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ON CALCULATION OF SENSITIVITY FOR NON-DEFECTIVE EIGENPROBLEMS WITH REPEATED ROOTS

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Methods of calculating eigensolution sensitivity have long been divided into two categories: the modal methods and the direct methods. This paper presents a unified theory for the calculation of derivatives of eigenvalues and eigenvectors, where the most general case, non-defective eigenproblems with repeated roots, is considered. The intrinsic relation between these two methods is exposed. The present modal method is shown to be actually the asymptotic expansion of a special direct method. A numerical example is given to verify the validity of the presented formulae, and the issue of computational efficiency is addressed.

1. INTRODUCTION

Calculation of eigensolution sensitivity plays an important role in the field of dynamics. The derivatives of eigenvalues and eigenvectors, which characterize the tendency of variation for frequencies and mode shapes with respect to design parameters, provide guidance in system identification, optimization and control. With the progress in structural dynamics, many algorithms for computing eigensolution sensitivity have been proposed during the recent two decades.

It is well known that the main difficulty of this problem lies in solving the eigenvector derivatives governed by singular matrix equations. Correspondingly, the various methods previously developed have been divided into two categories: the modal expansion methods and the direct methods. Several review papers [1–3] have excellently documented these methods, and comparisons of computational efficiency were also made in other literature. Briefly mentioned below are some algorithms believed to be pertinent to this paper, which is by no means a thorough survey.

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Fox and Kapoor [4] were the first to propose a simple modal method to compute the eigensolution sensitivity, where the exact eigenvector derivatives were expressed in terms of the full eigenvectors of the system. Juang *et al*. [5] proved the existence of the eigenvector derivatives associated with repeated eigenvalues for non-defective systems, and established a set of formulae for eigenvector derivatives also by using modal expansion. Since obtaining the full eigenvectors *a priori* cannot be the practical case and, usually, only a subset of them is known, Wang [6] developed a modified method to improve the accuracy of the truncated modal expansion by adding a static correction term. Akgün [7] presented a family of modal methods, including Fox and Kapoor's method and the modified modal method as special cases, to compute the eigenvector derivatives. When the higher order modal methods in the family are used, higher order corrections (relative to the static correction) can be expected to compensate for the information lost due to the exclusion of higher order eigenvectors. Bernard and Bronowick [8] improved the modal method to allow computation of derivatives of eigenvectors with repeated eigenvalues and repeated eigenvalue derivatives, and at the end of their paper they suggested using Wang's technique to increase accuracy. Lin and Lim [9] extended Wang's algorithm to systems with rigid body modes by means of frequency shifting. Yu *et al*. [10] examined the efficiency of some modal expansion methods through numerical examples. Also worthy of mentioning is the work by Zhang and Zerva [11, 12], where the iterative procedure essentially is based upon the modal expansion expression.

On the other hand, it was not until Nelson [13] presented his powerful algorithm preserving the band property of system matrices that the direct methods became competitive. In his method, a particular solution of an eigenvector derivative is solved by removing the singularity of the governing equation, i.e., a non-singular version of the governing equation is obtained by eliminating certain columns and rows of the singular coefficient matrix. The homogeneous solution of the eigenvector derivatives can then be solved by employing the eigenvectors' gauge condition. Mills-Curran [14] produced an extension to Nelson's method to deal with the repeated eigenvalue case for self-adjoint systems. Furthermore, Zhang and Wang [15] derived a direct method for computing eigenvector derivatives for the case in which repeated eigenvalues with repeated first order derivatives are present. Similar work was done by Friswell [16], where an in-depth discussion of the continuity of eigenvalues and eigenvectors was given.

This paper aims at developing a unified theory for the direct methods and the modal methods. The mathematical expression of the problem is given in the next section. Herein we are concerned with the calculation of eigensolution sensitivity associated with repeated eigenvalues for general non-defective systems. In the third section, an algorithm along similar lines as the previously developed direct methods is derived. It is shown that after pre-multiplying an arbitrary particular solution of the eigenvector derivatives by a so-called projection matrix [17], one will obtain a uniquely determined particular solution which falls into the complementary space with respect to the space where the homogeneous solution

falls. Since the latter space is spanned by the linearly independent eigenvectors to be differentiated, such a decomposition of the eigenvector derivatives enables us to present a most general modal method in the fourth section. The intrinsic relation between the direct method and the modal expansion method is explained. In the fifth section a numerical example is given, for which the computational efficiency will be discussed. Concluding remarks are given in the last section.

2. PROBLEM DEFINITION

Given $N \times N$ general complex matrices $\mathbf{A}(p)$ and $\mathbf{B}(p)$ which are continuous functions of a real parameter p, where $\mathbf{B}(p)$ is non-singular, the right and left eigenproblems associated with $\mathbf{A}(p)$ and $\mathbf{B}(p)$ are defined as

$$\mathbf{A}(p)\mathbf{x}(p) = \lambda(p)\mathbf{B}(p)\mathbf{x}(p), \quad \mathbf{A}^{\mathrm{T}}(p)\mathbf{y}(p) = \lambda(p)\mathbf{B}^{\mathrm{T}}(p)\mathbf{y}(p). \tag{1a, b}$$

This paper is concerned with the derivatives of eigenvalues and eigenvectors at $p = p_0$. Particularly, only the case that the eigenproblems (1a) and (1b) are non-defective is considered. Let the complete eigensolutions of (1a) and (1b) be represented by

$$\mathbf{\Lambda}(p) = \mathrm{diag}(\lambda_1(p), \lambda_2(p), \ldots, \lambda_N(p)), \tag{2a}$$

$$\mathbf{X}(p) = (\mathbf{x}_1(p)|\mathbf{x}_2(p)|\cdots|\mathbf{x}_N(p)), \quad \mathbf{Y}(p) = (\mathbf{y}_1(p)|\mathbf{y}_2(p)|\cdots|\mathbf{y}_N(p)), \tag{2b}$$

where the eigenvalues $\lambda_i(p)$ are so arranged that $0 \leqslant |\lambda_1(p_0)| \leqslant |\lambda_2(p_0)| \leqslant \cdots \leqslant |\lambda_N(p_0)|$.

It is well known that an eigenvector given by equations (1a) or (1b) is uncertain to the extent of a non-zero constant multiplier, and gauge condition should be imposed to result in unique eigenvectors and thereafter, for one to solve the corresponding eigenvector derivatives. However, as pointed out by Murthy and Haftka [2], confusion exists in previous literature regarding this point. For a non-self-adjoint system, one can no longer use

$$\mathbf{x}_i^{\mathrm{T}}(p)\mathbf{B}(p)\mathbf{x}_i(p) = 1 \tag{3}$$

which may fail even if $\mathbf{B}(p)$ is real, for example, if $\mathbf{B}(p)$ is skew-symmetric. In this paper an alternative condition is adopted which will be generally valid,

$$\mathbf{x}_i^{*}(p)\mathbf{x}_i(p) = 1, \tag{4a}$$

where "*" denotes a conjugate-transpose. For the purpose of comparison, the results corresponding to condition (3) will also be given. Care should be taken when using condition (4a), at that time we still have uncertainty for an eigenvector to the extent of a unit complex factor, and a complementary condition is proposed by setting

$$x_{iq}(p) = |x_{iq}(p)|, \tag{4b}$$

where, for an arbitrary eigenvector $\mathbf{x}_i(p)$, the subscript q is to be chosen so that

$$|x_{iq}(p_0)| = \max_{j=1\sim N} |x_{ij}(p_0)|.$$

In other words, for an arbitrary eigenvector $\mathbf{x}_i(p)$, its qth component is set to be a real number, where q is to be chosen so that $x_{iq}(p_0)$ has the largest modulus among all the components of $\mathbf{x}_i(p_0)$. Since $\mathbf{A}(p)$ and $\mathbf{B}(p)$ and hence the eigenvectors are continuous, in the vicinity of $p = p_0$ the eigenvector component $x_{iq}(p)$ (corresponding to the differentiated eigenvector component $x_{iq}(p_0)$) must be non-zero and therefore can be set to a real number by multiplying the eigenvector by a complex factor of unit modulus. After the right eigenvectors have been determined by using either one of the above gauge conditions, imposing the following biorthonormalization condition will uniquely determine the left eigenvectors:

$$\mathbf{Y}^{\mathrm{T}}(p)\mathbf{B}(p)\mathbf{X}(p) = \mathbf{I}_N, \tag{5}$$

where $\mathbf{I}_N$ is the Nth order identity matrix. Clearly, for self-adjoint eigenproblems, only using equation (5) is sufficient to render the eigenvectors unique because in that case the left and right eigenvectors are the same.

