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VIBRATIONS OF THICK ROTATING LAMINATED COMPOSITE CYLINDRICAL SHELLS

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Free vibrations of thick rotating cross-ply laminated composite cylindrical shells are studied based on the first order shear deformation shell theory (FSDT). The governing equations of five degrees of freedom with consideration of Coriolis accelerations and rotary inertias are established. Analytical solutions to the equations are obtained to calculate the frequencies of the shells. Numerical results are presented and compared with those available in the literature. In addition, frequency characteristics of thin and thick shells are investigated with respect to the variations of rotating speeds, circumferential wave numbers, and length and thickness ratio.

1. INTRODUCTION

The dynamic behavior of thin rotating shells has been studied for over a century. Analytical techniques have remained to be developed on the basis of classical thin shell theories for a long time, which assume that straight lines normal to the middle surface before deformation remain straight, inextensible and normal to the middle surface after deformation. Transverse shear and transverse normal effects have been neglected. There are a substantial number of books and papers on this subject [1–4].

Recently, with the development of aerospace technology, attention has been drawn to the theoretical development of thick shells and composite laminated shells. Researchers have found that application of classical thin shell theory to laminated thick shells could lead to as much as 30% or more errors in natural frequencies [5]. The accuracy of classical thin shell theory is thus not sufficient when applied to thick shells. The modern use of laminated composite thick shells and the inaccuracy of classical thin shell theory have prompted researchers to develop new theories to calculate frequency characteristics of thick shells.

In the literature, analyses of composite shells are carried out on the basis of two-dimensional (2-D) shear deformation laminate theories, layerwise theories and numerical methods. Barbero and Reddy [6] developed a general 2-D theory of laminated cylindrical shells. The theory accounts for a desired degree of approximation of the displacements through the thickness. Voyiadjis and Shi [7] presented a refined 2-D theory for thick cylindrical shells, which made a very good approximation for the shell constitutive equation and the non-linear distributions of in-plane stresses across the thickness of the shell. Jing and Tzeng [8] proposed a refined shear deformation theory of laminated shells. The effect of transverse shear deformation is included through an independently assumed transverse shear force field. Huang and Dasgupta [9] developed a layer-wise analysis for free vibration of thick composite cylindrical shells. The displacement field was modelled by finite-element interpolation shape functions along the thickness direction. Gautham and Ganesan [10] presented free-vibration analysis of orthotropic thick shells of revolution using discrete layer theory. A two-noded finite element was presented for the analysis of thick orthotropic laminated shells. Timarci and Soldatos [11] presented comparative dynamic studies for symmetrical cross-ply circular cylindrical shells on the basis of a unified shear-deformable shell theory. The theory used a general shape function, which takes into account the shear deformation effects. Lam and Loy [12, 13] considered Coriolis accelerations in the vibration analysis of rotating laminated composite cylindrical shells. The main difference between the present method and that of Werner's [14] is that present method considers the initial curvature Z/R in the stress–strain relationships. Generally speaking, some theories are too complicated to have analytical solutions for thick rotating laminated composite shells, and some still have the classical thin shell theory as their basis. Up to now, analytical solutions and frequency characteristics for the vibrations of rotating laminated composite cylindrical thick shells have been given very little attention.

In this paper, an analytical solution of frequency characteristics for the vibrations of rotating laminated composite cylindrical thick shells is presented by using the first order shear deformation theory. Compared with classical theory with higher order theory, the first order shear deformation theory combines higher accuracy and lower calculation efforts. The objective of this study is to examine the difference in frequency characteristics between thick shells and thin shells. Numerical results for a long non-rotating cylindrical thin shell, a long rotating cylindrical thick shell, a short rotating cylindrical laminated shell and a short non-rotating laminated thick cylindrical shell are presented to compare with those available in the literature. For reasons of simplicity, the boundary conditions are simply supported at both ends of the shells. Figures are given to show variation of frequency with the rotating speed, the circumfrential wave number n, the H/R ratio, the L/R ratio for a rotating laminated composite cylindrical thick shell. The formulation is general. Different boundary conditions, lamination schemes (which may be isotropic or othotropic), order of shear deformation theories, and even forms of assumed solutions can be easily accommodated into the analysis. This is the first time analytical solutions have been applied to parametric studies of thick rotating cross-ply laminated composite cylindrical shells.

2. THEORETICAL FORMULATION

The geometry of the shell and coordinate system are shown in Figure 1. The cylindrical shell is assumed to have length L, thickness H, and radius R, and is also

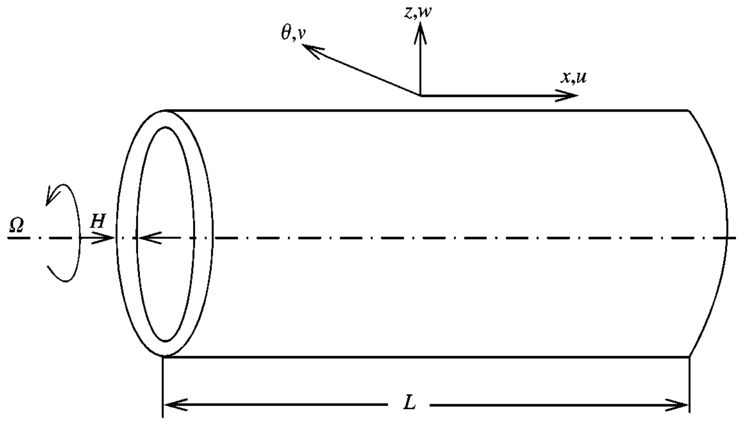

Figure 1. Geometry and co-ordinate system.

assumed to rotate about its horizontal axis with a constant velocity Ω. Both ends of the shell are simply supported. The orthogonal co-ordinate system (x, θ, z) is fixed at the mid surface of the cylindrical shell. X is the axial direction, θ is the circumferential direction and z is the radial direction. The deformations of the shell are defined by u, v, w, ϕ_x, and ϕ_θ, which are displacements of the point in x, θ, z and the rotations of the transverse normal about the θ and x axis respectively:

$$\partial u/\partial z = \phi_x, \qquad \partial v/\partial z = \theta_\theta. \tag{1}$$

First order theory is used to deal with the influence of shear forces on the frequencies of the shell. In the first order shear deformation laminated theory, the Kirchhoff hypothesis is relaxed by not constraining the transverse normals to remain perpendicular to the midsurface after deformation. This amounts to including transverse shear strains in the theory. The inextensiblity of transverse normals requires that w be independent of the thickness co-ordinate z.

