\partial x}\left[(H+h)\frac{\partial v}{\partial x}\right]+f_z\delta(x-x_c)+F_z=m\frac{\partial^2 v}{\partial t^2}+c_2\frac{\partial v}{\partial t}, \tag{3}$$

where x, y and z are the Cartesian co-ordinates in the horizontal, vertical and lateral directions respectively (see Figure 1); u, w and v are the cable dynamic displacement components in the x, y and z directions respectively, measured from

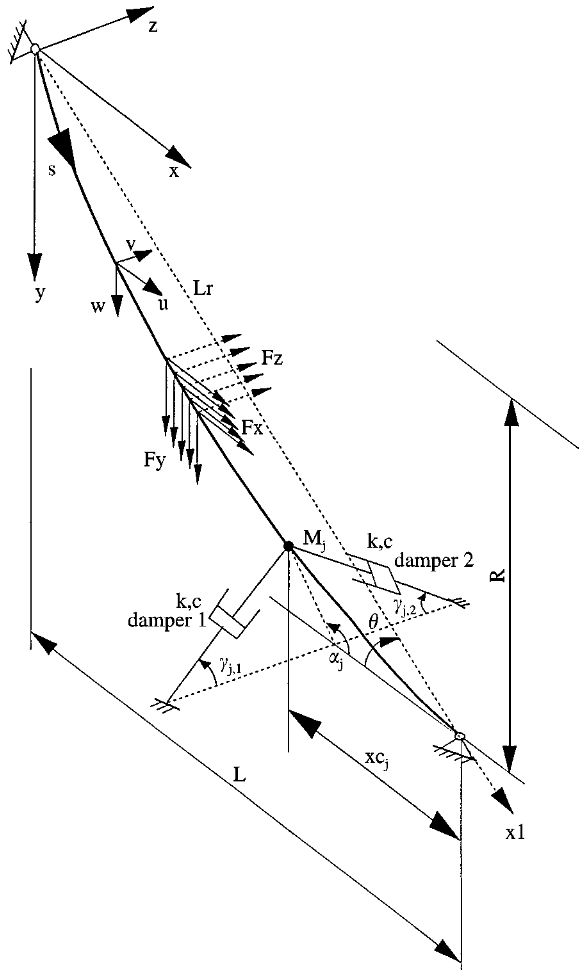

Figure 1. Schematic diagram of an inclined sag cable with oil dampers.

the position of static equilibrium of the cable; F_x, F_y and F_z are the external dynamic loading per unit length in the x, y and z directions, respectively; f_x, f_y and f_z are the forces exerted by the pair of oil dampers on the cable at the location of x_c in the x, y and z directions, respectively; δ is the Dirac's delta function; m the mass of the cable per unit length; t the time; c_1 and c_2 the in-plane and out-of-plane internal damping coefficients of the cable, respectively; H the horizontal component of the static cable tension T; h the horizontal component of the dynamic cable tension τ; y_x the first derivative of cable static equilibrium y with respect to x; and x_c is the co-ordinate of the pair of dampers measured from the right support of the cable.

The non-linear relationship between the horizontal dynamic cable tension and dynamic response is taken as

$$h = \frac{EA}{(1 + y_x^2)^{3/2}} \left\{ \frac{\partial u}{\partial x} + y_x \frac{\partial w}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] \right\}, \tag{4}$$

where E is the cable elastic modulus and A the cable cross-sectional area.

Assume that the pair of oil dampers have the same damping coefficient c, and define the direction of each damper from the ground to the cable as the positive direction. The direction cosines of each damper can be expressed in terms of two independent angular variables α and γ (see Figure 1). The relationship between the damper force components and cable displacement components is found to be

$$\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} -2\sin^2\gamma\cos^2\alpha & 2\sin^2\gamma\sin\alpha\cos\alpha & 0 \\ 2\sin^2\gamma\sin\alpha\cos\alpha & -2\sin^2\gamma\sin^2\alpha & 0 \\ 0 & 0 & -2\cos^2\gamma \end{bmatrix} \begin{bmatrix} c\dfrac{\partial u(x_c,t)}{\partial t} \\ c\dfrac{\partial w(x_c,t)}{\partial t} \\ c\dfrac{\partial v(x_c,t)}{\partial t} \end{bmatrix}. \tag{5}$$

The static profile of the inclined sag cable is of the form [11]

$$x(s) = \frac{Hs}{EA} + \frac{H}{mg}\left[\sinh^{-1}\left(\frac{V}{H}\right) - \sinh^{-1}\left(\frac{V-mgs}{H}\right)\right], \tag{6}$$

$$y(s) = \frac{mgL_0 s}{EA}\left(\frac{V}{mgL_0} - \frac{s}{2L_0}\right) + \frac{H}{mg}\left\{\left[1+\left(\frac{V}{H}\right)^2\right]^{1/2} - \left[1+\left(\frac{V-mgs}{H}\right)^2\right]^{1/2}\right\}, \tag{7}$$

where s is the arc-length co-ordinate associated with the cable static profile, L_0 the arc length of the cable under free load condition, g the acceleration due to gravity; and V the vertical component of the cable static tension at the left support of the cable.

The first derivative of the displacement y with respect to x in equations (1)–(4) can be then calculated by

$$y_x = \frac{\mathrm{d}y/\mathrm{d}s}{\mathrm{d}x/\mathrm{d}s}. \tag{8}$$

The boundary conditions of the cable considered here are

$$u(0,t) = u(L,t) = 0; \quad w(0,t) = w(L,t) = 0; \quad v(0,t) = v(L,t) = 0 \tag{9}$$

All these basic equations constitute a non-linear vibration problem of an inclined sag cable with a pair of oil dampers symmetrically installed. For the convenience of physical understanding, the basic equations described above are not non-dimensionalized.