Suppose that at $p = p_0$ one is given an arbitrary n-fold ($1 \leqslant n \leqslant N$) eigenvalue

$$\lambda_l = \lambda_{l+1} = \cdots = \lambda_h \stackrel{\text{def}}{=} \lambda_0, \quad |\lambda_0| \geqslant 0, \quad h - l + 1 = n, \tag{6}$$

with corresponding gauged eigenvector subsets $\hat{\mathbf{X}}_U = (\hat{\mathbf{x}}_l|\hat{\mathbf{x}}_{l+1}|\cdots|\hat{\mathbf{x}}_h)$ and $\hat{\mathbf{Y}}_U = (\hat{\mathbf{y}}_l|\mathbf{y}_{l+1}|\cdots|\hat{\mathbf{y}}_h)$; hereafter "$(p_0)$" is omitted for variables evaluated at $p = p_0$. Our task is to solve the derivatives of this eigensolution. When $n > 1$, i.e., the eigenvalue to be differentiated is repeated, the corresponding right and left eigenvectors are both degenerate and, mathematically speaking, for any non-singular $n \times n$ matrices α and β, $\mathbf{X}_U = \hat{\mathbf{X}}_U\alpha$ and $\hat{\mathbf{Y}}_U = \mathbf{Y}_U\beta$ will also be the right and left eigenvector subsets. However, it was concluded in previous literature that, if the derivatives of the repeated eigenvalue are all distinct, under proper gauge conditions (such as those given by equations (3) or (4a) and (4b), and (5)) only specific $\mathbf{X}_U$ and $\mathbf{Y}_U$ corresponding to unique α and β are differentiable and thus have derivatives. Notice that condition (5) imposes a restriction on α and β:

$$\beta^{\mathrm{T}}\alpha = \mathbf{I}_n.$$

Let $\mathbf{\Lambda}_U(p)$, $\mathbf{X}_U(p)$ and $\mathbf{Y}_U(p)$ denote the continuously differentiable subsets of the eigenvalues and the corresponding right and left eigenvectors, respectively. Taking the dth partial derivative of $\mathbf{A}(p)\mathbf{X}_U(p) = \mathbf{B}(p)\mathbf{X}_U(p)\mathbf{\Lambda}_U(p)$ and letting $p \to p_0$, one has

$$\mathring{\mathbf{A}}\mathbf{X}_U^{(d)} = \mathbf{F}_d, \tag{7}$$

where

$$\mathring{\mathbf{A}} \overset{\text{def}}{=} \mathbf{A} - \lambda_0 \mathbf{B}, \quad \mathbf{F}_d \overset{\text{def}}{=} \sum_{k=1}^{d} \frac{d!}{k!(d-k)!} \left[\sum_{j=1}^{k} \frac{k!}{j!(k-j)!} \mathbf{B}^{(k-j)} \mathbf{X}_U^{(d-k)} \mathbf{\Lambda}_U^{(j)} - \mathring{\mathbf{A}}^{(k)} \mathbf{X}_U^{(d-k)} \right],$$

$$\mathring{\mathbf{A}}^{(d)} \overset{\text{def}}{=} \mathbf{A}^{(d)} - \lambda_0 \mathbf{B}^{(d)}, \quad \mathbf{\Lambda}_U = \lambda_0 \mathbf{I}_n, \quad d = 1, 2, \ldots, \tag{8}$$

and where the parenthesized superscript "(d)" represents the dth partial derivative. The eigenvalue and eigenvector derivatives to be solved are governed by equation (7) and the eigenvectors' gauge condition.

3. A DIRECT METHOD WITH BIORTHOGONAL DECOMPOSITION

3.1. BIORTHOGONAL DECOMPOSITION OF EIGENVECTOR DERIVATIVES

Equation (5) results in

$$\mathbf{I}_N = \mathbf{B}\mathbf{X}\mathbf{Y}^{\mathrm{T}} = \mathbf{X}\mathbf{Y}^{\mathrm{T}}\mathbf{B}. \tag{9}$$

Define

$$\mathbf{\Lambda}_V = \operatorname{diag}(\lambda_1, \ldots, \lambda_{l-1}, \lambda_{h+1}, \ldots, \lambda_N),$$
$$\mathbf{X}_V = (\mathbf{x}_1 | \cdots | \mathbf{x}_{l-1} | \mathbf{x}_{h+1} | \cdots | \mathbf{x}_N), \quad \mathbf{Y}_V = (\mathbf{y}_1 | \cdots | \mathbf{y}_{l-1} | \mathbf{y}_{h+1} | \cdots | \mathbf{y}_N),$$

which are the complementary subsets with respect to the known eigensolution $\mathbf{\Lambda}_U$, $\hat{\mathbf{X}}_U$ and $\hat{\mathbf{Y}}_U$. Then one has

$$\mathbf{I}_N = \mathbf{B}(\hat{\mathbf{X}}_U \hat{\mathbf{Y}}_U^{\mathrm{T}} + \mathbf{X}_V \mathbf{Y}_V^{\mathrm{T}}) = (\hat{\mathbf{X}}_U \hat{\mathbf{Y}}_U^{\mathrm{T}} + \mathbf{X}_V \mathbf{Y}_V^{\mathrm{T}})\mathbf{B},$$

which yields

$$\mathbf{R}(\lambda_0) \overset{\text{def}}{=} \mathbf{I}_N - \mathbf{B}\hat{\mathbf{X}}_U \hat{\mathbf{Y}}_U^{\mathrm{T}} = \mathbf{B}\mathbf{X}_V \mathbf{Y}_V^{\mathrm{T}}, \quad \mathbf{L}(\lambda_0) \overset{\text{def}}{=} \mathbf{I}_N - \hat{\mathbf{X}}_U \hat{\mathbf{Y}}_U^{\mathrm{T}} \mathbf{B} = \mathbf{X}_V \mathbf{Y}_V^{\mathrm{T}} \mathbf{B}. \tag{10a, b}$$

Clearly, under the biorthonormalization condition (5), matrices $\mathbf{R}(\lambda_0)$ and $\mathbf{L}(\lambda_0)$ remain invariant though the eigenvector subsets $\hat{\mathbf{X}}_U$ and $\hat{\mathbf{Y}}_U$ are degenerate. In mechanics terminology, $\mathbf{R}(\lambda_0)$ and $\mathbf{L}(\lambda_0)$ are often referred to as the right and left projection matrices [17].

Notice that the column vectors of $\hat{\mathbf{X}}_U$ and $\hat{\mathbf{Y}}_U$ span the null spaces of matrices $\mathring{\mathbf{A}}$ and $\mathring{\mathbf{A}}^{\mathrm{T}}$, respectively, and hence the rank of $\mathring{\mathbf{A}}$ is $m \overset{\text{def}}{=} N - n$. The singular matrix equation (7) is consistent if and only if

$$\hat{\mathbf{Y}}_U^{\mathrm{T}} \mathbf{F}_d = \mathbf{0} \quad (\text{or equivalently, } \mathbf{Y}_U^{\mathrm{T}} \mathbf{F}_d = \mathbf{0}). \tag{11}$$

Provided that condition (11) is satisfied, the solution set of equation (7) is given by

$$\mathbf{X}_U^{(d)} = \mathbf{X}_U \mathbf{S}_d + \tilde{\mathbf{X}}_U^{(d)}. \tag{12}$$

In the above equation, $\mathbf{S}_d$ is an $n \times n$ coefficient matrix to be determined, and $\mathbf{X}_U \mathbf{S}_d$ represents the homogeneous solution part which is the linear combination

of the differentiated eigenvectors. The particular solution part in equation (12), $\tilde{\mathbf{X}}_U^{(d)}$, can be written as [18]

$$\tilde{\mathbf{X}}_U^{(d)} = \mathbf{C}(\lambda_0)\mathbf{F}_d, \quad \forall \mathbf{C}(\lambda_0) \in \mathring{\mathbf{A}}\{1\}, \tag{13}$$

where $\mathring{\mathbf{A}}\{1\} \stackrel{\text{def}}{=} \{\mathbf{C}(\lambda_0) | \mathring{\mathbf{A}}\mathbf{C}(\lambda_0)\mathring{\mathbf{A}} = \mathring{\mathbf{A}},\ \mathbf{C}(\lambda_0) \in \mathcal{C}^{N\times N}\}$, which is referred to as the generalized 1-inverse set of matrix $\mathring{\mathbf{A}}$ [18]. Notice that the generalized 1-inverse of $\mathring{\mathbf{A}}$, $\mathbf{C}(\lambda_0)$, is not unique, nor is the above particular solution $\tilde{\mathbf{X}}_U^{(d)}$.

Let $\mathbf{G}(\lambda_0) \stackrel{\text{def}}{=} \mathbf{L}(\lambda_0)\mathbf{C}(\lambda_0)\mathbf{R}(\lambda_0)$, $\forall \mathbf{C}(\lambda_0) \in \mathring{\mathbf{A}}\{1\}$. In virtue of

$$\mathring{\mathbf{A}}\mathbf{X}_V = (\mathbf{A} - \lambda_0\mathbf{B})\mathbf{X}_V = \mathbf{B}\mathbf{X}_V(\mathbf{\Lambda}_V - \lambda_0\mathbf{I}_m),$$

$$\mathring{\mathbf{A}}^{\mathrm{T}}\mathbf{Y}_V = (\mathbf{A}^{\mathrm{T}} - \lambda_0\mathbf{B}^{\mathrm{T}})\mathbf{Y}_V = \mathbf{B}^{\mathrm{T}}\mathbf{Y}_V(\mathbf{\Lambda}_V - \lambda_0\mathbf{I}_m)$$

it is easy to verify that

$$\begin{aligned}
\mathbf{G}(\lambda_0) &= \mathbf{X}_V\mathbf{Y}_V^{\mathrm{T}}\mathbf{B}\mathbf{C}(\lambda_0)\mathbf{B}\mathbf{X}_V\mathbf{Y}_V^{\mathrm{T}} \\
&= \mathbf{X}_V(\mathbf{\Lambda}_V - \lambda_0\mathbf{I}_m)^{-1}\mathbf{Y}_V^{\mathrm{T}}\mathring{\mathbf{A}}\mathbf{C}(\lambda_0)\mathring{\mathbf{A}}\mathbf{X}_V(\mathbf{\Lambda}_V - \lambda_0\mathbf{I}_m)^{-1}\mathbf{Y}_V^{\mathrm{T}} \\
&= \mathbf{X}_V(\mathbf{\Lambda}_V - \lambda_0\mathbf{I}_m)^{-1}\mathbf{Y}_V^{\mathrm{T}} = \sum_{\substack{j=1 \\ j\neq l\sim h}}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j - \lambda_0}
\end{aligned} \tag{14}$$

and hence

$$\begin{aligned}
\mathring{\mathbf{A}}\mathbf{G}(\lambda_0)\mathring{\mathbf{A}} &= \mathring{\mathbf{A}}\mathbf{X}_V(\mathbf{\Lambda}_V - \lambda_0\mathbf{I}_m)^{-1}\mathbf{Y}_V^{\mathrm{T}}\mathring{\mathbf{A}} \\
&= \mathbf{B}\mathbf{X}_V\mathbf{Y}_V^{\mathrm{T}}\mathring{\mathbf{A}} = \mathbf{R}(\lambda_0)\mathring{\mathbf{A}} \\
&= (\mathbf{I}_N - \mathbf{B}\hat{\mathbf{X}}_U\hat{\mathbf{Y}}_U^{\mathrm{T}})\mathring{\mathbf{A}} = \mathring{\mathbf{A}}.
\end{aligned} \tag{15}$$