According to the first order shear deformation laminated theory, the displacement fields are of the form (see reference [15])

$$\begin{aligned} u(x, \theta, z, t) &= u_0(x, \theta, t) + z\phi_x(x, \theta, t), \\ v(x, \theta, z, t) &= v_0(x, \theta, t) + z\phi_\theta(x, \theta, t), \\ w(x, \theta, z, t) &= w_0(x, \theta, t), \end{aligned} \tag{2}$$

where u_0, v_0, w_0, ϕ_x, and ϕ_θ are unknowns to be determined u_0, v_0, w_0, ϕ_x, and ϕ_θ are the displacements of a point on the surface $z = 0$ and the rotations of transverse normal about its θ and x-axis respectively.

Based on Hamilton's principle, with consideration of Coriolis accelerations, the equations of motion in terms of the forces and moment resultants can be written as

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{\theta x}}{R\partial\theta} = I_0\ddot{u}_0 + I_1\ddot{\phi}_x,$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_\theta}{R\partial\theta} = I_0\ddot{v} + I_1\ddot{\phi}_\theta + 2I_0\Omega\dot{w}_0 - I_0\Omega^2 v_0,$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_\theta}{R\partial\theta} - \frac{N_x}{R} = I_0\ddot{w}_0 - 2I_0\Omega\dot{v}_0 - I_0 w_0\Omega^2,$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{\theta x}}{R\partial\theta} - Q_x = I_1\ddot{u}_0 + I_2\ddot{\phi}_\theta,$$

$$\frac{\partial M_{x\theta}}{\partial x} + \frac{\partial M_\theta}{R\partial\theta} + Q_\theta = I_1\ddot{v}_0 + I_2\ddot{\phi}_\theta, \tag{3}$$

where Ω is the angular velocity. The force and moment resultants for a thick shell are defined respectively by

$$\{N_x, N_\theta, N_{x\theta}, N_{\theta x}\} = \sum_{k=1}^{N} \int_{Z_{k+1}}^{Z_k} \{\sigma_x(1 + z/R), \sigma_\theta, \sigma_{x\theta}(1 + z/R), \sigma_{x\theta}\}\, \mathrm{d}z,$$

$$\{M_x, M_\theta, M_{x\theta}, M_{\theta x}\} = \sum_{k=1}^{N} \int_{Z_{k+1}}^{Z_k} \{\sigma_x(1 + z/R), \sigma_\theta, \sigma_{x\theta}(1 + z/R), \sigma_{x\theta}\}\, \mathrm{d}z,$$

$$\{Q_x, Q_\theta\} = \sum_{k=1}^{N} \int_{Z_{k+1}}^{Z_k} \{\sigma_{xz}(1 + z/R), \sigma_{x\theta}\}\, z\, \mathrm{d}z \tag{4}$$

and the mass moments of inertia are

$$(I_0, I_1, I_2) = \sum_{K=1}^{N} \int_{Z_{k+1}}^{Z_k} \rho^k(1, z, z^2)\, \mathrm{d}z. \tag{5}$$

We should note that for a thick shell the term z/R in the force and moment resultants is so large that it cannot be neglected.

For orthotropic layers, the compliance matrix and stress–strain relation in the material co-ordinates are of the form

$$[S_{ij}] = \begin{bmatrix} 1/E_1 & -v_{21}/E_2 & -v_{31}/E_3 & 0 & 0 & 0 \\ -v_{12}/E_1 & 1/E_2 & -v_{32}/E_3 & 0 & 0 & 0 \\ -v_{13}/E_1 & -v_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix},$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}. \tag{6}$$

As the global co-ordinate system used in the solution of a problem does not coincide with the material co-ordinate system, coupled with the fact that this composite laminated shell has several layers each with different orientations, we need to establish the transformation relation

$$[\bar{Q}] = [T][Q][T]^{\mathrm{T}}, \tag{7}$$

where

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 2\sin\theta\cos\theta \\ 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & 0 & 0 & \cos^2\theta - \sin^2\theta \end{bmatrix}.$$

Now we obtain the stress–strain relations in the global co-ordinate:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_{\theta z} \\ \sigma_{xz} \\ \sigma_{x\theta} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & \bar{Q}_{26} \\ 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \gamma_{\theta z} \\ \gamma_{xz} \\ \gamma_{x\theta} \end{Bmatrix}. \tag{8}$$

The strains for shell are defined by

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \gamma_{\theta z} \\ \gamma_{xz} \\ \gamma_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{R\partial\theta} + \dfrac{w}{R} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{R\partial\theta} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{R\partial\theta} \end{Bmatrix}. \tag{9}$$

By using first order theory and substituting equation (2) into equation (9), we obtain

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \gamma_{\theta z} \\ \gamma_{xz} \\ \gamma_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_\theta^0 \\ \gamma_{\theta z}^0 \\ \gamma_{xz}^0 \\ \gamma_{x\theta}^0 \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_x^1 \\ \varepsilon_\theta^1 \\ \gamma_{\theta z}^1 \\ \gamma_{xz}^1 \\ \gamma_{x\theta}^1 \end{Bmatrix}, \qquad \{\varepsilon\} = \{\varepsilon^0\} + z\{\varepsilon^1\}, \tag{10}$$

where

$$\{\varepsilon^0\} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_\theta^0 \\ \gamma_{\theta z}^0 \\ \gamma_{xz}^0 \\ \gamma_{x\theta}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{w_0}{R} + \dfrac{\partial v_0}{R\partial\theta} \\ \phi_\theta + \dfrac{\partial w_0}{R\partial\theta} \\ \phi_x + \dfrac{\partial w_0}{\partial x} \\ \dfrac{\partial v_0}{\partial x} + \dfrac{\partial u_0}{R\partial\theta} \end{Bmatrix}, \qquad \{\varepsilon^1\} = \begin{Bmatrix} \varepsilon_x^1 \\ \varepsilon_\theta^1 \\ \gamma_{\theta z}^1 \\ \gamma_{xz}^1 \\ \gamma_{x\theta}^1 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_0}{R\partial\theta} \\ 0 \\ 0 \\ \dfrac{\partial \phi_\theta}{\partial x} + \dfrac{\partial \phi_x}{R\partial\theta} \end{Bmatrix}. \tag{11}$$

