Journal of Sound and Vibration (1999) **225**(4), 633–653
Article No. jsvi.1999.2269, available online at http://www.idealibrary.com on IDEAL®

A "WARPING-KIRCHHOFF" AND A "WARPING-MINDLIN" THEORY OF SHELL DEFORMATION

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(*Received* 11 *September* 1998, *and in final form* 2 *February* 1999)

The warping high-order theory of plate deformation developed in reference [1] is extended here to shells. A theory of shell deformation is derived which accounts for the effects of transverse shear deformation and a non-linear distribution of the in-surface displacements with respect to the thickness co-ordinate. This theory uses the normal modes associated to the normal fibre (considered as a geometrical beam) as basis functions. Using only the rigid body modes, we find the classical theory and using the deformation normal modes, a high order theory is constructed. This theory is compared to other theories and the exact solution through an application to a particular problem of shells.

1. INTRODUCTION

In Hassis [1] and in Lo *et al.* [2], it can be seen that sophisticated models, compared to classical theories, find applications to problems where classical plate theory is simply inadequate to describe the behaviour. Such examples concern contact or laminated problems involving plates and shells, or high-frequency analysis. The present work concerns the derivation and evaluation of a particular "warping" theory of shell behaviour.

Before describing the present theory, it is necessary briefly to review and comment upon the "recent" developments in the generalization of classical shell theory. The developments of Reissner [3, 4], generalizing the classical shell theory, incorporate the effect of shear deformation. The derivation given by Reissner resulting in displacements of the form

$$\mathbf{U}(M) = \begin{cases} U_1(x^1, x^2, x^3) = u_1(x^1, x^2) + x^3\beta_1(x^1, x^2), \\ U_2(x^1, x^2, x^3) = u_2(x^1, x^2) + x^3\beta_2(x^1, x^2), \\ U_3(x^1, x^2, x^3) = u_3(x^1, x^2), \end{cases} \tag{1}$$

where M is a point of the shell, (x^1, x^2) are the surface co-ordinates and x^3 is the normal co-ordinate to the surface. Relation (1) predicts a uniform shear stress distribution through the thickness of the plate. It is then necessary to introduce a correction factor into the shear stress resultant; this is incorrect and in general would violate the surface conditions.

The next higher order theory, for shells, involves displacement forms of type:

$$\mathbf{U}(M) = \begin{cases} U_1(x^1, x^2, x^3) = u_1(x^1, x^2) + x^3\beta_1(x^1, x^2), \\ U_2(x^1, x^2, x^3) = u_2(x^1, x^2) + x^3\beta_2(x^1, x^2), \\ U_3(x^1, x^2, x^3) = u_3(x^1, x^2) + x^3\beta_3(x^1, x^2) + (x^3)^2\zeta_3(x^1, x^2). \end{cases} \tag{2}$$

This theory includes the effect of transverse normal strain. Displacement assumption of the form of equation (2) has been used by Naghdi [5] to derive a general theory of shells and by Essenberg [6] to obtain the corresponding one-dimensional plate theory. They used a shear correction factor which is not appropriate for use with the displacement form of equation (2). This is because non-uniform shear stress is implied by equation (2) along with consequent possible satisfaction of top and bottom boundary conditions for shear traction; thus the rationale for a correction factor is obviated.

For plates, Nelson and Lorch [7], Reissner [8, 9] and Lo *et al.* [2] presented a different high order theory. Bhimaraddi [10] also used a high order theory for free vibration analysis for circular cylindrical shells.

The warping model for plates is associated to the following displacement field [1]:

$$U_1(x^1, x^2, x^3) = u_1(x^1, x^2) + x^3\beta_1(x^1, x^2) + \sum_n W_1^n(x^1, x^2)\phi_n(x^3),$$

$$U_2(x^1, x^2, x^3) = u_2(x^1, x^2) + x_3^3\beta_2(x^1, x^2) + \sum_n W_2^n(x^1, x^2)\phi_n(x^3)$$

$$U_3(x^1, x^2, x^3) = u_3(x^1, x^2) + \sum_k W_3^k(x^1, x^2)\Phi_k(x^3), \tag{3}$$

where $\{\phi_n\}$ and $\{\Phi_k\}$ denote respectively the nth transverse and the kth longitudinal modes (see Appendix A) inducing deformations of normal fibre which is considered as a geometrical beam. The functions (W_1^n, W_2^n, W_3^k) represent the participation of the deformation modes to warping of normal fibres. The functions ϕ_n and Φ_k are called warping co-ordinates.

This work is an extension of the warping theory for plate deformation presented by Hassis [1] to the shell structures.