Obviously $\mathbf{G}(\lambda_0)$ is an invariant generalized 1-inverse of $\mathring{\mathbf{A}}$. In the following part of this paper the notation $\tilde{\mathbf{X}}_U^{(d)}$ is assigned to the special particular solution given by

$$\tilde{\mathbf{X}}_U^{(d)} = \mathbf{G}(\lambda_0)\mathbf{F}_d \tag{16}$$

i.e.,

$$\tilde{\mathbf{X}}_U^{(d)} = \mathbf{L}(\lambda_0)\mathbf{C}(\lambda_0)\mathbf{R}(\lambda_0)\mathbf{F}_d \tag{17a}$$

$$= \mathbf{X}_V(\mathbf{\Lambda}_V - \lambda_0\mathbf{I}_m)^{-1}\mathbf{Y}_V^{\mathrm{T}}\mathbf{F}_d, \tag{17b}$$

which is uniquely determined and falls into the space spanned by the column vectors of $\mathbf{X}_V$. The biorthonormal condition then yields

$$\mathbf{Y}_U^{\mathrm{T}}\mathbf{B}\tilde{\mathbf{X}}_U^{(d)} = \mathbf{0}. \tag{18}$$

Such a biorthogonal decomposition of the eigenvector derivatives simplifies the

subsequent derivations as well as the practical computations and, more importantly, exposes the intrinsic relation between the so-called direct methods and the modal expansion methods. Actually, the major difference between these two kinds of methods is in the computation of $\tilde{\mathbf{X}}_U^{(d)}$. As one will see later on, $\tilde{\mathbf{X}}_U^{(d)}$ can be either solved by using equation (17a) (direct method, i.e., to directly solve the generalized 1-inverse from known information), or solved by using equation (17b) or its asymptotic expansion (modal method).

3.2. A DIRECT METHOD FOR GENERAL NON-SELF-ADJOINT SYSTEMS

First, the differentiable eigenvector subsets $\mathbf{X}_U$ and $\mathbf{Y}_U$ need to be determined from the given eigenvectors $\hat{\mathbf{X}}_U$ and $\hat{\mathbf{Y}}_U$. Setting $d = 1$ in equations (11) and (8), after simple derivations one has

$$\mathbf{\Lambda}_U^{(1)} = \mathbf{Y}_U^{\mathrm{T}} \mathring{\mathbf{A}}^{(1)} \mathbf{X}_U. \tag{19}$$

In virtue of $\mathbf{X}_U = \hat{\mathbf{X}}_U \alpha$, $\mathbf{Y}_U = \hat{\mathbf{Y}}_U \beta$ and $\beta^{\mathrm{T}} \alpha = \mathbf{I}_n$, one obtains an n-order standard eigenproblem and its adjoint problem as

$$\mathbf{D}_1 \alpha = \alpha \mathbf{\Lambda}_U^{(1)}, \quad \mathbf{D}_1^{\mathrm{T}} \beta = \beta \mathbf{\Lambda}_U^{(1)}, \tag{20a, b}$$

where $\mathbf{D}_1 \overset{\text{def}}{=} \hat{\mathbf{Y}}_U^{\mathrm{T}} \mathring{\mathbf{A}}^{(1)} \hat{\mathbf{X}}_U$. Equations (20a) and (20b) give the first order derivatives of the repeated eigenvalue, and the eigenvector matrices α and β will designate the differentiable right and left eigenvector subsets $\mathbf{X}_U$ and $\mathbf{Y}_U$. It is still worth mentioning the case in which repeated eigenvalues arise in eigenproblems (20a) and (20b). In that case some eigenvectors of equations (20a) and (20b) are degenerate and one will still be unable to determine the differentiable eigenvectors. In fact, eigensolution sensitivity reflects the eigensolution perturbation due to system parameter perturbation. Equal eigenvalue derivatives show that the perturbed eigenvalues are still repeated and they fail to separate the originally degenerate eigenvector subspace. As demonstrated by Zhang and Wang [15] and Friswell [16], in that case higher order eigenvalue derivatives need to be solved until all the perturbed eigenvalues become distinct. In this paper only the case in which all the eigenvalues of equations (20a) and (20b) are distinct is considered, i.e., $\lambda_j^{(1)} \neq \lambda_k^{(1)}$ (Xj, $k = 1, \ldots, n$, $j \neq k$). Then α and β are both non-degenerative and the differentiable eigenvectors $\mathbf{X}_U$ and $\mathbf{Y}_U$ can be uniquely determined by imposing the gauge conditions (3), (4) and (5).

Next to be solved is the particular solution $\tilde{\mathbf{X}}_U^{(1)}$. With $d = 1$ in equations (16) and (8) and recalling equation (19), one has

$$\begin{aligned} \tilde{\mathbf{X}}_U^{(1)} &= \mathbf{G}(\lambda_0)\mathbf{F}_1 = -\mathbf{G}(\lambda_0)(\mathbf{I}_N - \mathbf{B}\mathbf{X}_U\mathbf{Y}_U^{\mathrm{T}})\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U = -\mathbf{G}(\lambda_0)\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U \\ &= -\mathbf{L}(\lambda_0)\mathbf{C}(\lambda_0)\mathbf{R}(\lambda_0)\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U. \end{aligned} \tag{21}$$

It was Nelson [13] who first proposed an efficient algorithm for extracting a particular solution of the eigenvector derivative by exploiting the property that the null space of $\mathring{\mathbf{A}}$ is known *a priori*, and then a direct method for calculating the eigenvector derivatives was completed by employing the eigenvector's gauge

condition. Mills-Curran [14] extended Nelson's method to a self-adjoint system with repeated eigenvalues. A similar algorithm for solving a particular solution for a non-self-adjoint system with repeated eigenvalues was given by Tang *et al.* [19]. The algorithm is now modified to yield the particular solution in the form of (21):

(1) Find n linearly independent rows in $\mathbf{X}_U$ and eliminate the corresponding n columns of matrix $\mathring{\mathbf{A}}$ and n rows of the right-hand side $-\mathbf{R}(\lambda_0)\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U$.

(2) Find n linearly independent rows in $\mathbf{Y}_U$ and eliminate the corresponding n rows of matrix $\mathring{\mathbf{A}}$. It was proved [19] that the remainder of $\mathring{\mathbf{A}}$ must form a $(N-n)\times(N-n)$ non-singular coefficient matrix.

(3) Solve the $(N-n)$ non-singular simultaneous equations and then fill the entries corresponding to the eliminated rows with zeros.

(4) Pre-multiply the obtained result by $\mathbf{L}(\lambda_0)$ and one then has the solution given by equation (21).

Finally, one needs to find the coefficient matrix $\mathbf{S}_1$ in the homogeneous solution part. With $d=2$ in equations (11) and (8), after simple calculations one has

$$\mathbf{\Lambda}_U^{(2)} = \mathbf{Y}_U^{\mathrm{T}}(\mathring{\mathbf{A}}^{(2)}\mathbf{X}_U + 2\,\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U^{(1)} - 2\mathbf{B}^{(1)}\mathbf{X}_U\mathbf{\Lambda}_U^{(1)} - 2\mathbf{B}\mathbf{X}_U^{(1)}\mathbf{\Lambda}_U^{(1)}).$$

Setting $d=1$ in equation (12) and inserting it into the above equation, recalling equations (21) and (18), one obtains

$$\mathbf{S}_1\mathbf{\Lambda}_U^{(1)} - \mathbf{\Lambda}_U^{(1)}\mathbf{S}_1 + \tfrac{1}{2}\mathbf{\Lambda}_U^{(2)} = \tfrac{1}{2}\mathbf{Y}_U^{\mathrm{T}}(\mathring{\mathbf{A}}^{(2)} - 2\mathring{\mathbf{A}}^{(1)}\mathbf{G}\mathring{\mathbf{A}}^{(1)} - 2\mathbf{B}^{(1)}\mathbf{X}_U\mathbf{Y}_U^{\mathrm{T}}\mathring{\mathbf{A}}^{(1)})\mathbf{X}_U. \quad (22)$$