By substituting equation (11) into equation (4), the force and moment resultants can be obtained for cross-ply laminated shells:

$$N_x = A_{11}\frac{\partial u_0}{\partial x} + A_{12}\left(\frac{w_0}{R} + \frac{\partial v_0}{R\partial\theta}\right) + \frac{1}{R}D_{11}\frac{\partial\phi_x}{\partial x} + \frac{1}{R}D_{12}\frac{\partial\phi_\theta}{R\partial\theta},$$

$$N_\theta = A_{12}\frac{\partial u_0}{\partial x} + A_{22}\left(\frac{w_0}{R} + \frac{\partial v_0}{R\partial\theta}\right),$$

$$N_{x\theta} = A_{66}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{R\partial\theta}\right) + \frac{1}{R}D_{66}\left(\frac{\partial\phi_\theta}{\partial x} + \frac{\partial\phi_x}{R\partial\theta}\right),$$

$$N_{\theta x} = A_{66}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{R\partial\theta}\right),$$

$$M_x = D_{11}\frac{\partial\phi_x}{\partial x} + \frac{1}{R}D_{12}\frac{\partial\phi_\theta}{\partial\theta} + \frac{1}{R}D_{11}\frac{\partial u_0}{\partial x} + \frac{1}{R}D_{12}\left(\frac{w_0}{R} + \frac{\partial v_0}{R\partial\theta}\right)$$

$$M_\theta = D_{12}\frac{\partial\phi_x}{\partial x} + \frac{1}{R}D_{22}\frac{\partial\phi_\theta}{\partial\theta}$$

$$M_{x\theta} = D_{66}\left(\frac{\partial\phi_\theta}{\partial x} + \frac{\partial\phi_x}{R\partial\theta}\right) + \frac{1}{R}D_{66}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{R\partial\theta}\right)$$

$$M_{\theta x} = D_{66}\left(\frac{\partial\phi_\theta}{\partial x} + \frac{\partial\phi_x}{R\partial\theta}\right)$$

$$Q_x = KA_{55}\left(\phi_x + \frac{\partial w_0}{\partial x}\right)$$

$$Q_\theta = KA_{44}\left(\phi_\theta + \frac{\partial w_0}{R\partial\theta}\right) \tag{12}$$

where the extension and bending stiffness are defined by

$$A_{ij} = \sum_{K=1}^{N}\int_{Z_{k+1}}^{Z_k}\bar{Q}_{ij}\,\mathrm{d}z, \qquad D_{ij} = \sum_{K=1}^{N}\int_{Z_{k+1}}^{Z_k}\bar{Q}_{ij}z^2\,\mathrm{d}z \tag{13}$$

where Z_k, Z_{k+1} denote the distances of the Kth layer and $(K+1)$th layer from the shell reference surface. Substitution of equation (11) into equation (3) yields five equations of motion in matrix form, namely,

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix}\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \phi_0 \\ \phi_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{14}$$

where $L_{ij}\,(i,\ j = 1, 5)$ are the differential operators in terms of five unknowns u_0, v_0, w_0, ϕ_x, ϕ_θ (see Appendix A).

Up to now, the analysis has been general without reference to the boundary conditions. For reasons of simplicity, only simply supported boundary condition (S3 in accordance with Nosier and Reddy [16]) are considered along all edges for the rotating shell, namely, $w(0, L) = v(0, L) = N_x(0, L) = M_x = \phi_x(0, L)$.

The displacement fields which satisfy the above boundary conditions can be written as

$$u_0 = A\cos\left(\frac{m\pi x}{l}\right)\cos(n\theta + \omega t),$$

$$v_0 = B\sin\left(\frac{m\pi x}{l}\right)\sin(n\theta + \omega t),$$

$$w_0 = C\sin\left(\frac{m\pi x}{l}\right)\cos(n\theta + \omega t),$$

$$\phi_x = D\cos\left(\frac{m\pi x}{l}\right)\cos(n\theta + \omega t),$$

$$\phi_\theta = E\sin\left(\frac{m\pi x}{l}\right)\sin(n\theta + \omega t). \tag{15}$$

where A, B, C, D and E are displacement amplitudes, m and n are the axial and circumferential wave numbers respectively, and ω is the natural frequency (rad/s).

Substituting these displacement fields into equation (13) yields

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \\ E \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{16}$$

where $C_{ij}\,(i, j = 1, 5)$ are the operators in terms of five unknowns u_0, v_0, w_0, ϕ_x, and ϕ_θ (see Appendix A).

Since the matrix equation is, in general, satisfied only if the determinant of the matrix is zero, we obtain a 10th-order equation in ω. When rotating speed is zero, it reduces to a fifth-order algebraic equation in ω^2, i.e., there are five paris of distinct frequencies for every m and n combination. The roots of these equations are the natural frequencies. The positive and negative roots of each pair of frequencies are defined by forward- and backward-travelling frequencies respectively. From the five pairs of roots, we find modes, which correspond to the deflections u_0, v_0, w_0, ϕ_x, and ϕ_θ. In addition, we find that the lowest of the pairs of frequencies is associated with the mode where the transverse displacement component W_0 dominates.

3. NUMERICAL RESULTS AND DISCUSSION

To examine the present analysis, numerical results are presented to compare with those available in the literature. The comparison for a long isotropic cylindrical thin shell is presented in Table 1 against results presented by Chen *et al.* [17] and Werner's [14] simplified theory. The parameters for this cylindrical shell are $m = 1$, $L/R = 200$, $K = 5/6$, $G_{12} = 0{\cdot}5E_2$, $V_{12} = 0{\cdot}3$ and $H/R = 0{\cdot}002$. The frequency parameter is $\sqrt{\rho R^2(1 - v_{12}^2)/E_2}$.