3. REDUCED FORM OF BASIC EQUATIONS

The cable–damper system being studied is a non-linear non-classically damped system. Following the way suggested by Foss [12], Caughey and O'Kelly [13], and Veletsos and Ventura [14] for linear non-classically damped systems, one can reduce the second order non-linear partial differential equations (1)–(3) into the first order non-linear partial differential equations:

$$[M^{in}]\left\{\frac{\partial Z^{in}}{\partial t}\right\} + [\tilde{K}^{in}]\{Z^{in}\} = \{F^{in}\}, \tag{10}$$

$$[M^{out}]\left\{\frac{\partial Z^{out}}{\partial t}\right\} + [\tilde{K}^{out}]\{Z^{out}\} = \{F^{out}\}, \tag{11}$$

where

$$[M^{in}] = \begin{bmatrix} [0] & [m^{in}] \\ [m^{in}] & [c^{in}] \end{bmatrix}, \tag{12}$$

$$\{Z^{in}(x,t)\} = \begin{bmatrix} \frac{\partial u}{\partial t} & \frac{\partial w}{\partial t} & u & w \end{bmatrix}^{\mathrm{T}}, \tag{13}$$

$$[\tilde{K}^{in}] = \begin{bmatrix} -[m^{in}] & [0] \\ [0] & [\tilde{k}^{in}] \end{bmatrix}, \tag{14}$$

$$\{F^{in}\} = [0 \quad 0 \quad \sqrt{1+y_x^2}F_x \quad \sqrt{1+y_x^2}F_y]^{\mathrm{T}}, \tag{15}$$

$$[M^{out}] = \begin{bmatrix} 0 & m^{out} \\ m^{out} & c^{out} \end{bmatrix}, \tag{16}$$

$$\{Z^{out}(x,t)\} = \begin{bmatrix} \frac{\partial v}{\partial t} & v \end{bmatrix}^{\mathrm{T}}, \tag{17}$$

$$[\tilde{K}^{out}] = \begin{bmatrix} -m^{out} & 0 \\ 0 & \tilde{k}^{out} \end{bmatrix}, \tag{18}$$

$$\{F^{out}\} = [0 \quad \sqrt{1+y_x^2}F_z]^{\mathrm{T}} \tag{19}$$

in which the superscript "in" means the in plane while "out" indicates the out of plane. The elements in equations (12), (14), (16), and (18) are further expressed as follows:

$$[m^{in}] = \sqrt{1+y^{2x}}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \tag{20}$$

$$[c^{in}] = \sqrt{1+y_x^2}\begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} 2c\sin^2\gamma\cos^2\alpha & -2c\sin^2\gamma\sin\alpha\cos\alpha \\ -2c\sin^2\gamma\sin\alpha\cos\alpha & 2c\sin^2\gamma\sin^2\alpha \end{bmatrix}$$

$$\times\,\delta(x - x_c), \tag{21}$$

$$[\tilde{k}^{in}] = -\begin{bmatrix} \dfrac{\partial}{\partial x}\left[(H+h)\left(1+\dfrac{\partial}{\partial x}\right)\right] & 0 \\ 0 & \dfrac{\partial}{\partial x}\left[(H+h)\dfrac{\partial}{\partial x}+hy_x\right] \end{bmatrix}, \tag{22}$$

$$m^{out} = \sqrt{1+y_x^2}m, \tag{23}$$

$$c^{out} = \sqrt{1+y_x^2}c_2 + 2c\cos^2\gamma\cdot\delta(x-x_c), \tag{24}$$

$$\tilde{k}^{out} = -\frac{\partial}{\partial x}\left[(H+h)\frac{\partial}{\partial x}\right], \tag{25}$$

where $[\tilde{k}^{in}]$ and $\tilde{k}^{out}$ are the partial differential operators.

4. ORTHOGONALITY PROPERTIES

To apply the generalized modal superposition method to the reduced form of equations of the non-linear cable–damper system, it is necessary to find the orthogonality properties of the complex vibration modes of the linear cable–damper system. For the linear cable–damper system, the reduced form of equations of motion of the system is as follows:

$$[M^{in}]\left\{\frac{\partial Z^{in}}{\partial t}\right\} + [K^{in}]\{Z^{in}\} = 0, \tag{26}$$

$$[M^{out}]\left\{\frac{\partial Z^{out}}{\partial t}\right\} + [K^{out}]\{Z^{out}\} = 0. \tag{27}$$

All other quantities in the above two equations are the same as those described in the last section, except for the stiffness matrices$[K^{in}]$ and $[K^{out}]$:

$$[K^{in}] = \begin{bmatrix} -[m^{in}] & [0] \\ [0] & [k^{in}] \end{bmatrix}, \tag{28}$$

$$[K^{out}] = \begin{bmatrix} -m^{out} & 0 \\ 0 & k^{out} \end{bmatrix}, \tag{29}$$

$$[k^{in}] = -\begin{bmatrix} \dfrac{\partial}{\partial x}\left[G_{11}(x)\dfrac{\partial}{\partial x}\right] & \dfrac{\partial}{\partial x}\left[G_{12}(x)\dfrac{\partial}{\partial x}\right] \\ \dfrac{\partial}{\partial x}\left[G_{21}(x)\dfrac{\partial}{\partial x}\right] & \dfrac{\partial}{\partial x}\left[G_{22}(x)\dfrac{\partial}{\partial x}\right] \end{bmatrix}, \tag{30}$$

$$k^{out} = -\frac{\partial}{\partial x}\left[H\frac{\partial}{\partial x}\right], \tag{31}$$

where

$$G_{11}(x) = H + \frac{EA}{(1 + y_x^2)^{3/2}}; \quad G_{12}(x) = G_{21}(x) = \frac{EAy_x}{(1 + y_x^2)^{3/2}};$$

$$G_{22}(x) = H + \frac{EAy_x^2}{(1 + y_x^2)^{3/2}}. \tag{32}$$

By setting

$$u(x, t) = \Phi(x)\mathrm{e}^{\Omega t}, \tag{33}$$

$$w(x, t) = \Theta(x)\mathrm{e}^{\Omega t}, \tag{34}$$

$$v(x, t) = \Psi(x)\mathrm{e}^{\Omega^* t} \tag{35}$$

the complex eigenfunctions $\Phi(x)$, $\Theta(x)$, $\Psi(x)$ and complex eigenvalues Ω, Ω^* can be determined from equations (26) and (27) in terms of the hybrid method developed by the writers [5]. Here, $\Phi(x)$ and $\Theta(x)$ are the components of the in-plane eigenfunction in horizontal and vertical directions respectively, $\Psi(x)$ is the out-of-plane eigenfunction of the cable, Ω is the in-plane eigenvalue of the cable, and Ω^* is the out-of-plane eigenvalue of the cable. The eigenvalues occur in complex conjugate pairs, and to each pair there corresponds a complex conjugate pair of eigenfunctions. Thus, one can constitute the kth generalized eigenfunctions as