Therefore, the off-diagonal elements of matrix $\mathbf{S}_1$ and the second order derivatives of the repeated eigenvalues are, respectively,

$$S_{1jk} = \frac{\mathbf{y}_j^{\mathrm{T}}(\mathring{\mathbf{A}}_2 - 2\mathring{\mathbf{A}}^{(1)}\mathbf{G}\mathring{\mathbf{A}}^{(1)} - 2\mathbf{B}^{(1)}\mathbf{X}_U\mathbf{Y}_U^{\mathrm{T}}\mathring{\mathbf{A}}^{(1)})\mathbf{x}_k}{2(\lambda_k^{(1)} - \lambda_j^{(1)})}, \quad j\neq k, \quad j,k = 1,2\ldots,n, \quad (23)$$

$$\lambda_j^{(2)} = \mathbf{y}_j^{\mathrm{T}}(\mathring{\mathbf{A}}_2 - 2\mathring{\mathbf{A}}^{(1)}\mathbf{G}\mathring{\mathbf{A}}^{(1)} - 2\mathbf{B}^{(1)}\mathbf{X}_U\mathbf{Y}_U^{\mathrm{T}}\mathring{\mathbf{A}}^{(1)})\mathbf{x}_j, \quad j = 1,2,\ldots,n, \quad (24)$$

where temporarily we let $\mathbf{y}_j$ and $\mathbf{x}_j$ represent the jth column vectors of $\mathbf{X}_U$ and $\mathbf{Y}_U$, respectively. The diagonal elements of matrix $\mathbf{S}_1$ will be solved by using the gauge conditions given by equation (3) or (4). Differentiating equations (3) and (4a) with respect to p and then letting $p\to p_0$ results in

$$\mathbf{x}_i^{\mathrm{T}}(\mathbf{B}+\mathbf{B}^{\mathrm{T}})\mathbf{x}_i^{(1)} = -\mathbf{x}_i^{\mathrm{T}}\mathbf{B}^{(1)}\mathbf{x}_i, \quad \mathrm{Re}(\mathbf{x}_i^{*}\mathbf{x}_i^{(1)}) = 0. \quad (25a, b)$$

Let $\mathbf{x}_i^{(1)}$ denote the ith column vector $\tilde{\mathbf{X}}_U^{(1)}$. Recalling equations (12), (3) and (4a),

from equations (25a) and (25b) one has, for $i = 1, 2, \cdots, n$,

$$S_{1ii} = -\frac{1}{2}\left[\mathbf{x}_i^{\mathrm{T}}\mathbf{B}^{(1)}\mathbf{x}_i + \mathbf{x}_i^{\mathrm{T}}(\mathbf{B}+\mathbf{B}^{\mathrm{T}})\tilde{\mathbf{x}}_i^{(1)} + \sum_{\substack{j=1\\ j\neq i}}^{n}\mathbf{x}_i^{\mathrm{T}}(\mathbf{B}+\mathbf{B}^{\mathrm{T}})\mathbf{x}_j S_{1ji}\right], \tag{26a}$$

$$\mathrm{Re}(S_{1ii}) = -\mathrm{Re}\left(\mathbf{x}_i^{*}\tilde{\mathbf{x}}_i^{(1)} + \sum_{\substack{j=1\\ j\neq i}}^{n}\mathbf{x}_i^{*}\mathbf{x}_j S_{1ji}\right). \tag{26b}$$

Notice that in equation (26b) corresponding to condition (4a), only the real parts of S_{1ii} can be solved. Indeed, this is due to the fact that condition (4a) can only gauge the modulus of a differentiated eigenvector. Recalling that the complementary condition (4b) requires that the maximum component of $\mathbf{x}_i(p)$, denoted by $x_{iq}(p)$, remains real in the vicinity of $p = p_0$, which implies that $x_{iq}^{(1)}$ is also real, one can obtain the imaginary part of S_{1ii} as

$$\mathrm{Im}(S_{1ii}) = -\mathrm{Im}\left(\tilde{x}_{iq}^{(1)} + \sum_{\substack{j=1\\ j\neq i}}^{n} x_{jq}S_{1ji}\right)/x_{iq}. \tag{27}$$

Now the homogeneous and particular solution parts of equation (12) with $d = 1$ are both solved; see equation (18). A direct method with biorthogonal decomposition is then completed.

4. MODAL METHOD OBTAINED BY ASYMPTOTIC EXPANSION

In virtue of equations (12), (21) and (17b), the first order eigenvector derivatives can be alternatively expressed as

$$\begin{aligned}\mathbf{X}_U^{(1)} &= \tilde{\mathbf{X}}_U^{(1)} + \mathbf{X}_U\mathbf{S}_1 = -\mathbf{G}(\lambda_0)\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U + \mathbf{X}_U\mathbf{S}_1\\ &= \mathbf{X}_V(\mathbf{\Lambda}_V - \lambda_0\mathbf{I}_m)^{-1}\mathbf{Y}_V^{\mathrm{T}}\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U + \hat{\mathbf{X}}_U\alpha\mathbf{S}_1,\end{aligned} \tag{28}$$

which coincides with the modal expansion result given by Juang [5]. In fact, the only difference between the direct method presented in the preceding section and the modal expansion method is the solving strategy for the particular solution $\tilde{\mathbf{X}}_U^{(1)}$ which falls into the space spanned by the column vectors of $\mathbf{X}_V$. In view of modal expansion, an exact solution for the eigenvector derivatives requires that all the eigenvectors of the original system are known *a priori*, which cannot be the case in practice. This has resulted in some approximate techniques by means of modal truncation and then compensating. In this section, an asymptotic

expansion of the invariant generalized 1-inverse $\mathbf{G}(\lambda_0)$ is derived and then used to obtain approximate eigenvector derivatives. It is shown that, for the expansion to be convergent, one needs to know all the lower order eigenvectors at least up to the one corresponding to λ_{h+1} (see equation (6) for the definition of h).

Suppose equation (1) has r-fold ($r \geqslant 0$) zero eigenvalue, and the corresponding right and left eigenvector subsets are denoted by $\mathbf{X}_R$ and $\mathbf{Y}_R$. Recalling equation (14), analogously one can obtain

$$\mathbf{G}(0) = \begin{cases} (\mathbf{I}_N - \mathbf{X}_R\mathbf{Y}_R^{\mathrm{T}}\mathbf{B})\mathbf{C}(0)(\mathbf{I}_N - \mathbf{B}\mathbf{X}_R\mathbf{Y}_R^{\mathrm{T}}), \\ \displaystyle\sum_{j=r+1}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j}, \end{cases} \tag{29}$$

where the first expression suggests a practical method for the computation of $\mathbf{G}(0)$ which is often called the elastic flexibility matrix in mechanics terminology [17], and the second expression will be used in the following derivation. From the biorthonormalization condition (5), it is easy to verify that, for $k = 0, 1, 2, \cdots$,

$$[\mathbf{G}(0)\mathbf{B}]^k\mathbf{G}(0) = \sum_{j=r+1}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j^{k+1}}.$$

Generally, in a practical dynamic analysis, the number of degrees of freedom of the system is very large but the eigenvalue λ_0 of interest often falls into the lower-frequency segment. Therefore, one can always find an integer $g(\lambda_0) > h$ such that $\forall j > g(\lambda_0)$, $|\lambda_0/\lambda_j| < 1$. Recalling equation (14), one can express $\mathbf{G}(\lambda_0)$ in terms of a convergent power series,

$$\begin{aligned} \mathbf{G}(\lambda_0) = & \sum_{\substack{j=1 \\ j\neq l\sim h}}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j - \lambda_0} = \sum_{\substack{j=1 \\ j\neq l\sim h}}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j - \lambda_0} \\ & + \sum_{j=g(\lambda_0)+1}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j}\left[1 + \frac{\lambda_0}{\lambda_j} + \left(\frac{\lambda_0}{\lambda_j}\right)^2 + \cdots + \left(\frac{\lambda_0}{\lambda_j}\right)^k + \cdots\right] \\ = & \mathbf{H}_{-1} + \mathbf{H}_0 + \mathbf{H}_1 + \mathbf{H}_2 + \cdots + \mathbf{H}_k + \cdots, \end{aligned} \tag{30}$$

where

$$\mathbf{H}_{-1} = \sum_{\substack{j=1 \\ j\neq l\sim h}}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j - \lambda_0} = -\frac{\mathbf{X}_R\mathbf{Y}_R^{\mathrm{T}}}{\lambda_0} + \sum_{\substack{j=r+1 \\ j\neq l\sim h}}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j - \lambda_0},$$

$$\mathbf{H}_0 = \sum_{j=g(\lambda_0)+1}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j} = \sum_{j=r+1}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j} - \sum_{j=r+1}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j} = \mathbf{G}(0) - \left(\frac{\mathbf{X}_U\mathbf{Y}_U^{\mathrm{T}}}{\lambda_0} + \sum_{\substack{j=r+1 \\ j\neq l\sim h}}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j} \right),$$

$$\mathbf{H}_1 = \lambda_0 \sum_{j=g(\lambda_0)+1}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j^2} = \lambda_0 \sum_{j=r+1}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j^2} - \lambda_0 \sum_{j=r+1}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j^2}$$

$$= [\lambda_0\mathbf{G}(0)\mathbf{B}]\mathbf{G}(0) - \left(\frac{\mathbf{X}_U\mathbf{Y}_U^{\mathrm{T}}}{\lambda_0} + \lambda_0 \sum_{\substack{j=r+1 \\ j\neq l\sim h}}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j^2} \right),$$

$$\mathbf{H}_k = \lambda_0^k \sum_{j=g(\lambda_0)+1}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j^{k+1}} = \lambda_0^k \sum_{j=r+1}^{N} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j^{k+1}} - \lambda_0^k \sum_{j=r+1}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j^{k+1}}$$

$$= [\lambda_0\mathbf{G}(0)\mathbf{B}]^k\mathbf{G}(0) - \left(\frac{\mathbf{X}_U\mathbf{Y}_U^{\mathrm{T}}}{\lambda_0} + \lambda_0^k \sum_{\substack{j=r+1 \\ j\neq l\sim h}}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j^{k+1}} \right).$$