The results of comparison with a very long rotating cylindrical thick shell with simply supported boundary conditions are presented in Table 2, where the non-dimensional frequency parameter is $\sqrt{\rho R^2(1 - v_{12}^2)/E_2}$. The angular speed is $\Omega = 0{\cdot}01$ rad/s. The parameters for this shell are $m = 1$, $L/R = 200$, $K = 5/6$, $G_{12} = 0{\cdot}5E_2$, $V_{12} = 0{\cdot}3$, and $H/R = 0{\cdot}2$.

From the results presented in the above two tables, it can be seen that when the circumferential wave number n becomes large, the results tend to agree with those available in the literature. But when the circumferential wave number n is equal to

TABLE 1

Comparison of the frequency parameter ($\omega' = \omega\sqrt{\rho R^2(1-v_{12}^2)/E_2}$) *for a long non-rotating cylindrical thin shell* ($m = 1$, $H/R = 0{\cdot}002$, $L/R = 200$, $v = 0{\cdot}3$)

n	Chen	Werner simplified	Present
2	0·00154919	0·00200014	0·00206636
3	0·00438178	0·00489912	0·00492965
4	0·00840168	0·00894441	0·00896183
5	0·0135873	0·0141423	0·0141533
6	0·0199323	0·020494	0·0205014
7	0·0274343	0·0280001	0·028005
8	0·0360922	0·0366607	0·0366635

The equation of Chen *et al.* gives

$$\omega_f = \frac{2n}{n^2+1}\Omega + \sqrt{\frac{n^2(n^2-)^2}{n^2+1}\frac{Eh^2}{\rho(1-v^2)12r^2} + \frac{n^4+3}{(n^2+1)^2}\Omega^2},$$

$$\omega_b = \frac{2n}{n^2+1}\Omega - \sqrt{\frac{n^2(n^2-1)^2}{n^2+1}\frac{Eh^2}{\rho(1-v^2)12r^2} + \frac{n^4+3}{(n^2+1)^2}\Omega^2},$$

Subscripts b and f are the backward and forward waves respectively.
The equation of Werner gives

$$\omega_{mn} = \sqrt{\frac{E}{12\rho(1-v^2)}\frac{h^2}{r^2}\frac{1}{r^2}\left[\left(\frac{m\pi r}{L}\right)^2 + n^2\right]\left[\left(\frac{m\pi r}{L}\right)^2 + n^2 - 1\right]}.$$

TABLE 2

Comparison of the frequency parameter ($\omega' = \omega\sqrt{\rho R^2(1-v_{12}^2)/E_2}$) *for a long rotating cylindrical thick shell* ($m = 1$, $\Omega = 0{\cdot}01$, $H/R = 0{\cdot}2$, and $L/R = 200$)

n	Werner simplified	Werner's theory		Present	
		ω'_b	ω'_f	ω'_b	ω'_f
1	0·000950806	0·000428968	0·000438266	0·0425013	0·0425106
2	0·209672	0·157745	0·157752	0·210515	0·210523
3	0·513568	0·431536	0·431542	0·486108	0·486114
4	0·937629	0·793428	0·793432	0·847591	0·847595
5	1·22303	1·22303	1·22304	1·27603	1·27603
6	2·14836	1·70367	1·70368	1·75521	1·75522
7	2·93521	2·2221	2·2221	2·2721	2·2721

1, the results are different. As we know, many factors influence of frequencies. The final frequencies are obtained by a 5×5 matrix. However, circumferential wave number n plays a key role in frequency characteristics. Frequencies are very sensitive to the change of circumferential wave number n. When circumferential wave number n is large, circumferential wave number n is dominant compared with other factors. When circumferential wave number n is small, the effects of other factors may be the same order as circumferential wave number n. Other factors may vary from theory to theory, depending on the assumptions made. For instance, Werner's theory does not take the shell's initial curvature into account. This makes the two results different. The above point is also evident in Chen's expression attained in Table 1.

In order to prove the validity of this analysis in short ($L/R < 5$) shells, the results are presented in Tables 3 and 4. Table 3 compares the results for a short rotating cylindrical laminated shell with simply supported boundary conditions with the results of Lam and Loy [12]. The non-dimensional frequency parameter is $\sqrt{\rho R^2(1-v_{12}^2)/E_2}$. The angular speed $\Omega = 0{\cdot}1$ rad/s. The parameters for this shell are $m=1$, $L/R=1$ and 5, $K=5/6$, $V_{12}=0{\cdot}26$. $H/R=0{\cdot}002$, $G_{13}=G_{12}$, $G_{23}=0{\cdot}2E_2$, and $E_1=2{\cdot}5E_2$.

A comparison of the frequencies for a short non-rotating laminated thick cylindrical shell with simply supported boundary conditions is presented in Table 4. PAR (parabolic) HYP (hyperbolic), and UNI (uniform) are shear deformation shape functions, and PSDT is parabolic shear deformable shell theory (see reference [11]). The frequency parameter is $L^2\sqrt{\rho E_2/h}$. The parameters for this shell are $m=1$, $n=2$, $H/R=0{\cdot}2$, $E_1=E_2^*40$, $K=5/6$. $G_{23}=0{\cdot}5E_2$, and $G_{12}=0{\cdot}6E_2$.

From the results in Tables 3 and 4, we can see that the present analysis is consistent with many other theories for a short laminated thick cylindrical shell with simply supported boundary conditions. In other words, the boundary conditions have been correctly taken into consideration.

In summary, comparing Tables 1–4, we can see that very good agreement is obtained even for short thick laminated composite cylindrical shells. As we know, the thicker the shell, the larger the influence of shear forces on frequencies. Meanwhile, the shorter the shell, the larger the influence of boundary conditions on the natural frequencies. Furthermore, when the circumferential wave number n is