$$\{\Gamma_{(k)}^{in}(x)\} = [\Omega_{(k)}\Phi^{(k)}(x) \quad \Omega_{(k)}\Theta^{(k)}(x) \quad \Phi^{(k)}(x) \quad \Theta^{(k)}(x)]^{\mathrm{T}}, \tag{36}$$

$$\{\Gamma_{(k)}^{out}(x)\} = [\Omega_{(k)}^{*}\Psi^{(k)}(x) \quad \Psi^{(k)}(x)]^{\mathrm{T}}, \tag{37}$$

where $\Phi^{(k)}(x)$ and $\Theta^{(k)}(x)$ are the normalized horizontal and vertical components of the kth in-plane complex eigenfunction of the cable–damper linear system; $\Psi^{(k)}(x)$ is the normalized kth out-of-plane complex eigenfunction of the cable–damper linear system, and $\Omega_{(k)}$ and $\Omega_{(k)}^{*}$ are the kth in-plane and out-of-plane eigenvalues, respectively.

It can be proved that the generalized eigenfunctions have the following orthogonality properties:

$$\int_0^L \{\Gamma_{(k)}^{in}\}^{\mathrm{T}}[M^{in}]\{\Gamma_{(j)}^{in}\}\,\mathrm{d}x = 0, \quad \int_0^L \{\Gamma_{(k)}^{in}\}^{\mathrm{T}}[K^{in}]\{\Gamma_{(j)}^{in}\}\,\mathrm{d}x = 0, \quad j \neq k, \tag{38}$$

$$\int_0^L \{\Gamma_{(k)}^{out}\}^{\mathrm{T}}[M^{out}]\{\Gamma_{(j)}^{out}\}\,\mathrm{d}x = 0, \quad \int_0^L \{\Gamma_{(k)}^{out}\}^{\mathrm{T}}[K^{out}]\{\Gamma_{(j)}^{out}\}\,\mathrm{d}x = 0, j \neq k. \tag{39}$$

For the cable without oil damper, the linear system can be seen as a classically damped system. The orthogonality properties of the modes of vibration can be reduced to

$$\int_0^L \sqrt{1+y_x^2}\,[\Phi^{(j)}(x)\Phi^{(k)}(x)+\Theta^{(j)}(x)\Theta^{(k)}(x)]\,\mathrm{d}x=0, \tag{40}$$

$$\int_0^L \sqrt{1+y_x^2}\,\Psi^{(j)}(x)\Psi^{(k)}(x)\,\mathrm{d}x=0. \tag{41}$$

5. GENERALIZED MODAL SUPERPOSITION METHOD

The approximate solutions of equations (10) and (11) can be obtained by expanding $\{Z^{in}\}$ and $\{Z^{out}\}$ into modal series:

$$\{Z^{in}\}=\sum_{k=1,2}^{\infty}\{\Gamma_{(k)}^{in}\}q_1^{(k)}=\sum_{k=1,2}^{\infty}\begin{bmatrix}\Omega_{(k)}\Phi^{(k)}\\ \Omega_{(k)}\Theta^{(k)}\\ \Phi^{(k)}\\ \Theta^{(k)}\end{bmatrix}q_1^{(k)}, \tag{42}$$

$$\{Z^{out}\}=\sum_{k=1,2}^{\infty}(\{\Gamma_{(k)}^{out}\}q_2^{(k)})=\sum_{k=1,2}^{\infty}\begin{bmatrix}\Omega_{(k)}^{*}\Psi^{(k)}\\ \Psi^{(k)}\end{bmatrix}q_2^{(k)}. \tag{43}$$

Referring to the definitions of $\{Z^{in}\}$ and $\{Z^{out}\}$ in equations (13) and (17), $u(x,t)$, $w(x,t)$ and $v(x,t)$ can be then written as

$$u(x,t)=\sum_{k=1,2}^{\infty}\Phi^{(k)}(x)q_1^{(k)}(t), \tag{44}$$

$$w(x,t)=\sum_{k=1,2}^{\infty}\Theta^{(k)}(x)q_1^{(k)}(t), \tag{45}$$

$$v(x,t)=\sum_{k=1,2}^{\infty}\Psi^{(k)}(x)q_2^{(k)}(t), \tag{46}$$

where $q_1^{(k)}(t)$ and $q_2^{(k)}(t)$ are the unknown complex time functions associated with the kth in-plane and out-of-plane modes respectively.

Note that the complex eigenvalues and complex eigenfunctions of a non-classically damped system occur as complex conjugate pairs. To ensure the responses $\{Z^{in}\}$ and $\{Z^{out}\}$ or u, w and v are real values, the solutions represented in equations (42) and (43) must be the linear combinations of complex conjugate pairs, which means that $\Phi^{(2j)}(x)$, $\Theta^{(2j)}(x)$, $\Psi^{(2j)}(x)$, $q_1^{(2j)}(t)$ and $q_2^{(2j)}(t)$ are the conjugate functions of $\Phi^{(2j-1)}(x)$, $\Theta^{(2j-1)}(x)$, $\Psi^{(2j-1)}(x)$, $q_1^{(2j-1)}(t)$ and $q_2^{(2j-1)}(t)$, $j=1, 2, \dots,$ respectively. This has been emphasized by using $k=1,2$ in equations (42)–(46).

By introducing equations (42) and (43) into equations (10) and (11) and then applying the Galerkin method in terms of derived orthogonality properties of complex vibrational modes, the equations about the time functions can be obtained as follows:

$$\frac{\mathrm{d}q_1^{(j)}}{\mathrm{d}t}\int_0^L \{\Gamma_{(j)}^{in}\}^{\mathrm{T}}[M^{in}]\{\Gamma_{(j)}^{in}\}\,\mathrm{d}x + \sum_{k=1,2}^{\infty} q_1^{(k)}\int_0^L \{\Gamma_{(j)}^{in}\}^{\mathrm{T}}[\tilde{K}^{in}]\{\Gamma_{(k)}^{in}\}\,\mathrm{d}x$$

$$= \int_0^L \{\Gamma_{(j)}^{in}\}^{\mathrm{T}}\{F^{in}\}\,\mathrm{d}x \tag{47}$$