Let $\mathbf{E}_k$ be the partial sum of the series; one has

$$\mathbf{E}_k = \sum_{j=-1}^{k} \mathbf{H}_k = -\frac{\mathbf{X}_R\mathbf{Y}_R^{\mathrm{T}}}{\lambda_0} - \frac{(k+1)\mathbf{X}_U\mathbf{Y}_U^{\mathrm{T}}}{\lambda_0} + \sum_{j=0}^{k} [\lambda_0\mathbf{G}(0)\mathbf{B}]^j\mathbf{G}(0)$$

$$+ \sum_{\substack{j=r+1 \\ j\neq l\sim h}}^{g(\lambda_0)} \mathbf{x}_j\mathbf{y}_j^{\mathrm{T}} \left[\frac{1}{\lambda_j - \lambda_0} - \left(\frac{1}{\lambda_j} + \frac{\lambda_0}{\lambda_j^2} + \frac{\lambda_0^2}{\lambda_j^3} + \cdots + \frac{\lambda_0^k}{\lambda_j^{k+1}} \right) \right]$$

$$= -\frac{\mathbf{X}_R\mathbf{Y}_R^{\mathrm{T}}}{\lambda_0} - \frac{(k+1)\mathbf{X}_U\mathbf{U}_U^{\mathrm{T}}}{\lambda_0} + \sum_{j=0}^{k} [\lambda_0\mathbf{G}(0)\mathbf{B}]^j\mathbf{G}(0) + \sum_{\substack{j=r+1 \\ j\neq l\sim h}}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j - \lambda_0} \left(\frac{\lambda_0}{\lambda_j} \right)^{k+1}. \quad (31)$$

Hence, provided all the lower order eigenvectors (up to the one corresponding to $\lambda_{g(\lambda_0)}$) are known, the kth approximation for the particular solution of the eigenvector derivatives has the form

$$\tilde{\mathbf{X}}_U^{(1)} \approx \mathbf{E}_k(-\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U)$$

$$= \frac{\mathbf{X}_R\mathbf{Y}_R^{\mathrm{T}}\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U}{\lambda_0} + \frac{(k+1)\mathbf{X}_U\mathbf{\Lambda}_U^{(1)}}{\lambda_0} + \sum_{j=0}^{k}[\lambda_0\mathbf{G}(0)\mathbf{B}]^j\mathbf{G}(0)(-\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U)$$

$$+ \sum_{\substack{j=r+1 \\ j\neq l\sim h}}^{g(\lambda_0)} \frac{\mathbf{x}_j\mathbf{y}_j^{\mathrm{T}}}{\lambda_j - \lambda_0}\left(\frac{\lambda_0}{\lambda_j}\right)^{k+1}(-\mathring{\mathbf{A}}^{(1)}\mathbf{X}_U). \tag{32}$$

After obtaining the approximate particular solution of the eigenvector derivatives, one can solve for the homogeneous solution part by using equations (22)–(27) derived in the preceding section. Notice that with sufficiently large k, the error or of the present modal method will diminish to zero.

The present modal method virtually covers all the previously developed modal-expansion based methods as special cases. If the eigenvalue to be differentiated is a distinct one, combining equations (32), (26a) and (12) and letting $n = 1$ one can get the so-called family of modal methods proposed by Akgün [7]. As noted by Akgün [7], if one lets $k = -$, i.e., excluding all the higher eigenvectors without compensating, one will have the truncated modal method result. If one lets $k = 0$, one can get Wang's modified modal method (or referred to as the modal acceleration method). Higher order compensation can be achieved if one applied larger k. On the other hand, no additional effort is needed if the original system has rigid-body modes. Equation (32) also implies a scheme of explicit iteration, which is similar to the iterative procedure proposed by Zhang and Zerva [11, 12]. In summary, the present modal method, which can be used to treat any non-defective eigenproblem with repeated eigenvalues, is the most general modal method so far developed.

5. ILLUSTRATIVE EXAMPLE

In this section a numerical example is given to demonstrate the validity of the presented formulae. The direct (exact) and modal (approximate) methods are compared, and the issue of computational efficiency is addressed. All results are obtained by using MATLAB5.1.

Suppose one is given the following system,

$$\mathbf{A}(p_0) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 10 & -1 & 5 & 0 & 0 \\ 2 & 6 & 3 & 4 & 10 & 3 & 0 & 10 \\ 1 & 1 & 1 & 1 & 1 & 10 & 0 & 10 \\ 20 & 0 & 0 & 0 & 0 & 0 & q & 20 \\ 3 & 7 & 4 & 5 & 11 & 13 & 0 & 20 \end{bmatrix}, \quad \mathbf{B}(p_0) = \mathbf{I}_8,$$

$$\mathbf{A}^{(1)}(p_0) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\mathbf{B}^{(i)}(p_0) = \mathbf{0}, \quad i = 1, 2, \ldots; \quad \mathbf{A}^{(i)}(p_0) = \mathbf{0}, \quad i = 2, 3, \ldots,$$

where q is an adjustable parameter and at this moment is set to $q = 5$. The original system has a two-fold eigenvalue $\lambda_0 = 1{\cdot}0$, whose corresponding biorthonormalized right and left eigenvectors are, respectively,

$$\hat{\mathbf{X}}_U = \begin{bmatrix} -2{\cdot}1192650\text{e}-01 & -2{\cdot}2191360\text{e}-01 \\ 6{\cdot}1062716\text{e}-03 & 7{\cdot}8322447\text{e}-02 \\ 2{\cdot}1623978\text{e}-02 & -2{\cdot}2191360\text{e}-01 \\ 1{\cdot}8419626\text{e}-03 & 3{\cdot}6550475\text{e}-03 \\ 1{\cdot}6577663\text{e}-02 & 3{\cdot}2895428\text{e}-02 \\ -8{\cdot}0758984\text{e}-17 & -2{\cdot}1051200\text{e}-16 \\ 9{\cdot}7674421\text{e}-01 & 9{\cdot}4509086\text{e}-01 \\ 1{\cdot}6577663\text{e}-02 & 3{\cdot}2895428\text{e}-02 \end{bmatrix},$$

$$\hat{\mathbf{Y}}_U = \begin{bmatrix} -4{\cdot}2817296\text{e}+00 & -2{\cdot}2130128\text{e}-01 \\ 3{\cdot}1086245\text{e}-15 & 3{\cdot}3307127\text{e}-00 \\ 4{\cdot}2817296\text{e}+00 & -3{\cdot}1094114\text{e}+00 \\ 8{\cdot}0987389\text{e}-16 & -4{\cdot}7536980\text{e}-17 \\ 1{\cdot}2173526\text{e}-15 & -5{\cdot}6939838\text{e}-16 \\ 6{\cdot}8585474\text{e}-16 & -1{\cdot}4599762\text{e}-15 \\ 0 & 0 \\ -8{\cdot}5798454\text{e}-16 & 4{\cdot}4007863\text{e}-16 \end{bmatrix},$$

The derivatives of this eigenpair will be computed by using the two different approaches proposed in this paper.

First, the direct method developed in the third section is used. In the direct method, the above information is sufficient for sensitivity analysis, i.e., only the eigenvalues and the eigenvectors to be differentiated are needed in this approach. Using equations (20)–(27) in sequence, one can obtain the differentiable right and left eigenvectors,

$$\mathbf{X}_U =$$

$$\begin{bmatrix} -2{\cdot}1422098\text{e}-01+1{\cdot}7469047\text{e}-02\text{i} & -2{\cdot}1422098\text{e}-01-1{\cdot}7469047\text{e}-02\text{i} \\ 8{\cdot}3370652\text{e}-02-7{\cdot}5052202\text{e}-02\text{i} & 8{\cdot}3370652\text{e}-02+7{\cdot}5052202\text{e}-02\text{i} \\ -2{\cdot}4061602\text{e}-01+2{\cdot}5168368\text{e}-01\text{i} & -2{\cdot}4061602\text{e}-01-2{\cdot}5168368\text{e}-01\text{i} \\ 3{\cdot}7146635\text{e}-03-1{\cdot}9410052\text{e}-03\text{i} & 3{\cdot}7146635\text{e}-03+1{\cdot}9410052\text{e}-03\text{i} \\ 3{\cdot}3431972\text{e}-02-1{\cdot}7469047\text{e}-02\text{i} & 3{\cdot}3431972\text{e}-02+1{\cdot}7469047\text{e}-02\text{i} \\ -2{\cdot}1680813\text{e}-16+1{\cdot}3719260\text{e}-16\text{i} & -2{\cdot}1680813\text{e}-16-1{\cdot}3719260\text{e}-16\text{i} \\ 9{\cdot}0394506\text{e}-01 & 9{\cdot}0394506\text{e}-01 \\ 3{\cdot}3431972\text{e}-02-1{\cdot}7469047\text{e}-02\text{i} & 3{\cdot}3431972\text{e}-02+1{\cdot}7469047\text{e}-02\text{i} \end{bmatrix},$$