TABLE 3

Comparison of the non-dimensional frequency parameter $\sqrt{\rho R^2(1-v_{12}^2)/E_2}$) *for a short rotating cylindrical laminated shell* ($m=1$, $\Omega=0{\cdot}1$)

n	Loy $L/R=1$		Present $L/R=1$		Loy $L/R=5$		Present $L/R=5$	
	ω_b'	ω_f'	ω_f'	ω_b'	ω_f'	ω_b'	ω_f'	ω_b'
1	1·061429	1·061140	1·06126	1·06131	0·248917	0·248917	0·24859	0·24868
2	0·804214	0·803894	0·804033	0·804084	0·107436	0·106972	0·107181	0·107254
3	0·598476	0·598187	0·598325	0·598371	0·055267	0·054916	0·0551545	0·0552104
4	0·450270	0·450021	0·450171	0·45021	0·033945	0·033669	0·0341573	0·0342011
5	0·345363	0·345356	0·345348	0·345382	0·25943	0·025718	0·0268394	0·0268753
6	0·270852	0·270667	0·270988	0·271018	0·026026	0·025836	0·0278363	0·0278665
7	0·217651	0·217489	0·218059	0·218085	0·031089	0·030925	0·0337889	0·033815

TABLE 4

Comparison of the frequency parameter $\omega' = \omega L^2\sqrt{\rho E_2}/h$ for a short non-rotating laminated thick cylindrical shell ($m = 1$, $H/R = 0{\cdot}2$, $E1 = E2*40$, $K = 5/6$, $n = 2$, $G_{12} = 0{\cdot}6E_2$, and $G_{23} = 0{\cdot}5E_2$)

Thoery	$L/R = 1$, $n = 2$	$L/R = 2$, $n = 2$
Present	10·1438	18·8438
PAR	9·97	17·16
PSDT	10·07	17·77
HYP	9·99	17·16
UNI	9·99	17·16
CST	14·77	20·17

TABLE 5

Properties of a laminated composite cylindrical shell

Material properties	Layer thickness
$E_2 = 7{\cdot}6 \times 10^9$ n/m^2, $E_1 = 2{\cdot}5E_2$	Inner layer thickness = $H/3$
$\rho = 1643$ kg/m^3, $v_{12} = 0{\cdot}26$	Middle layer thickness = $H/3$
$G_{12} = G_{13}$, $G_{23} = 0{\cdot}2G_{12}$	Outer layer thickness = $H/3$
$k = 5/6$	0/90/0

small, such as $n = 1$, the influence of boundary conditions on frequencies may be dominant. The combination of the three factors creates a difficult situation for frequency calculation, that is, the shell is short and thick, and the circumferential wave number n is small.

Many theories used to calculate frequencies of a thick short shell diverge when the circumferential wave number n is 1. Choosing different orders of shear deformation theories and different forms of an assumed solution is planned for our further work and it may lead to more accurate solutions, but first order theory is still one of the simplest ways to solve this kind of problem.

In this paper, parametric studies are focused on the differences in the frequency characteristics between thin and thick shells. Properties of the laminated composite cylindrical shell are listed in Table 5. In order to study frequency characteristics, the combination $(m, n) = (1, 1)$ is chosen to compare with thin shells and thick shells. The combination of $(m, n) = (1, 1)$ may not correspond to the fundamental frequencies for a thin shell.

Figures 2, 3(a), (b) show the variation of fundamental frequencies with the rotating speed Ω and circumferential wave number n. The influence of geometric properties (H/R and L/R) on the fundamental frequencies is presented in Figures 4 and 5.

Figure 2 shows the variation of fundamental frequencies with the rotating speed for thin and thick rotating laminated composite cylindrical shells. The fundamental

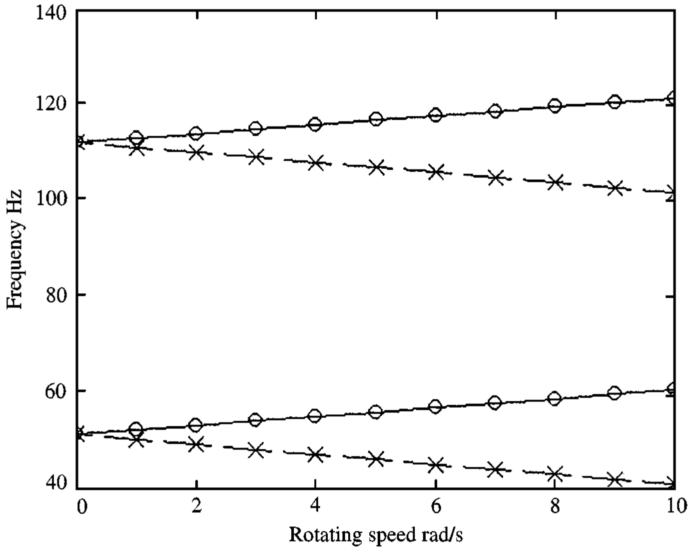

Figure 2. Variation of frequency ($(m, n) = (1, 1)$) with the rotating speed for thin and thick rotating laminated composite cylindrical shells ($H/R = 0{\cdot}002$, $H/R = 0{\cdot}2$, $L/R = 20$). --- Forward thin shell; --- Backword thin shell.

frequencies of the backward waves for a thick shell decrease monotonically at a constant rate when the rotating speed increases, while those of the forward wave also increase at the same rate. In addition, the figure shows that the frequencies of a thick shell are larger than those of a thin shell with the other parameters remaining the same. The figure also shows the forward wave frequencies are always larger than the backward wave frequencies. This is attributed to the influence of Coriolis force.

Figures 3(a), (b) show the variation of the non-dimensional frequency parameter with the circumferential wave number n for a thin ($H/R = 0{\cdot}002$) and thick ($H/R = 0{\cdot}2$) rotating laminated composite cylindrical shell respectively.

In Figure 3(a), the general behavior of a frequency curve for a rotating cylindrical thin shell is observed. It first drops, and rises with an increase of the circumferential wave number n. This is because when the circumferential wave number n is small, the boundary conditions are dominant compared with the contribution of the circumferential wave number n on the frequencies and hence the curve increases. With the increase of circumferential wave number n, circumferential wave number n is dominant and increases the frequency. Meanwhile, the boundary conditions become less important, and the curve ascends.

For a thick shell in Figure 3(b), it can be seen that the frequency curve ascends monotonically. In addition, with an increase in the circumferential wave number n, the curve of the frequency parameter rises at a faster rate than that of the thin shell. This is because boundary conditions increase the rigidity of the rotating body but not the mass of the rotating body, and the rigidity of thick shell is larger than that of the thin shell. In other words, the influence of boundary conditions on a thin shell is greater than that on a thick shell when other parameters are the same.