$$\frac{\mathrm{d}q_2^{(j)}}{\mathrm{d}t}\int_0^L \{\Gamma_{(j)}^{out}\}^{\mathrm{T}}[M^{out}]\{\Gamma_{(j)}^{out}\}\,\mathrm{d}x + \sum_{k=1,2}^{\infty} q_2^{(k)}\int_0^L \{\Gamma_{(j)}^{out}\}^{\mathrm{T}}[\tilde{K}^{out}]\{\Gamma_{(k)}^{out}\}\,\mathrm{d}x$$

$$= \int_0^L \{\Gamma_{(j)}^{out}\}^{\mathrm{T}}\{F^{out}\}\,\mathrm{d}x \tag{48}$$

where $j = 1, 2, \ldots$

The complex eigenvalues and eigenfunctions of the cable–damper linear system obtained by the hybrid method rather than sine or cosine functions should be used in the above equations for two main reasons. One is because the existence of oil dampers causes the eigenfunctions to have abrupt change at damper locations. The other is because the performance of the oil damper is sensitive to the frequency crossover or frequency avoidance phenomenon [6]. Consequently, the integrals involved in equations (47) and (48) should be evaluated numerically in a consistent way with the hybrid method.

In the hybrid method [5], the global eigenfunctions of the cable–damper system are obtained by assembling the local eigenfunctions for each small cable segment,

$$\Phi^{(k)}(x) = \sum_{i=1}^{N} \phi_i^{(k)}(x)|_{x\in[x_i,x_{i+1})}, \tag{49}$$

$$\Theta^{(k)}(x) = \sum_{i=1}^{N} \varphi_i^{(k)}(x)|_{x\in[x_i,x_{i+1})}, \tag{50}$$

$$\Psi^{(k)}(x) = \sum_{i=1}^{N} \psi_i^{(k)}(x)|_{x\in[x_i,x_{i+1})}, \tag{51}$$

where $\phi_i^{(k)}(x)$ and $\varphi_i^{(k)}(x)$ are the horizontal and vertical components of the kth in-plane complex eigenfunction associated with the ith cable segment, and $\psi_i^{(k)}(x)$ is the kth out-of-plane complex eigenfunction associated with the ith cable segment.

The connective conditions between two segments are the continuity of displacement and the equilibrium of nodal forces. That is, if node i is not equal to the nth node where the damper is installed, the connective conditions are

$$u_{i-1}(x_i^-) = u_i(x_i^+), \tag{52}$$

$$w_{i-1}(x_i^-) = w_i(x_i^+), \tag{53}$$

$$v_{i-1}(x_i^-) = v_i(x_i^+), \tag{54}$$

$$[H + h_i(x_i^+)]\frac{\partial u_i(x_i^+)}{\partial x} + h_i(x_i^+) - [H + h_{i-1}(x_i^-)]\frac{\partial u_{i-1}(x_i^-)}{\partial x} - h_{i-1}(x_i^-) = 0, \tag{55}$$

$$[H + h_i(x_i^+)]\frac{\partial w_i(x_i^+)}{\partial x} + h_i(x_i^+)y_{i,x} - [H + h_{i-1}(x_i^-)]\frac{\partial w_{i-1}(x_i^-)}{\partial x}$$

$$- h_{i-1}(x_i^-)y_{i-1,x} = 0, \tag{56}$$

$$[H + h_i(x_i^+)]\frac{\partial v_i(x_i^+)}{\partial x} - [H + h_{i-1}(x_i^-)]\frac{\partial v_{i-1}(x_i^-)}{\partial x} = 0. \tag{57}$$

If the ith node is equal to the nth node, the continuity conditions of displacement, equations (52)–(54), remain the same but the equilibrium conditions of forces at the nth node should be

$$[H + h_n(x_n^+)]\frac{\partial u_n(x_n^+)}{\partial x} + h_n(x_n^+) - [H + h_{n-1}(x_n^-)]\frac{\partial u_{n-1}(x_n^-)}{\partial x}$$

$$- h_{n-1}(x_n^-) = -f_x, \tag{58}$$

$$[H + h_n(x_n^+)]\frac{\partial w_n(x_n^+)}{\partial x} + h_n(x_n^+)y_{n,x} - [H + h_{n-1}(x_n^-)]\frac{\partial w_{n-1}(x_n^-)}{\partial x}$$

$$- h_{n-1}(x_n^-)y_{n-1,x} = -f_y, \tag{59}$$

$$[H + h_n(x_i^+)]\frac{\partial v_n(x_n^+)}{\partial x} - [H + h_{n-1}(x_i^-)]\frac{\partial v_{n-1}(x_n^-)}{\partial x} = -f_z. \tag{60}$$

In the above equations, the non-linear dynamic tension h should be used:

$$h_i = \frac{EA}{(1 + y_{i,x}^2)^{3/2}}\left[\frac{\partial u_i}{\partial x} + y_{i,x}\frac{\partial w_i}{\partial x} + \frac{1}{2}\left(\left(\frac{\partial u_i}{\partial x}\right)^2 + \left(\frac{\partial w_i}{\partial x}\right)^2 + \left(\frac{\partial v_i}{\partial x}\right)^2\right)\right]. \tag{61}$$

The subscript i indicates the ith segment except for the x-coordinate in which the subscript i indicates the ith node. The superscripts $-$ and $+$ mean x approaches x_i from the left and right sides of x_i respectively, and $y_{i,x}$ is the derivative of y with respect to x in the ith segment and is assumed a constant for each small segment. The assumption that $y_{i,x}$ is constant in each element has been checked through the numerical convergence study of the hybrid method.