2. MOTIVATION

For plates, in the case of top and bottom boundary conditions of shear traction and of some loads, or in higher dynamic analysis, classical theories, except the higher theory of Lo [2] and Hassis [1], are unsatisfactory for predicting the non-uniform shear and the normal stress, and displacement distribution. In reference [1], it has been seen that for plate bending problems where the loading characteristics possess a high degree of asymmetry with respect to middle plane, a higher theory (Lo or Hassis) is required.

Also, in the case of laminated plates and shells, a high order must be used because of the likely strong non-linear thickness distribution of stress and displacement.

The warping theory for plates is here extended to shells. This theory is based on the non-uniform distribution of in-plane displacement: it is called the "warping" phenomenon. In the present high order theory, non-uniformity of in-surface displacement of shells is considered by a linear combination of normal modes of the normal fibre to the mid-surface. When only the first six normal modes (rigid-body modes) are considered, this theory corresponds to the Reissner–Mindlin lower order theory. In the present work, the transverse normal deformation modes, of the normal fibre, are considered.

3. DISPLACEMENT FIELD

Let ϕ_n be the nth transverse mode (see Appendix A) inducing deformations of normal fibre which is considered as a geometrical beam. Each natural frequency associated with transverse normal modes corresponds to two transverse normal modes: one is in the "first" direction of the tangent plane and the other is in the "second" direction of the tangent plane [11].

The shell is characterized by the surface and the normal fibre. The surface is defined by the vectorial function $\mathbf{0m}(x^1, x^2)$. x^1 and x^2 are two parameters which characterize the surface. x^3 is the normal co-ordinate to the surface. The surface is also characterized by the local natural base $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{n})$ defined by

$$\mathbf{a}_1 = \frac{\partial \mathbf{0m}}{\partial x^1}, \quad \mathbf{a}_2 = \frac{\partial \mathbf{0m}}{\partial x^2}, \quad \mathbf{n} = \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|}.$$

The following local natural base $(\mathbf{a}^1, \mathbf{a}^2, \mathbf{n})$ is also used:

$$\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta, \quad \mathbf{a}_\alpha = a_{\alpha\beta}\mathbf{a}^\beta \quad \text{with } a_{\alpha\beta} = \mathbf{a}_\alpha \,.\, \mathbf{a}_\beta;\ a^{\alpha\beta} = \mathbf{a}^\alpha \,.\, \mathbf{a}^\beta.$$

$a_{\alpha\beta}$ is the covariant metric coefficient; Greek indices run from 1 to 2 (the standard summation convention for repeated indices will be used).

We propose here two displacement fields which combine the *warping theory and the Kirchhoff field or the warping theory and the Mindlin field*. The present theory is appropriate to the following displacement forms

- For the Warping–Kirchhoff theory,

$$\mathbf{U}(M) = \begin{Bmatrix} U_\lambda(x^1, x^2, x^3) = u_\lambda - x^3(u_{3,\lambda} + C^\gamma_\lambda u_\gamma) + \mathbf{W}^{\mathbf{n}}_\lambda \boldsymbol{\phi}_{\mathbf{n}}(x^3) \\ U_3(x^1, x^2, x^3) = u_3 \end{Bmatrix}_{(\mathbf{a}^\lambda, \mathbf{n})}, \tag{4a}$$

- For the Warping–Mindlin theory,

$$\mathbf{U}(M) = \begin{Bmatrix} U_\lambda(x^1, x^2, x^3) = u_\lambda + x^3\beta_\lambda + \mathbf{W}^{\mathbf{n}}_\lambda \boldsymbol{\phi}_{\mathbf{n}}(x^3) \\ U_3(x^1, x^2, x^3) = u_3 \end{Bmatrix}_{(\mathbf{a}^\lambda, \mathbf{n})}, \tag{4b}$$

where (u_λ, u_3) are the displacements of the surface written in the local surface co-ordinate, (U_λ, U_3) are the displacements of a point of the shell written in the local surface co-ordinate, $C_{\alpha\beta}$ is the coefficient of the curvature tensor, β_λ is the rotation of the normal fibre.

The following notation will be used:

$$\sum_n W^n_1\phi_n \equiv W^n_1\phi_n, \quad \mathbf{u}_\omega = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \pi(\mathbf{u}_\omega) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathbf{W}^n = \begin{pmatrix} W^n_1 \\ W^n_2 \\ 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \end{pmatrix}.$$

This displacement field includes both in-surface and out-of-surface deformation modes. The vector $\mathbf{W}^n$ represents the participation of the deformation modes to the warping of normal fibre. The functions ϕ_n are called warping co-ordinates. Figure 1 shows the first two transverse modes (h is the thickness of the shell). The number of modes used depends on the order of the theory required.