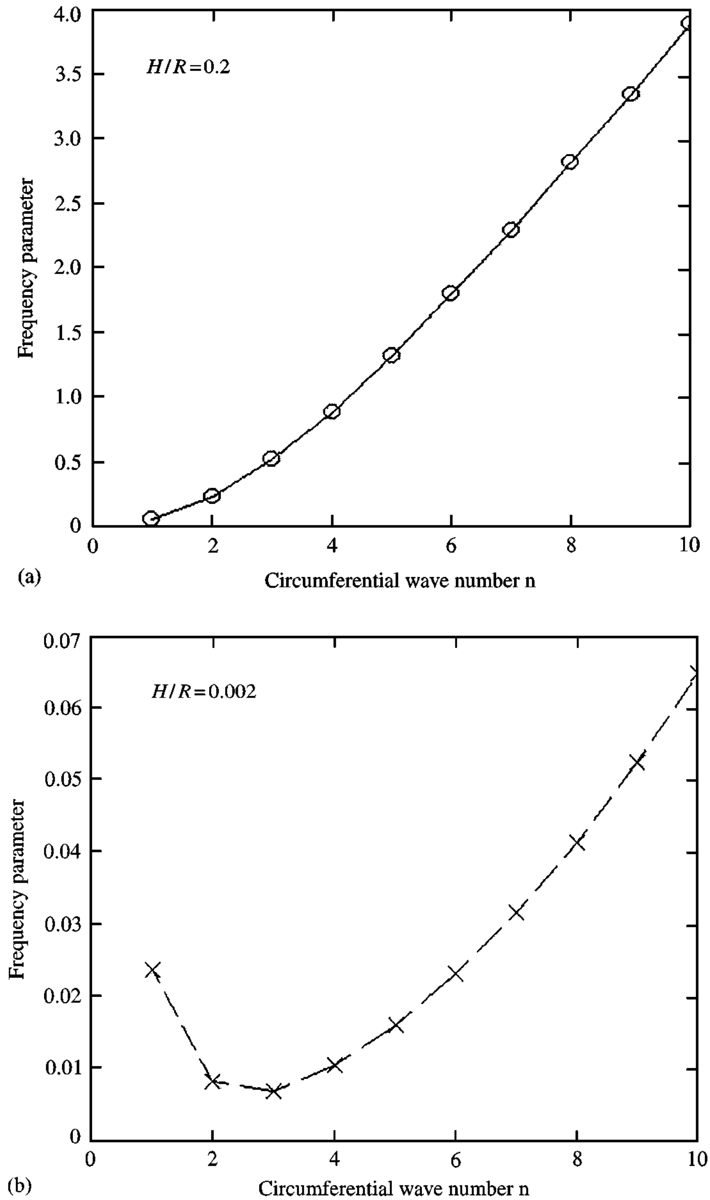

Figure 3. Variation of of fundamental frequency parameter $\omega' = \omega\sqrt{\rho R^2(1 - v_{12}^2)/E_2}$ $[(m, n) = (1, 1)]$ with the circumferential wave number n for a (a) thin; (b) thick rotating laminated composite cylindrical shell ($H/R = 0{\cdot}2$, $L/R = 20$).

Figure 4 shows the variation of fundamental frequency parameter $\sqrt{\rho R^2(1 - v_{12}^2)/E_2}$ with the H/R ratio for a rotating laminated composite cylindrical shell. Only the forward wave frequency parameter is shown in the figure since the behavior of the backward wave is the same as that of the forward wave. The fundamental frequency parameters increase monotonically with the increase of H/R ratio. This is because the thicker shell has greater rigidity. From Figure 4, we also can see that the curve line of $L/R = 5$ is above the curve line of $L/R = 20$. This

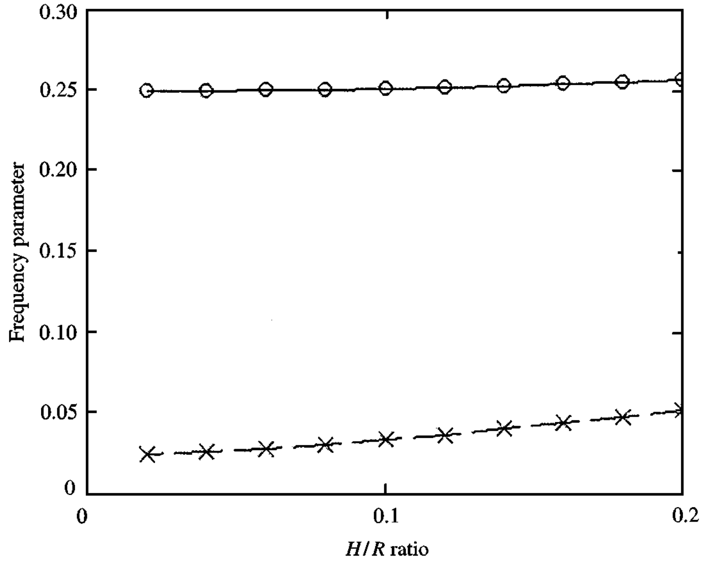

Figure 4. Variation of fundamental frequency parameter $\omega' = \omega\sqrt{\rho R^2(1 - v_{12}^2)/E_2}$ with the H/R ratio for a rotating laminated composite cylindrical shell ($L/R = 20$, $L/R = 5$ and $\Omega = 0{\cdot}1$ rad/s). --- $L/R = 5$; --- $K/R = 20$.