In consideration of the connective conditions at each node and using local eigenfunctions and integration-by-parts formula, the integrals in equations (47) and (48) can be evaluated, and eventually these two equations become the ordinary differential equations governing the in-plane and out-of-plane non-linear vibration of the cable–damper system:

$$m_1^{(j)}\dot{q}_1^{(j)} + k_1^{(j)}q_1^{(j)} + \sum_{k=1,2}^{\infty}\sum_{k1=1,2}^{\infty} a_1^{(j,k,k1)}q_1^{(k)}q_1^{(k1)} + \sum_{k=1,2}^{\infty}\sum_{k1=1,2}^{\infty} a_2^{(j,k,k1)}q_2^{(k)}q_2^{(k1)}$$

$$+ \sum_{k=1,2}^{\infty}\sum_{k1=1,2}^{\infty}\sum_{k2=1,2}^{\infty} a_3^{(j,k,k1,k2)}q_1^{(k)}q_1^{(k1)}q_1^{(k2)}$$

$$+ \sum_{k=1,2}^{\infty}\sum_{k1=1,2}^{\infty}\sum_{k2=1,2}^{\infty} a_4^{(j,k,k1,k2)}q_2^{(k)}q_2^{(k1)}q_1^{(k2)} = Q_1^{(j)}, \tag{62}$$

$$m_2^{(j)}\dot{q}_2^{(j)} + k_2^{(j)}q_2^{(j)} + \sum_{k=1,2}^{\infty}\sum_{k1=1,2}^{\infty} b_1^{(j,k,k1)}q_1^{(k)}q_2^{(k1)}$$

$$+ \sum_{k=1,2}^{\infty}\sum_{k1=1,2}^{\infty}\sum_{k2=1,2}^{\infty} b_3^{(j,k,k1,k2)}q_2^{(k)}q_2^{(k1)}q_2^{(k2)}$$

$$+ \sum_{k=1,2}^{\infty}\sum_{k1=1,2}^{\infty}\sum_{k2=1,2}^{\infty} b_4^{(j,k,k1,k2)}q_1^{(k)}q_1^{(k1)}q_2^{(k2)} = Q_2^{(j)}, \tag{63}$$

where $j = 1, 2, \ldots,$

$$m_1^{(j)} = (2m\Omega_{(j)} + c_1)\sum_{i=1}^{N}\sqrt{1 + y_{i,x}^2}\int_{x_i}^{x_{i+1}}(\phi_i^{(j)}\phi_i^{(j)} + \varphi_i^{(j)}\varphi_i^{(j)})\,\mathrm{d}x$$

$$+ 2c\sin^2\gamma\{\phi_n^{(j)}(x_c)\phi_n^{(j)}(x_c)\cos^2\alpha - [\phi_n^{(j)}(x_c)\phi_n^{(j)}(x_c) + \phi_n^{(j)}(x_c)\varphi_n^{(j)}(x_c)]$$

$$\times\sin\alpha\cos\alpha + \varphi_n^{(j)}(x_c)\varphi_n^{(j)}(x_c)\sin^2\alpha\}, \tag{64}$$

$$m_2^{(j)} = (2m\Omega_{(j)}^{*} + c_2)\sum_{i=1}^{N}\sqrt{1 + y_{i,x}^2}\int_{x_i}^{x_{i+1}}\psi_i^{(j)}\psi_i^{(j)}\,\mathrm{d}x + 2c\psi_n^{(j)}(x_c)\psi_n^{(j)}(x_c)\cos^2\gamma, \tag{65}$$

$$k_1^{(j)} = \sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}[(H + \hat{k}_i)\phi_{i,x}^{(j)}\phi_{i,x}^{(j)} + 2\hat{k}_i y_{i,x}\phi_{i,x}^{(j)}\phi_{i,x}^{(j)} + (H + \hat{k}_i y_{i,x}^2)\varphi_{i,x}^{(j)}\varphi_{i,x}^{(j)}]\,\mathrm{d}x$$

$$- m\Omega_{(j)}^2\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\sqrt{1 + y_{i,x}^2}\,[\phi_{i,x}^{(j)}\phi_{i,x}^{(j)} + \varphi_{i,x}^{(j)}\varphi_{i,x}^{(j)}], \tag{66}$$

$$k_2^{(j)} = \sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}H\psi_{i,x}^{(j)}\psi_{i,x}^{(j)}\,\mathrm{d}x - m\Omega_{(j)}^{*2}\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\sqrt{1 + y_{i,x}^2}\psi_{i,x}^{(j)}\psi_{i,x}^{(j)}\,\mathrm{d}x, \tag{67}$$

$$a_1^{(j,k,k1)} = \sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}[0{\cdot}5\hat{k}_i(\phi_{i,x}^{(k)}\phi_{i,x}^{(k1)} + \varphi_{i,x}^{(k)}\varphi_{i,x}^{(k1)})(\phi_{i,x}^{(j)} + y_{i,x}\varphi_{i,x}^{(j)})$$

$$+ \hat{k}_i(\phi_{i,x}^{(k)} + y_{i,x}\varphi_{i,x}^{(k)})(\phi_{i,x}^{(j)}\phi_{i,x}^{(k1)} + \varphi_{i,x}^{(j)}\varphi_{i,x}^{(k1)})]\,\mathrm{d}x, \tag{68}$$

$$b_1^{(j,k,k1)} = \sum_{i=1}^{N} \hat{k}_i \int_{x_i}^{x_{i+1}} [(\phi_{i,x}^{(k)} + y_{i,x}\varphi_{i,x}^{(k)})\psi_{i,x}^{(k1)}\psi_{i,x}^{(j)}]\,\mathrm{d}x, \tag{69}$$

$$a_2^{(j,k,k1)} = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} 0{\cdot}5\hat{k}_i\psi_{i,x}^{(k)}\psi_{i,x}^{(k1)}(\phi_{i,x}^{(j)} + y_{i,x}\varphi_{i,x}^{(j)})\,\mathrm{d}x, \tag{70}$$

$$a_3^{(j,k,k1,k2)} = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} 0{\cdot}5\hat{k}_i(\phi_{i,x}^{(k)}\phi_{i,x}^{(k1)} + \varphi_{i,x}^{(k)}\varphi_{i,x}^{(k1)})(\phi_{i,x}^{(j)}\phi_{i,x}^{(k2)} + \varphi_{i,x}^{(j)}\varphi_{i,x}^{(k2)})\,\mathrm{d}x, \tag{71}$$

$$b_3^{(j,k,k1,k2)} = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} 0{\cdot}5\hat{k}_i\psi_{i,x}^{(k)}\psi_{i,x}^{(k1)}\psi_{i,x}^{(k2)}\psi_{i,x}^{(j)}\,\mathrm{d}x, \tag{72}$$

$$a_4^{(j,k,k1,k2,)} = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} 0{\cdot}5\hat{k}_i\psi_{i,x}^{(k)}\psi_{i,x}^{(k1)}(\phi_{i,x}^{(j)}\phi_{i,x}^{(k2)} + \varphi_{i,x}^{(j)}\varphi_{i,x}^{(k2)})\,\mathrm{d}x, \tag{73}$$