$$\mathbf{Y}_U =$$

$$\begin{bmatrix} -2{\cdot}4289664\text{e}+00+2{\cdot}4085287\text{e}+00\text{i} & -2{\cdot}4289664\text{e}+00-2{\cdot}4085287\text{e}+00\text{i} \\ 1{\cdot}7411601\text{e}+00-1{\cdot}9622168\text{e}-01\text{i} & 1{\cdot}7411601\text{e}+00+1{\cdot}9622168\text{e}-01\text{i} \\ 6{\cdot}8780633\text{e}-01-2{\cdot}2123070\text{e}+00\text{i} & 6{\cdot}8780633\text{e}-01+2{\cdot}2123070\text{e}+00\text{i} \\ 4{\cdot}1269812\text{e}-16-4{\cdot}5029809\text{e}-16\text{i} & 4{\cdot}1269812\text{e}-16+4{\cdot}5029809\text{e}-16\text{i} \\ 3{\cdot}6003785\text{e}-16-6{\cdot}4752511\text{e}-16\text{i} & 3{\cdot}6003785\text{e}-16+6{\cdot}4752511\text{e}-16\text{i} \\ -3{\cdot}9267073\text{e}-16-2{\cdot}9770255\text{e}-16\text{i} & -3{\cdot}926707\text{e}-16+2{\cdot}9770255\text{e}-16\text{i} \\ 0 & 0 \\ -2{\cdot}3348606\text{e}-16+4{\cdot}5408874\text{e}-16\text{i} & -2{\cdot}3348606\text{e}-16-4{\cdot}5408875\text{e}-16\text{i} \end{bmatrix},$$

and then the first and second order eigenvalue derivatives,

$$\mathbf{\Lambda}_U^{(1)} = \begin{bmatrix} 0{\cdot}3460869565217411+0{\cdot}3010956677440148\text{i} & \\ & 0{\cdot}3460869565217411-0{\cdot}3010956677440148\text{i} \end{bmatrix},$$

$$\mathbf{\Lambda}_U^{(2)} = \begin{bmatrix} -1{\cdot}993970323662362+1{\cdot}6293906337786 91\text{i} & \\ & -1{\cdot}993970323662362-1{\cdot}6293906337786 91\text{i} \end{bmatrix},$$

and finally the right eigenvector derivatives

$$\mathbf{X}_U^{(1)} =$$

$$\begin{bmatrix}
4{\cdot}2046858\text{e}-01+9{\cdot}1150201\text{e}-02\text{i} & 4{\cdot}2046858\text{e}-01-9{\cdot}1150201\text{e}-02\text{i} \\
-9{\cdot}9724949\text{e}-02-1{\cdot}7656000\text{e}-01\text{i} & -9{\cdot}9724949\text{e}-02+1{\cdot}7656000\text{e}-01\text{i} \\
1{\cdot}1328952\text{e}+00+7{\cdot}4747461\text{e}-01\text{i} & 1{\cdot}1328952\text{e}+00-7{\cdot}4747461\text{e}-01\text{i} \\
-2{\cdot}1527791\text{e}-01+1{\cdot}0081471\text{e}-02\text{i} & -2{\cdot}1527791\text{e}-01-1{\cdot}0081471\text{e}-02\text{i} \\
7{\cdot}0603519\text{e}-02-7{\cdot}6232028\text{e}-02\text{i} & 7{\cdot}0603519\text{e}-02+7{\cdot}6232028\text{e}-02\text{i} \\
3{\cdot}5387175\text{e}-01+1{\cdot}7033024\text{e}-02\text{i} & 3{\cdot}5387174\text{e}-01-1{\cdot}7033024\text{e}-02\text{i} \\
1{\cdot}9769911\text{e}-01 & 1{\cdot}9769911\text{e}-01 \\
-4{\cdot}4603782\text{e}-01-7{\cdot}6668051\text{e}-02\text{i} & -4{\cdot}4603782\text{e}-01+7{\cdot}6668051\text{e}-02\text{i}
\end{bmatrix}.$$

To verify the obtained eigenvalue and eigenvector derivatives, they are used to approximate the eigensolution of the perturbed matrix pair ($\mathbf{A}(p)$, $\mathbf{B}(p)$). In fact, $\mathbf{\Lambda}_U(p)$ and $\mathbf{X}_U(p)$ can be approximated by the truncated Taylor series in the following manner:

$$\mathbf{\Lambda}_U(p)|\text{approx} \approx \lambda_0 \mathbf{I}_2 + \varepsilon \mathbf{\Lambda}_U^{(1)} + \frac{\varepsilon^2}{2}\mathbf{\Lambda}_U^{(2)}, \quad \mathbf{X}_U(p)|\text{approx} \approx \mathbf{X}_U + \varepsilon \mathbf{X}_U^{(1)},$$

where $\varepsilon \stackrel{\text{def}}{=} p - p_0$. Figure 1 shows the difference between the exact eigensolution and the approximate one obtained by way of sensitivity analysis. Since the original repeated eigenvalue splits into a pair of complex conjugated eigenvalues, only the results of one perturbed eigenvalue and its eigenvector are plotted here.

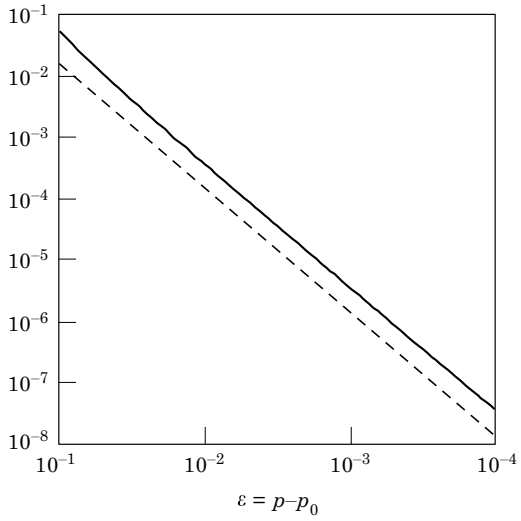

Figure 1. Difference between the exact perturbed eigensolution and the approximate one obtained by sensitivity analysis: ----, $|\lambda(p)_{\text{exact}} - \lambda(p)_{\text{approx}}|/|\lambda(p)_{\text{exact}}|$; ——, $\|\mathbf{x}(p)_{\text{exact}} - \mathbf{x}(p)_{\text{approx}}\|_2/\|\mathbf{x}(p)_{\text{exact}}\|_2$.

Clearly, the difference diminishes rapidly as ε decreases. The validity of the presented direct method is then justified.

The modal method, on the other hand, requires the knowledge of all the lower order eigenvectors at least up to the one corresponding to λ_{h+1} (the eigenvalue next to λ_0). Actually, in this example the complete eigenvalue set of the original system is

$$\mathbf{\Lambda} = \mathrm{diag}(0{\cdot}0,\ 1{\cdot}0,\ 1{\cdot}0,\ 1{\cdot}6028948,\ q,\ 8{\cdot}3210326 \pm 2{\cdot}5437262\mathrm{i},\ 32{\cdot}7550401).$$

Here it is assumed only the four lowest order eigenvectors are available, and firstly it is assumed $q = 5$ is the lowest unavailable eigenvalue. The first and fourth eigenvectors are, respectively,

$$\mathbf{x}_1 = \begin{bmatrix} 2{\cdot}5684747\mathrm{e}-01 \\ 2{\cdot}5684747\mathrm{e}-01 \\ 2{\cdot}5684747\mathrm{e}-01 \\ -2{\cdot}0215643\mathrm{e}-01 \\ 1{\cdot}8182625\mathrm{e}-01 \\ 4{\cdot}4067810\mathrm{e}-01 \\ 5{\cdot}1770371\mathrm{e}-01 \\ -5{\cdot}1569933\mathrm{e}-01 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} -4{\cdot}4090189\mathrm{e}-16 \\ -1{\cdot}5153578\mathrm{e}-15 \\ 1{\cdot}7998107\mathrm{e}-15 \\ 2{\cdot}0687479\mathrm{e}-16 \\ 8{\cdot}7857737\mathrm{e}-01 \\ 8{\cdot}7857737\mathrm{e}-01 \\ 0 \\ -8{\cdot}7857737\mathrm{e}-01 \end{bmatrix},$$

$$\mathbf{x}_4 = \begin{bmatrix} -4{\cdot}2097315\mathrm{e}-01 \\ -4{\cdot}2097315\mathrm{e}-01 \\ -4{\cdot}2097315\mathrm{e}-01 \\ 1{\cdot}1328386\mathrm{e}-01 \\ 3{\cdot}1675014\mathrm{e}-01 \\ -1{\cdot}2690127\mathrm{e}-01 \\ 5{\cdot}5048561\mathrm{e}-01 \\ 1{\cdot}8984888\mathrm{e}-01 \end{bmatrix}, \quad \mathbf{y}_4 = \begin{bmatrix} -1{\cdot}3849739\mathrm{e}-01 \\ -1{\cdot}9692185\mathrm{e}+00 \\ -5{\cdot}9617766\mathrm{e}-01 \\ 7{\cdot}5664826\mathrm{e}-02 \\ 2{\cdot}1040979\mathrm{e}-01 \\ 9{\cdot}5470964\mathrm{e}-01 \\ 0 \\ -4{\cdot}8634286\mathrm{e}-01 \end{bmatrix}.$$