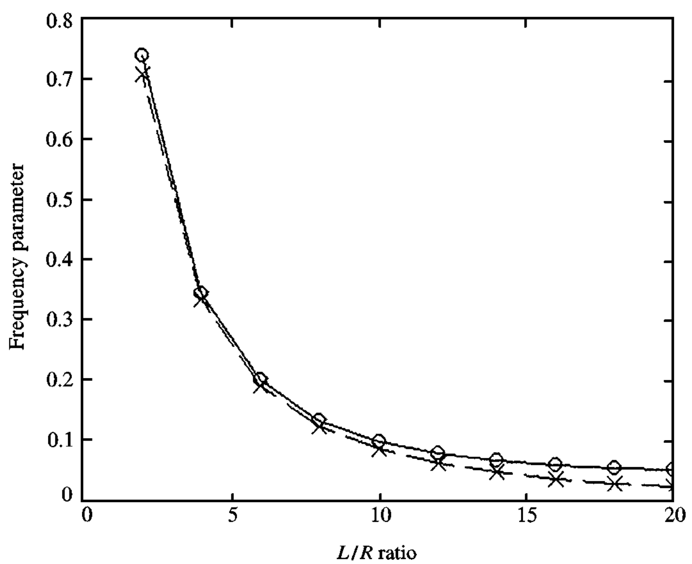

Figure 5. Variation of fundamental frequency parameter $\omega' = \omega\sqrt{\rho R^2(1 - v_{12}^2)/E_2}$ $[(m, n) = (1, 1)]$ with L/R ratio for a rotating laminated composite cylindrical shell ($H/R = 0{\cdot}2$, $H/R = 0{\cdot}002$). --- $H/R = 0{\cdot}002$; --- $H/R = 0{\cdot}2$.

means that the longer the shell, the less the influence of boundary conditions. In addition, since we find that the two curves are almost parallel to each other, we may conclude empirically that the influence of circumferential wave number n and boundary conditions on frequencies are almost independent of each other.

Figure 5 shows the variation of fundamental frequency parameter $\sqrt{\rho R^2(1-v_{12}^2)/E_2}$ with L/R ratio for rotating laminated composite cylindrical shells. The frequencies go down with an increase in the L/R ratio. The reason is that the boundary conditions influence the shorter shell more than the longer one. The fundamental frequency parameter of thin shell is always larger than the thick shell. That is because the boundary conditions affect the frequency of the thin shell more than that of the thick shell when the circumferential wave number n is constant. We can also see that when the circumferential wave number n is constant, without the effect of boundary conditions, the fundamental frequency parameter of the thin shell is larger than that of the thick shell.

4. CONCLUSIONS

A theoretical analysis and analytical solution for vibrations of thick rotating laminated composite shells are presented in the paper. The objective of the study is to present a general method for calculating the frequencies and to study frequency characteristics of short thick rotating laminated composite shells. It is found that it is most difficult to calculate frequencies for thick rotating laminated composite shells when the shell is short and thick, especially when the circumferential wave number n is small. Many theories used to calculate frequencies of a thick short shell diverge when the circumferential wave number n approaches 1. In the theoretical analysis, we also find that the lowest of five pairs of frequencies is associated with the mode where the transverse component dominates w_0. In addition, the differences in frequency characteristics between thin shells and thick shells are also presented in this paper. Many useful conclusions are obtained through this parametric study. The frequency of a thick shell is larger than that of a thin shell when the other parameters are the same. The influence of boundary condition of a thin shell is greater than that of a thick shell when other parameters are the same. The influence of circumferential wave number n and boundary conditions on frequencies are almost independent of each other. The boundary conditions influence the frequencies of the short shell more than those of the longer one.

The present formulation is general. Different boundary conditions, different numbers of layers (which may be isotropic or orthotropic), higher order shear deformation theories, and even different forms of assumed solutions can be easily accommodated in the formulation.

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APPENDIX A

L_{ij} $(i, j = 1, 5)$ are the differential operators shown as follows:

$$L_{11} = A_{11}u_{0,xx} + 1/R^2 A_{66}u_{0,\theta\theta} + A_{16}\frac{1}{R}u_{0,x\theta} + \frac{1}{R}A_{16}u_{0,x\theta} - I_0\ddot{u}_0,$$

$$L_{12} = 1/RA_{12}v_{0,x\theta} + 1/RA_{66}v_{0,x\theta} + A_{16}v_{0,xx} + \frac{1}{R^2}A_{26}v_{0,\theta\theta},$$

$$L_{13} = 1/RA_{12}w_{0,x} + \frac{1}{R^2}A_{25}w_{0,\theta},$$

$$L_{14} = \frac{1}{R}D_{11}\phi_{x,xx} + \frac{1}{R^2}D_{16}\phi_{x,x\theta},$$

$$L_{15} = D_{12}\frac{1}{R^2}\phi_{\theta,x\theta} + \frac{1}{R}D_{16}\phi_{\theta,xx},$$

$$L_{21} = A_{66}\frac{1}{R}u_{0,x\theta} + A_{16}u_{0,xx} + \frac{1}{R}A_{12}u_{0,x\theta} + \frac{1}{R^2}A_{16}u_{0,\theta\theta},$$

$$L_{22} = A_{66}v_{0,xx} + A_{26}\frac{1}{R}v_{0,x\theta} + A_{22}\frac{1}{R^2}v_{0,\theta\theta} + \frac{1}{R^2}A_{16}v_{0,x\theta} - I_0\ddot{v}_0,$$

$$L_{23} = \frac{1}{R^2}A_{22}w_{0,\theta} + \frac{1}{R}A_{26}w_{0,x} + 2I_0\Omega\dot{w}_0$$

$$L_{24} = \frac{1}{R^2}D_{66}\phi_{x,x\theta} + \frac{1}{R}D_{16}\phi_{x,xx},$$

$$L_{25} = \frac{1}{R}D_{66}\phi_{\theta,xx} + D_{26}\frac{1}{R^2}\phi_{\theta,x\theta},$$

$$L_{31} = -\frac{1}{R}A_{12}u_{0,x} - \frac{1}{R^2}A_{26}u_{0,\theta},$$

$$L_{32} = -\frac{1}{R^2}A_{22}v_{0,\theta} - \frac{1}{R}A_{26}v_{0,x} - 2I_0\Omega\dot{v}_0,$$

$$L_{33} = KA_{45}\frac{1}{R}w_{0,x\theta} + K_{55}w_{0,xx}$$

$$+ \frac{1}{R^2}KA_{44}w_{0,\theta\theta} + \frac{1}{R}KA_{45}w_{0,x\theta} - \frac{1}{R^2}A_{22}w_0 - I_0\ddot{w}_0,$$