$$b_4^{(j,k,k1,k2)} = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} 0{\cdot}5\hat{k}_i(\phi_{i,x}^{(k)}\phi_{i,x}^{(k1)} + \varphi_{i,x}^{(k)}\varphi_{i,x}^{(k1)})\psi_{i,x}^{(k2)}\psi_{i,x}^{(j)}\,\mathrm{d}x, \tag{74}$$

$$Q_1^{(j)} = \sum_{i=1}^{N} \sqrt{1 + y_{i,x}^2}\int_{x_i}^{x_{i+1}} (f_x\phi_i^{(j)} + f_y\varphi_i^{(j)})\,\mathrm{d}x, \tag{75}$$

$$Q_2^{(j)} = \sum_{i=1}^{N} \sqrt{1 + y_{i,x}^2}\int_{x_i}^{x_{i+1}} f_z\psi_i^{(j)}\,\mathrm{d}x \tag{76}$$

in which

$$\hat{k}_i = \frac{EA}{(1 + y_{i,x}^2)^{3/2}}. \tag{77}$$

6. HARMONIC BALANCE METHOD

To find the solutions about the unknown time functions in equations (62) and (63), the harmonic balance method is employed. Assume that the external harmonic loads can be expressed as

$$Q_1^{(j)}(t) = A_{in}^{(j)}\sin(\omega t), \tag{78}$$

$$Q_2^{(j)}(t) = A_{out}^{(j)}\sin(\omega t), \tag{79}$$

and the time functions are of the form

$$q_1^{(j)} = R_{j,0} + \sum_{n=1}^{4} (R_{j,1}^{(n)}\cos n\omega t + R_{j,2}^{(n)}\sin n\omega t) \tag{80}$$

$$q_2^{(j)} = S_{j,0} + \sum_{n=1}^{4} (S_{j,1}^{(n)}\cos n\omega t + S_{j,2}^{(n)}\sin n\omega t) \tag{81}$$

where $R_{j,0}$, $R_{j,1}^{(n)}$, $R_{j,2}^{(n)}$, $S_{j,0}$, $S_{j,1}^{(n)}$ and $S_{j,2}^{(n)}$ are complex constants to be determined and satisfied with $R_{2m,0} = \bar{R}_{2m-1,0}$, $R_{2m,1}^{(n)} = \bar{R}_{2m-1,1}^{(n)}$, $R_{2m,2}^{(n)} = \bar{R}_{2m-1,2}^{(n)}$, $S_{2m,0} = \bar{S}_{2m-1,0}$, $S_{2m,1}^{(n)} = \bar{S}_{2m-1,1}^{(n)}$, $S_{2m,2}^{(n)} = \bar{S}_{2m-1,2}^{(n)} (m = 1, 2, \ldots)$. The superscript $^-$ is the conjugate symbol of a complex function.

Substituting equations (78)–(81) into equations (62) and (63) and equating the constants and the coefficients of $\cos(n\omega t)$and $\sin(n\omega t)$ yield

$$m_1^{(j)} g1_L^{jn} + k_1^{(j)} g2_L^{jn} + \sum_{k=1,2}^{\infty} \sum_{k1=1,2}^{\infty} a_1^{(j,k,k1)} g3_{L,k,k1}^{n} + \sum_{k=1,2}^{\infty} \sum_{k1=1,2}^{\infty} a_2^{(j,k,k1)} g4_{L,k,k1}^{n}$$
$$+ \sum_{k=1,2}^{\infty} \sum_{k1=1,2}^{\infty} \sum_{k2=1,2}^{\infty} a_3^{(j,k,k1,k2)} g5_{L,k,k1,k2}^{n}$$
$$+ \sum_{k=1,2}^{\infty} \sum_{k1=1,2}^{\infty} \sum_{k2=1,2}^{\infty} a_4^{(j,k,k1,k2)} g6_{L,k,k1,k2}^{n} = g7_L^{j}, \tag{82}$$

$$m_2^{(j)} p1_L^{jn} + k_2^{(j)} p2_L^{jn} + \sum_{k=1,2}^{\infty} \sum_{k1=1,2}^{\infty} b_1^{(j,k,k1)} p3_{L,k,k1}^{n}$$
$$+ \sum_{k=1,2}^{\infty} \sum_{k1=1,2}^{\infty} \sum_{k2=1,2}^{\infty} b_3^{(j,k,k1,k2)} p4_{L,k,k1,k2}^{n}$$
$$+ \sum_{k=1,2}^{\infty} \sum_{k1=1,2}^{\infty} \sum_{k2=1,2}^{\infty} b_4^{(j,k,k1,k2)} p5_{L,k,k1,k2}^{n} = p6_L^{j}, \tag{83}$$

where $j = 1, 2, \ldots$ and $L = 0, 1, 2$, corresponding to the equations about the constant, $\cos(n\omega t)$ and $\sin(n\omega t)$, respectively. Thus, for the jth time functions, equations (82) and (83) imply actually six complex equations or 12 real equations. $g1_L^{jn}$, $g2_L^{jn}$, $g3_{L,k,k1}^{n}$, $g4_{L,k,k1}^{n}$, $g5_{L,k,k1,k2}^{n}$, $g6_{L,k,k1,k2}^{n}$, $g7_L^{j}$, $p1_L^{jn}$, $p2_L^{jn}$, $p3_{L,k,k1}^{n}$, $p4_{L,k,k1,k2}^{n}$, $p5_{L,k,k1,k2}^{n}$ and $p6_L^{j}$ are functions of $R_{j,0}$, $R_{j,1}$, $R_{j,2}$, $S_{j,0}$, $S_{j,1}$ and $S_{j,2}$ listed in Appendix A.

Equations (82) and (83) are non-linear algebraic equations and can be solved by the Newton–Raphson method. Then, through equations (80) and (81) and (44)–(46) the non-linear vibration response of the cable–damper system can be finally determined.

The approach suggested here for a non-linear vibration analysis of the cable–damper system has been examined by checking the orthogonality properties of the complex modes of vibration of the cable–damper systems obtained by the hybrid method and by comparing the various coefficients in equations (64)–(76) with Irvine's theory [15] for the case of a linear system without oil damper. A convergence study on the required number of segments was also carried out to ensure that the constant assumption of $y_{i,x}$ in each element is valid and the complex eigenfunctions in the subsequent non-linear vibration study have a high accuracy. The results were found to be satisfactory and the details can be found in Yu's PhD thesis [16]. Further verification of the suggested approach is presented in Part II of this paper.