Notice the original system has a rigid body mode. Using equations (29)–(32) and then equations (22)–(27), one can solve for the approximate eigenvector derivatives. The kth ($k = -1,\ 0,\ 1,\ 5$) approximation of the right eigenvector derivatives and the corresponding relative error are, respectively,

$$\mathbf{X}_U^{(1)}|_{k=-1} =$$

$$\begin{bmatrix} 3{\cdot}9075913\text{e}-01+6{\cdot}5878510\text{e}-02\text{i} & 3{\cdot}9075913\text{e}-01-6{\cdot}5878510\text{e}-02\text{i} \\ -8{\cdot}1568828\text{e}-02-1{\cdot}7733796\text{e}-01\text{i} & -8{\cdot}1568828\text{e}-02+1{\cdot}7733796\text{e}-01\text{i} \\ 1{\cdot}0716723\text{e}+00+7{\cdot}4790469\text{e}-01\text{i} & 1{\cdot}0716723\text{e}+00-7{\cdot}4790469\text{e}-01\text{i} \\ -1{\cdot}6090894\text{e}-01-1{\cdot}6625710\text{e}-02\text{i} & -1{\cdot}6090894\text{e}-01+1{\cdot}6625710\text{e}-02\text{i} \\ 7{\cdot}1089663\text{e}-02-6{\cdot}8107287\text{e}-02\text{i} & 7{\cdot}1089663\text{e}-02+6{\cdot}8107287\text{e}-02\text{i} \\ 3{\cdot}3499602\text{e}-01+1{\cdot}5937830\text{e}-02\text{i} & 3{\cdot}3499602\text{e}-01-1{\cdot}5937830\text{e}-02\text{i} \\ 1{\cdot}7366558\text{e}-01 & 1{\cdot}7366558\text{e}-01 \\ -4{\cdot}6442740\text{e}-01-6{\cdot}9638504\text{e}-02\text{i} & -4{\cdot}6442740\text{e}-01+6{\cdot}9638504\text{e}-02\text{i} \end{bmatrix},$$

$$\text{error}|_{k=-1} = \frac{\| \mathbf{x}_2^{(1)}|_{k=-1} - \mathbf{x}_2^{(1)}|_{\text{exact}} \|^2}{\| \mathbf{x}_2^{(1)}|_{\text{exact}} \|^2} = 6{\cdot}5358385\text{e}-02,$$

$$\mathbf{X}_U^{(1)}|_{k=0} =$$

$$\begin{bmatrix} 4{\cdot}1608745\text{e}-01+8{\cdot}5626028\text{e}-02\text{i} & 4{\cdot}1608745\text{e}-01-8{\cdot}5626028\text{e}-02\text{i} \\ -9{\cdot}6045488\text{e}-02-1{\cdot}7613975\text{e}-01\text{i} & -9{\cdot}6045488\text{e}-02+1{\cdot}7613975\text{e}-01\text{i} \\ 1{\cdot}1219471\text{e}+00+7{\cdot}4572545\text{e}-01\text{i} & 1{\cdot}1219471\text{e}+00-7{\cdot}4572545\text{e}-01\text{i} \\ -2{\cdot}0956226\text{e}-01+7{\cdot}9010801\text{e}-03\text{i} & -2{\cdot}0956226\text{e}-01-7{\cdot}9010801\text{e}-03\text{i} \\ 7{\cdot}0636467\text{e}-02-7{\cdot}4355391\text{e}-02\text{i} & 7{\cdot}0636467\text{e}-02+7{\cdot}4355391\text{e}-02\text{i} \\ 3{\cdot}5320655\text{e}-01+1{\cdot}6397787\text{e}-02\text{i} & 3{\cdot}5320655\text{e}-01-1{\cdot}6397787\text{e}-02\text{i} \\ 1{\cdot}9409018\text{e}-01 & 1{\cdot}9409018\text{e}-01 \\ -4{\cdot}4667007\text{e}-01-7{\cdot}5426651\text{e}-02\text{i} & -4{\cdot}4667007\text{e}-01+7{\cdot}5426651\text{e}-02\text{i} \end{bmatrix},$$

$$\text{error}|_{k=0} = \frac{\| \mathbf{x}_2^{(1)}|_{k=0} - \mathbf{x}_2^{(1)}|_{\text{exact}} \|^2}{\| \mathbf{x}_2^{(1)}|_{\text{exact}} \|^2} = 9{\cdot}8708899\text{e}-03,$$

$$\mathbf{X}_U^{(1)}|_{k=1} =$$

$$\begin{bmatrix} 4{\cdot}1969650\text{e}-01+9{\cdot}0026313\text{e}-02\text{i} & 4{\cdot}1969650\text{e}-01-9{\cdot}0026313\text{e}-02\text{i} \\ -9{\cdot}9000001\text{e}-02-1{\cdot}7641961\text{e}-01\text{i} & -9{\cdot}9000001\text{e}-02+1{\cdot}7641961\text{e}-01\text{i} \\ 1{\cdot}1307925\text{e}+00+7{\cdot}4700310\text{e}-01\text{i} & 1{\cdot}1307925\text{e}+00-7{\cdot}4700310\text{e}-01\text{i} \\ -2{\cdot}1474295\text{e}-01+9{\cdot}9567968\text{e}-03\text{i} & -2{\cdot}1474295\text{e}-01-9{\cdot}9567968\text{e}-03\text{i} \\ 7{\cdot}0595376\text{e}-02-7{\cdot}5907796\text{e}-02\text{i} & 7{\cdot}0595376\text{e}-02+7{\cdot}5907796\text{e}-02\text{i} \\ 3{\cdot}5392371\text{e}-01+1{\cdot}6901918\text{e}-02\text{i} & 3{\cdot}5392371\text{e}-01-1{\cdot}6901918\text{e}-02\text{i} \\ 1{\cdot}9706044\text{e}-01 & 1{\cdot}9706044\text{e}-01 \\ -4{\cdot}4599402\text{e}-01-7{\cdot}6474924\text{e}-02\text{i} & -4{\cdot}4599402\text{e}-01+7{\cdot}6474924\text{e}-02\text{i} \end{bmatrix},$$

$$\text{error}|_{k=1} = \frac{\| \mathbf{x}_2^{(1)}|_{k=1} - \mathbf{x}_2^{(1)}|_{\text{exact}} \|^2}{\| \mathbf{x}_2^{(1)}|_{\text{exact}} \|^2} = 1{\cdot}7818865\text{e}-03,$$

$$\mathbf{X}_U^{(1)}|_{k=5} =$$

$$\begin{bmatrix} 4{\cdot}2046736\text{e}-01+9{\cdot}1148497\text{e}-02\text{i} & 4{\cdot}2046736\text{e}-01-9{\cdot}1148497\text{e}-02\text{i} \\ -9{\cdot}9723881\text{e}-02-1{\cdot}7655970\text{e}-01\text{i} & -9{\cdot}9723881\text{e}-02+1{\cdot}7655970\text{e}-01\text{i} \\ 1{\cdot}1328919\text{e}+00+7{\cdot}4747388\text{e}-01\text{i} & 1{\cdot}1328919\text{e}+00-7{\cdot}4747388\text{e}-01\text{i} \\ -2{\cdot}1527793\text{e}-01+1{\cdot}0081552\text{e}-02\text{i} & -2{\cdot}1527793\text{e}-01-1{\cdot}0081552\text{e}-02\text{i} \\ 7{\cdot}0603765\text{e}-02-7{\cdot}6231803\text{e}-02\text{i} & 7{\cdot}0603765\text{e}-02+7{\cdot}6231803\text{e}-02\text{i} \\ 3{\cdot}5387181\text{e}-01+1{\cdot}7032985\text{e}-02\text{i} & 3{\cdot}5387181\text{e}-01-1{\cdot}7032985\text{e}-02\text{i} \\ 1{\cdot}9769809\text{e}-01 & 1{\cdot}9769809\text{e}-01 \\ -4{\cdot}4603751\text{e}-01-7{\cdot}6667865\text{e}-02\text{i} & -4{\cdot}4603751\text{e}-01+7{\cdot}6667865\text{e}-02\text{i} \end{bmatrix},$$

$$\text{error}|_{k=5} = \frac{\| \mathbf{x}_2^{(1)}|_{k=5} - \mathbf{x}_2^{(1)}|_{\text{exact}} \|^2}{\| \mathbf{x}_2^{(1)}|_{\text{exact}} \|^2} = 2{\cdot}7161320\text{e}-06.$$

The eigenvector derivatives obtained by the direct method are now referred to as the exact solution. Only one error index is needed because the two eigenvector derivatives are complex conjugates. Clearly, when the first unavailable eigenvalue is $q = 5$, the 5th approximation of the eigenvector derivatives already has satisfactory accuracy.

The computational efficiency of the present direct and modal methods still needs to be discussed. The advantageous feature of the modal method is obvious. The most time-consuming steps in the above two methods are solving $\mathbf{G}(\lambda_0)$ (in the direct method) and $\mathbf{G}(0)$ (in the modal method), respectively. Observe equations (21) and (32). While the direct method requires solving for different $\mathbf{G}(\lambda_0)$ if a number of eigenvectors corresponding to different eigenvalues are to be differentiated, only one $\mathbf{G}(0)$ is needed for the modal method in that case. The disadvantage of the modal method is the error induced by approximation. As pointed out in the preceding section, the modal method uses $\mathbf{E}_k$ to approximate the generalized 1-inverse $\mathbf{G}(\lambda_0)$. With k increasing, $\mathbf{E}_k$ will eventually converge to $\mathbf{G}(\lambda_0)$. Observing equations (30) and (31), one can find that the convergence rate of the modal method depends on the ratios of the differentiated eigenvalue to the unavailable eigenvalues, and primarily depends on the ratio of the differentiated eigenvalue to the lowest unavailable eigenvalue. Actually, when k increases, the error between $\mathbf{E}_k$ and $\mathbf{G}(\lambda_0)$ will decrease mainly at the rate of $(\lambda_0/\lambda_{g(\lambda_0)+1})^{k+1}$. This effect is simulated in Figure 2, where we let the adjustable parameter q in the present numerical example be 2, 5 and 8, respectively. In each case q is the lowest unavailable eigenvalue, and the error between $\mathbf{E}_k$ and $\mathbf{G}(\lambda_0)$ is plotted. The error reduces roughly in the order of $(1/q)^{k+1}$, in this specific example ($\lambda_0 = 1{\cdot}0$), and the convergence rate increases quickly as q increases. The ratios of the differentiated eigenvalue to other unavailable eigenvalues, on the other hand, have less effect on the convergence rate. Figure 3 shows different convergence rates corresponding to $q = 9$, 10, 30 which represent different second lowest unavailable eigenvectors. Though q differs significantly, the convergence rates are roughly the same. In all, the ratio