$$L_{34} = KA_{55}\phi_{x,x} + \frac{1}{R}KA_{45}\phi_{x,\theta},$$

$$L_{35} = KA_{45}\phi_{\theta,x} + \frac{1}{R}KA_{44}\phi_{\theta,\theta},$$

$$L_{41} = \frac{1}{R}D_{11}u_{0,xx} + \frac{1}{R^2}D_{16}u_{0,x\theta},$$

$$L_{42} = \frac{1}{R^2}D_{12}v_{0,x\theta} + \frac{1}{R}D_{16}v_{0,xx},$$

$$L_{43} = \frac{1}{R^2}D_{12}w_{0,x} - \frac{1}{R}KA_{45}w_{0,\theta} - KA_{55}\phi_x - KA_{55}w_{0,x},$$

$$L_{44} = D_{11}\phi_{x,xx} + \frac{1}{R}D_{16}\phi_{x,x\theta} + \frac{1}{R^2}D_{66}\phi_{x,\theta\theta} + \frac{1}{R}D_{16}\phi_{x,x\theta} - KA_{55}\phi_x - I_2\ddot{\phi}_x,$$

$$L_{45} = \frac{1}{R}D_{12}\phi_{\theta,x\theta} + D_{16}\phi_{\theta,xx} + \frac{1}{R}D_{66}\phi_{\theta,x\theta} + \frac{1}{R^2}D_{26}\phi_{\theta,\theta\theta} - KA_{45}\phi_\theta,$$

$$L_{51} = \frac{1}{R^2} D_{66} u_{0,x\theta} + \frac{1}{R} D_{16} u_{0,xx},$$

$$L_{52} = \frac{1}{R} D_{66} v_{0,xx} + \frac{1}{R^2} D_{26} v_{0,x\theta},$$

$$L_{53} = \frac{1}{R^2} D_{26} w_{0,x} - \frac{1}{R} K A_{44} w_{0,\theta} - K A_{45} w_{0,x},$$

$$L_{54} = \frac{1}{R} D_{66} \phi_{x,x\theta} + D_{16} \phi_{x,xx} + \frac{1}{R} D_{12} \phi_{x,x\theta} + \frac{1}{R} D_{26} \phi_{x,\theta\theta} - K A_{45} \phi_x,$$

$$L_{55} = D_{66} \phi_{\theta,xx} + \frac{1}{R^2} D_{26} \phi_{\theta,x\theta} + \frac{1}{R^2} D_{22} \phi_{\theta,\theta\theta} + \frac{1}{R} D_{26} \phi_{\theta,x\theta} - K A_{44} \phi_\theta - I_2 \ddot{\phi}.$$

C_{ij} $(i, j = 1, 5)$ are the operators in terms of five unknowns u_0, v_0, w_0, ϕ_x, ϕ_θ:

$$C_{11} = -A_{11} p^2 - 1/R^2 A_{66} n^2 - 2A_{16} \frac{1}{R} pn + I_0 \omega^2,$$

$$C_{12} = 1/R A_{12} pn + 1/R A_{66} pn + A_{16} p^2 + \frac{1}{R^2} A_{26} n^2,$$

$$C_{13} = 1/R A_{12} p + \frac{1}{R^2} A_{26} n,$$

$$C_{14} = -\frac{1}{R} D_{11} p^2 - \frac{1}{R^2} D_{16} pn,$$

$$C_{15} = D_{12} \frac{1}{R^2} pn - \frac{1}{R} D_{16} p^2,$$

$$C_{21} = -A_{66} \frac{1}{R} pn - A_{16} p^2 - \frac{1}{R} A_{12} pn - \frac{1}{R^2} A_{16} n^2$$

$$C_{22} = A_{66} p^2 + A_{26} \frac{1}{R} pn + A_{22} \frac{1}{R^2} n^2 + \frac{1}{R} A_{16} pn - I_0 \omega^2,$$

$$C_{23} = \frac{1}{R^2} A_{22} n + \frac{1}{R} A_{26} p + 2 I_0 \Omega \omega,$$

$$C_{24} = -\frac{1}{R^2} D_{66} pn - \frac{1}{R} D_{16} p^2,$$

$$C_{25} = \frac{1}{R} D_{66}p^2 + D_{26} \frac{1}{R^2} pn,$$

$$C_{31} = -\frac{1}{R} A_{12}p - \frac{1}{R^2} A_{26}n,$$

$$C_{32} = \frac{1}{R^2} A_{22}n + \frac{1}{R} A_{26}p + 2I_0\Omega\omega,$$

$$C_{33} = KA_{45} \frac{1}{R} pn + KA_{55}p^2 + \frac{1}{R^2} KA_{44}n^2 + \frac{1}{R} KA_{45}pn + \frac{1}{R^2} A_{22}w_0 - I_0\omega^2,$$

$$C_{34} = KA_{55}p + \frac{1}{R} KA_{45}n,$$

$$C_{35} = -KA_{45}p - \frac{1}{R} KA_{44}n,$$

$$C_{41} = -\frac{1}{R} D_{11}p^2 - \frac{1}{R^2} D_{16}pn,$$

$$C_{42} = \frac{1}{R^2} D_{12}pn + \frac{1}{R} D_{16}p^2,$$

$$C_{43} = \frac{1}{R^2} D_{12}p - \frac{1}{R} KA_{45}n - KA_{55} - KA_{55}p,$$

$$C_{44} = -D_{11}p^2 - \frac{1}{R} D_{16}pn - \frac{1}{R^2} D_{66}n^2 - \frac{1}{R} D_{16}pn - KA_{55} + I_2\omega^2,$$

$$C_{45} = \frac{1}{R} D_{12}pn + D_{16}p^2 + \frac{1}{R} D_{66}pn + \frac{1}{R^2} D_{26}n^2 - KA_{45},$$

$$C_{51} = -\frac{1}{R^2} D_{66}pn - \frac{1}{R} D_{16}p^2,$$

$$C_{52} = \frac{1}{R} D_{66}p^2 + \frac{1}{R^2} D_{26}pn,$$

$$C_{53} = \frac{1}{R^2} D_{26}p - \frac{1}{R} KA_{44}n - KA_{45}p,$$

$$C_{54} = -\frac{1}{R} D_{66}pn - D_{16}p^2 - \frac{1}{R} D_{12}pn - \frac{1}{R} D_{26}n^2 - KA_{45},$$

$$C_{55} = D_{66}p^2 + \frac{1}{R^2} D_{26}pn + \frac{1}{R^2} D_{22}n^2 + \frac{1}{R} D_{26}pn + KA_{44} - I_2\omega^2.$$