7. CONCLUSIONS

A formulation for determining the non-linear vibration response of a non-classically damped cable–damper system has been presented in this paper. The second order partial non-linear differential equations of motion of the system were first reduced to first-order non-linear partial differential equations. The orthogonality properties between the complex modes of vibration of the linear cable–damper system were then derived. Based on the derived orthogonality properties and the complex modes of vibration achieved by the hybrid method, the generalized modal superposition method was then applied to the first order non-linear partial differential equations to obtain the first order non-linear ordinary differential equations with respect to time functions only. The harmonic balance method was finally applied to convert the non-linear ordinary differential equations to non-linear algebraic equations, from which the real solutions for the time functions and non-linear cable dynamic responses were found. The formulation will be verified in Part II of this paper through a comparison with experimental results.

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APPENDIX A: PARAMETERS IN EQUATIONS (82) and (83)

The following are the parameters in equations (82) and (83) when n equal to 1.
(1) $L = 0$:

$$g1_0^{j1} = 0,$$

$$g2_0^{j1} = R_{j,0},$$

$$g3^1_{0,k,k1} = R_{k,0}R_{k1,0} + \frac{R^{(1)}_{k,1}R^{(1)}_{k1,1} + R^{(1)}_{k,2}R^{(1)}_{k1,2}}{2},$$

$$g4^1_{0,k,k1} = S_{k,0}S_{k1,0} + \frac{S^{(1)}_{k,1}S^{(1)}_{k1,1} + S^{(1)}_{k,2}S^{(1)}_{k1,2}}{2},$$

$$g5^1_{0,k,k1,k2} = R_{k,0}R_{k1,0}R_{k2,0} + \frac{R^{(1)}_{k1,1}R^{(1)}_{k2,1} + R^{(1)}_{k1,2}R^{(1)}_{k2,2}}{2}R_{k,0} + \frac{R^{(1)}_{k,1}R^{(1)}_{k2,1} + R^{(1)}_{k,2}R^{(1)}_{k2,2}}{2}R_{k1,0} + \frac{R^{(1)}_{k,1}R^{(1)}_{k1,1} + R^{(1)}_{k,2}R^{(1)}_{k1,2}}{2}R_{k2,0},$$

$$g6^1_{0,k,k1,k2} = S_{k,0}S_{k1,0}R_{k2,0} + \frac{S^{(1)}_{k1,1}R^{(1)}_{k2,1} + S^{(1)}_{k1,2}R^{(1)}_{k2,2}}{2}S_{k,0} + \frac{S^{(1)}_{k,1}R^{(1)}_{k2,1} + S^{(1)}_{k,2}R^{(1)}_{k2,2}}{2}S_{k1,0} + \frac{S^{(1)}_{k,1}S^{(1)}_{k1,1} + S^{(1)}_{k,2}S^{(1)}_{k1,2}}{2}R_{k2,0},$$

$$g7_0^j = 0,$$

$$p1_0^{j1} = 0,$$

$$p2_0^{j1} = S_{j,0},$$

$$p3^1_{0,k,k1} = R_{k,0}S_{k1,0} + \frac{R^{(1)}_{k,1}S^{(1)}_{k1,1} + R^{(1)}_{k,2}S^{(1)}_{k1,2}}{2},$$

$$p4^1_{0,k,k1,k2} = S_{k,0}S_{k1,0}S_{k2,0} + \frac{S^{(1)}_{k1,1}S^{(1)}_{k2,1} + S^{(1)}_{k1,2}S^{(1)}_{k2,2}}{2}S_{k,0}$$

$$+ \frac{S^{(1)}_{k,1}S^{(1)}_{k2,1} + S^{(1)}_{k,2}S^{(1)}_{k2,2}}{2}S_{k1,0} + \frac{S^{(1)}_{k,1}S^{(1)}_{k1,1} + S^{(1)}_{k,2}S^{(1)}_{k1,2}}{2}S_{k2,0},$$

$$p5^1_{0,k,k1,k2} = R_{k,0}R_{k1,0}S_{k2,0} + \frac{R^{(1)}_{k1,1}S^{(1)}_{k2,1} + R^{(1)}_{k1,2}S^{(1)}_{k2,2}}{2}R_{k,0}$$

$$+ \frac{R^{(1)}_{k,1}S^{(1)}_{k2,1} + R^{(1)}_{k,2}S^{(1)}_{k2,2}}{2}R_{k1,0} + \frac{R^{(1)}_{k,1}R^{(1)}_{k1,1} + R^{(1)}_{k,2}R^{(1)}_{k1,2}}{2}S_{k2,0},$$

$$p6^1_0 = 0.$$

(2) $L = 1$:

$$g1^{j1}_1 = \omega R^{(1)}_{j,2},$$

$$g2^{j1}_1 = R^{(1)}_{j,1},$$

$$g3^1_{1,k,k1} = R^{(1)}_{k,1}R_{k1,0} + R_{k,0}R^{(1)}_{k1,1},$$

$$g4^1_{1,k,k1} = S^{(1)}_{k,1}S_{k1,0} + S_{k,0}S^{(1)}_{k1,1},$$

$$g5^1_{1,k,k1,k2} = R_{k,0}R_{k1,0}R^{(1)}_{k2,1} + R_{k,0}R^{(1)}_{k1,1}R_{k2,0} + R^{(1)}_{k,1}R_{k1,0}R_{k2,0} + \frac{3R^{(1)}_{k,1}R^{(1)}_{k1,1}R^{(1)}_{k2,1}}{4}$$

$$+ \frac{R^{(1)}_{k,1}R^{(1)}_{k1,2}R^{(1)}_{k2,2} + R^{(1)}_{k,2}R^{(1)}_{k1,1}R^{(1)}_{k2,2} + R^{(1)}_{k,2}R^{(1)}_{k1,2}R^{(1)}_{k2,1}}{4},$$

$$g6^1_{1,k,k1,k2} = S_{k,0}S_{k1,0}R^{(1)}_{k2,1} + S_{k,0}S^{(1)}_{k1,1}R_{k2,0} + S^{(1)}_{k,1}S_{k1,0}R_{k2,0} + \frac{3S^{(1)}_{k,1}S^{(1)}_{k1,1}R^{(1)}_{k2,1}}{4}$$