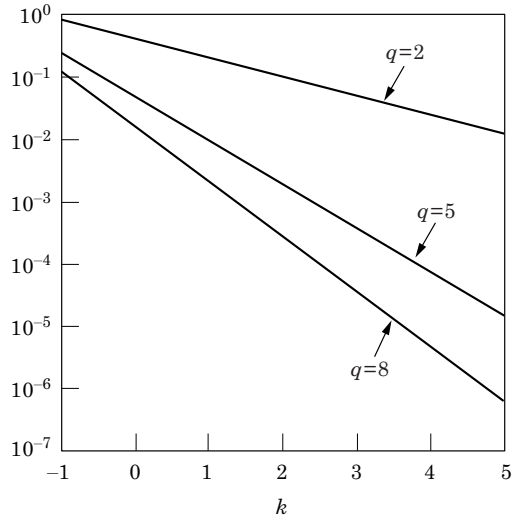

Figure 2. Relative error between the exact generalized 1-inverse $\mathbf{G}(\lambda_0)$ and its kth approximation $\mathbf{E}_k$, $\|\mathbf{G}(\lambda_0) - \mathbf{E}_k\|_2 / \|\mathbf{G}(\lambda_0)\|_2$, versus k ($= -1, 0, 1, 2, 3, 4, 5$). Here q is the lowest unavailable eigenvalue.

$\lambda_0/\lambda_{g(\lambda_0)+1}$ plays a key role in eigensolution sensitivity calculation. It decides the convergence rate if the modal method is used, and therefore decides which one between the direct and modal methods is preferred when a real system is to be analyzed.

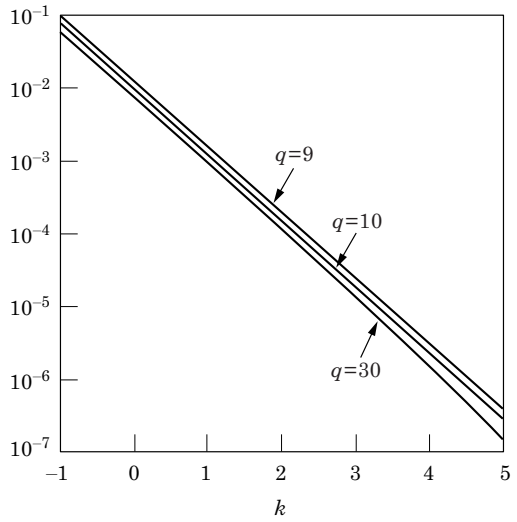

Figure 3. Relative error between the exact generalized 1-inverse $\mathbf{G}(\lambda_0)$ and its kth approximation $\mathbf{E}_k$, $\|\mathbf{G}(\lambda_0) - \mathbf{E}_k\|_2 / \|\mathbf{G}(\lambda_0)\|_2$, versus k ($= -1, 0, 1, 2, 3, 4, 5$). Here q is the second lowest unavailable eigenvalue.

6. CONCLUDING REMARKS

A unified theory for the calculation of eigensolution sensitivity is developed for general non-defective eigenproblems with repeated roots. A direct method with biorthogonal decomposition is first presented. The asymptotic expansion of this direct method then results in a modal-expansion based method. The intrinsic relation between these two kinds of methods is exposed.

Generally speaking, if only one eigenvalue and its eigenvectors set are to be differentiated, the direct method should be used as it requires least computational effort and gives accurate results. When a group of eigenvalues and their eigenvectors are to be differentiated, the modal method presented in this paper becomes more and more preferable as the number of (distinct) eigenvalues to be differentiated increases, provided all the lower order eigenvectors are available. The convergence rate of the present modal method is shown to mainly depend on the ratio of the differentiated eigenvalue to the lowest unavailable eigenvalue. The error between the approximate modal method solution and the exact solution diminishes roughly at the rate of $(\lambda_0/\lambda_{g(\lambda_0)+1})^{k+1}$. If the lowest unavailable eigenvalue $\lambda_{g(\lambda_0)+1}$ is not close to λ_0, the modal method can easily achieve satisfactory accuracy.

REFERENCES

1. H. M. Adelman and R. T. Haftka 1986 *AIAA Journal* **5**, 823–832. Sensitivity analysis of discrete structural systems.
2. D. V. Murthy and R. T. Haftka 1988 *International Journal for Numerical Methods in Engineering* **16**, 293–311. Derivatives of eigenvalues and eigenvectors of a general complex matrix.
3. T. R. Sutter, C. J. Camarda, J. L. Walsh and H. M. Adelman 1988 *AIAA Journal* **26**, 1506–1511. Comparison of several methods for calculating vibration mode shape derivatives.
4. R. L Fox and M. P. Kapoor 1968 *AIAA Journal* **6**, 2426–2429. Rates of changes of eigenvalues and eigenvectors.
5. J.-N. Juang, P. Ghaemmaghami and K. B. Lim 1989 *Journal of Guidance, Control and Dynamics* **12**, 480–486. Eigenvalue and eigenvector derivatives of a nondefective matrix.
6. B. P. Wang 1991 *AIAA Journal* **29**, 1018–1020. Improved approximate methods for computing eigenvector derivatives in structural dynamics.
7. M. A. Akgün 1994 *AIAA Journal* **32**, 379–386. New family of modal methods for calculating eigenvector derivatives.
8. M. L. Bernard and A. J. Bronowick 1994 *AIAA Journal* **32**, 1500–1506. Modal expansion method for eigensensitivity with repeated roots.
9. R. M. Lin and M. K. Lim 1996 *AIAA Journal* **34**, 1083–1085. Eigenvector derivatives of structures with rigid body modes.
10. M. Yu, S.-S. Liu and D.-J. Wang 1996 *Computers and Structures* **62**, 381–393. Comparison of several approximate modal methods for computing mode shape derivatives.
11. O. Zhang and A. Zerva 1996 *AIAA Journal* **34**, 1088–1090. Iterative method for calculating derivatives of eigenvectors.
12. O. Zhang and A. Zerva 1997 *AIAA Journal* **35**, 340–348. Accelerated iterative procedure for calculating eigenvector derivatives.

13. R. B. NELSON 1976 *AIAA Journal* **14**, 1201–1205. Simplified calculation of eigenvector derivatives.
14. W. C. MILLS-CURRAN *AIAA Journal* **26**, 867–871. Calculation of eigenvector derivatives for structures with repeated eigenvalues.
15. Y.-Q. ZHANG and W.-L. WANG 1995 *Transactions of the ASME, Journal of Gas Turbine and Power* **117**, 207–212. Eigenvector derivatives of generalized nondefective eigenproblems with repeated eigenvalues.
16. M. I. FRISWELL 1996 *Transactions of the ASME, Journal of Vibration and Acoustics* **118**, 390–397. The derivatives of repeated eigenvalues and their associated eigenvectors.
17. R. R. CRAIG, Jr. 1981 *Structural Dynamics—An Introduction to Computer Methods*. New York: Wiley.
18. A. BEN-ISRAEL 1974 *Generalized Inverses: Theory and Applications*. New York: Wiley.
19. J. TANG, W.-M. NI and W.-L. WANG 1996 *Journal of Sound and Vibration* **196**, 179–188. Sensitivity calculation for quadratic eigenproblems.

APPENDIX: NOMENCLATURE

$\mathbf{A}(p)$ and $\mathbf{B}(p)$	parameter-dependent complex matrices
$\mathring{\mathbf{A}}$	see equation (8)
$\mathring{\mathbf{A}}^{(d)}$	dth derivative of $\mathring{\mathbf{A}}$, see equation (8)
$\mathbf{F}_d$	see equation (8)
$\mathbf{G}(\lambda_0)$	invariant generalized 1-inverse of $\mathring{\mathbf{A}}$
$\mathbf{I}_N$	Nth order identity matrix
$\mathbf{L}(\lambda_0)$	left projection matrix corresponding to λ_0, see equation (10b)
$\mathbf{R}(\lambda_0)$	right projection matrix corresponding to λ_0, see equation (10a)
$\mathbf{S}_d$	coefficient matrix in the homogeneous solution part of the eigenvector derivative
$\mathbf{X}$ and $\mathbf{Y}$	complete right and left eigenvector sets
$\mathbf{X}_U$ and $\mathbf{Y}_U$	differentiable right and left eigenvector subsets corresponding to $\hat{\mathbf{X}}_U$ and $\hat{\mathbf{Y}}_U$
$\mathbf{X}_U^{(d)}$ and $\mathbf{Y}_U^{(d)}$	dth derivative of $\mathbf{X}_U$ and $\mathbf{Y}_U$
$\tilde{\mathbf{X}}_U^{(d)}$	particular solution part of the eigenvector derivative, see equations (13) and (16)
$\hat{\mathbf{X}}_U$ and $\hat{\mathbf{Y}}_U$	eigenvectors to be differentiated
$\mathbf{X}_V$ and $\mathbf{Y}_V$	complementary eigenvector subsets with respect to $\hat{\mathbf{X}}_U$ and $\hat{\mathbf{Y}}_U$
$\mathbf{x}(p)$ and $\mathbf{y}(p)$	right and left eigenvectors
$\mathbf{\Lambda}_U = \lambda_0 \mathbf{I}_n$	the n-fold eigenvalue to be differentiated
$\mathbf{\Lambda}_U^{(d)}$	dth derivative of $\mathbf{\Lambda}_U$
$\mathbf{\Lambda}_V$	complementary eigenvalue subsets with respect to $\mathbf{\Lambda}_U$