$$+ \frac{S^{(1)}_{k,1}S^{(1)}_{k1,2}R^{(1)}_{k2,2} + S^{(1)}_{k,2}S^{(1)}_{k1,1}R^{(1)}_{k2,2} + S^{(1)}_{k,2}S^{(1)}_{k1,2}R^{(1)}_{k2,1}}{4},$$

$$g7^j_1 = 0,$$

$$p1^{j1}_1 = \omega S^{(1)}_{j,2},$$

$$p2^{j1}_1 = S^{(1)}_{j,1},$$

$$p3^1_{1,k,k1} = R^{(1)}_{k,1}S_{k1,0} + R_{k,0}S^{(1)}_{k1,1},$$

$$p4^1_{1,k,k1,k2} = S_{k,0}S_{k1,0}S^{(1)}_{k2,1} + S_{k,0}S^{(1)}_{k1,1}S_{k2,0} + S^{(1)}_{k,1}S_{k1,0}S_{k2,0} + \frac{3S^{(1)}_{k,1}S^{(1)}_{k1,1}S^{(1)}_{k2,1}}{4}$$

$$+ \frac{S^{(1)}_{k,1}S^{(1)}_{k1,2}S^{(1)}_{k2,2} + S^{(1)}_{k,2}S^{(1)}_{k1,1}S^{(1)}_{k2,2} + S^{(1)}_{k,2}S^{(1)}_{k1,2}S^{(1)}_{k2,1}}{4},$$

$$p5^1_{1,k,k1,k2} = R_{k,0}R_{k1,0}S^{(1)}_{k2,1} + R_{k,0}R^{(1)}_{k1,1}S_{k2,0} + R^{(1)}_{k,1}R_{k1,0}S_{k2,0} + \frac{3R^{(1)}_{k,1}R^{(1)}_{k1,1}S^{(1)}_{k2,1}}{4}$$

$$+ \frac{R^{(1)}_{k,1}R^{(1)}_{k1,2}S^{(1)}_{k2,2} + R^{(1)}_{k,2}R^{(1)}_{k1,1}S^{(1)}_{k2,2} + R^{(1)}_{k,2}R^{(1)}_{k1,2}S^{(1)}_{k2,1}}{4},$$

$$p6^j_1 = 0.$$

(3) $L = 2$:

$$g1^{j1}_2 = -\omega R^{(1)}_{j,1},$$

$$g2^{j1}_2 = R^{(1)}_{j,2},$$

$$g3^1_{2,k,k1} = R_{k,0}R^{(1)}_{k1,2} + R_{k1,0}R^{(1)}_{k,2},$$

$$g4^1_{2,k,k1} = S_{k.0}S^{(1)}_{k1,2} + S_{k1,0}S^{(1)}_{k,2},$$

$$g5^1_{2,k,k1,k2} = R_{k,0}R_{k1,0}R^{(1)}_{k2,2} + R_{k,0}R^{(1)}_{k1,2}R_{k2,0} + R^{(1)}_{k,2}R_{k1,0}R_{k2.0} + \frac{3R^{(1)}_{k,2}R^{(1)}_{k1,2}R^{(1)}_{k2,2}}{4}$$

$$+ \frac{R^{(1)}_{k,1}R^{(1)}_{k1,1}R^{(1)}_{k2,2} + R^{(1)}_{k,1}R^{(1)}_{k1,2}R^{(1)}_{k2,1} + R^{(1)}_{k,2}R^{(1)}_{k1,1}R^{(1)}_{k2,1}}{4},$$

$$g6^1_{2,k,k1,k2} = S_{k,0}S_{k1,0}R^{(1)}_{k2,2} + S_{k,0}S^{(1)}_{k1,2}R_{k2,0} + S^{(1)}_{k,2}S_{k1,0}R_{k2,0} + \frac{3S^{(1)}_{k,2}S^{(1)}_{k1,2}R^{(1)}_{k2,2}}{4}$$

$$+ \frac{S^{(1)}_{k,1}S^{(1)}_{k1,1}R^{(1)}_{k2,2} + S^{(1)}_{k,1}S^{(1)}_{k1,2}R^{(1)}_{k2,1} + S^{(1)}_{k,2}S^{(1)}_{k1,1}R^{(1)}_{k2,1}}{4},$$

$$g7^j_2 = A^{(j)}_{in},$$

$$p1^{j1}_2 = -\omega S^{(1)}_{j,1},$$

$$p2^{j1}_2 = S^{(1)}_{j,2},$$

$$p3^1_{2,k,k1} = R^{(1)}_{k,2}S_{k1,0} + R_{k,0}S^{(1)}_{k1,2},$$

$$p4^1_{2,k,k1,k2} = S_{k,0}S_{k1,0}S^{(1)}_{k2,2} + S_{k,0}S^{(1)}_{k1,2}S_{k2,0} + S^{(1)}_{k,2}S_{k1,0}S_{k2,0} + \frac{3S^{(1)}_{k,2}S^{(1)}_{k1,2}S^{(1)}_{k2,2}}{4}$$

$$+ \frac{S^{(1)}_{k,1}S^{(1)}_{k1,1}S^{(1)}_{k2,2} + S^{(1)}_{k,1}S^{(1)}_{k1,2}S^{(1)}_{k2,1} + S^{(1)}_{k,2}S^{(1)}_{k1,1}S^{(1)}_{k2,1}}{4},$$

$$p5^1_{2,k,k1,k2} = R_{k,0}R_{k1,0}S^{(1)}_{k2,2} + R_{k,0}R^{(1)}_{k1,2}S_{k2,0} + R^{(1)}_{k,2}R_{k1,0}S_{k2,0} + \frac{3R^{(1)}_{k,2}R^{(1)}_{k1,2}S^{(1)}_{k2,2}}{4}$$

$$+ \frac{R^{(1)}_{k,1}R^{(1)}_{k1,1}S^{(1)}_{k2,2} + R^{(1)}_{k,1}R^{(1)}_{k1,2}S^{(1)}_{k2,1} + R^{(1)}_{k,2}R^{(1)}_{k1,1}S^{(1)}_{k2,1}}{4},$$

$$p6^j_2 = A^{(j)}_{out}.$$