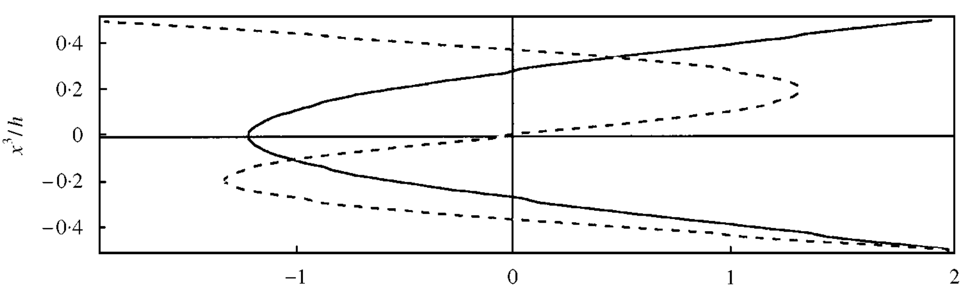

Figure 1. The two first transverse modes. —— Mode 1; - - - Mode 2.

4. DEFORMATION TENSOR–STRESS TENSOR–STRESS RESULTANT

4.1. DEFORMATION TENSOR

The un-deformed and deformed shell is characterized by as shown in Table 1.

Using the previous derivation, the deformation tensor associated to equation (4) is written as

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left\{\mathbf{A}_\alpha \cdot \frac{\partial \mathbf{U}}{\partial x^\beta} + \mathbf{A}_\beta \cdot \frac{\partial \mathbf{U}}{\partial x^\alpha}\right\},$$

$$2\varepsilon_{\alpha 3} = \left\{\mathbf{n} \cdot \frac{\partial \mathbf{U}}{\partial x^\alpha} + \frac{\partial \mathbf{U}}{\partial x^3}(\mathbf{a}_\alpha - x^3 C^\lambda_\alpha \mathbf{a}_\lambda)\right\} \tag{5a}$$

TABLE 1

Un-deformed shell	Deformed shell
Surface: $\mathbf{0m}(x^1, x^2)$; $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{n})$ Shell: $\mathbf{0M} = \mathbf{0m} + x^3\mathbf{n}$ Natural base: $(\mathbf{A}_\alpha, \mathbf{n})$	Surface: $\mathbf{0m}' = \mathbf{0m} + \mathbf{u}_\omega$; $(\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{n}')$ Shell: $\mathbf{0M}' = \mathbf{0m} + \mathbf{x}^3\mathbf{n} + \mathbf{U}$ Natural base: $(\mathbf{A}'_\alpha, \mathbf{A}'_3)$
$\mathbf{A}_\alpha = \dfrac{\partial \mathbf{0M}}{\partial x^\alpha} = \mathbf{a}_\alpha + x^3\dfrac{\partial \mathbf{n}}{\partial x^\alpha} = \mathbf{a}_\alpha - x^3 C_\alpha^\lambda \mathbf{a}_\lambda$	$\mathbf{A}'_\alpha = \dfrac{\partial \mathbf{0M}'}{\partial x^\alpha} = \mathbf{A}_\alpha + \dfrac{\partial \mathbf{U}}{\partial x^\alpha}$ $\mathbf{A}'_3 = \dfrac{\partial \mathbf{0M}'}{\partial x^3} = \mathbf{n} + \dfrac{\partial \mathbf{U}}{\partial x^3}$
Coefficient of metric tensor: $\begin{bmatrix} A_{\alpha\beta} & 0 \\ 0 & 1 \end{bmatrix}$	Coefficient of metric tensor: $\begin{bmatrix} A'_{\alpha\beta} & A'_{\alpha 3} \\ A'_{3\alpha} & 1 \end{bmatrix}$ $A'_{\alpha\beta} = A_{\alpha\beta} + \mathbf{A}_\alpha \cdot \dfrac{\partial \mathbf{U}}{\partial x^\beta} + \mathbf{A}_\beta \cdot \dfrac{\partial \mathbf{U}}{\partial x^\alpha}$ $A'_{\alpha 3} = \left(\mathbf{n} + \dfrac{\partial \mathbf{U}}{\partial x^3}\right) \cdot \left(\mathbf{A}_\alpha + \dfrac{\partial \mathbf{U}}{\partial x^\alpha}\right)$

Note: $\dfrac{\partial \mathbf{U}}{\partial u^\alpha} = [U_{\lambda\perp\alpha} - U^3 C_{\lambda\alpha}]\mathbf{a}^\lambda + [U^3_{,\alpha} + U_\lambda C^\lambda_\alpha]\mathbf{n}$; $U_{\lambda\perp\alpha} = \dfrac{\partial U_\lambda}{\partial u^\alpha} - U_\gamma \Gamma^\gamma_{\alpha\lambda}$ and $U^3_{,\alpha} = \dfrac{\partial U^3}{\partial u^\alpha}$; $U_{\alpha\perp\beta}$ is the covariant derivate; $\Gamma^\lambda_{\alpha\beta}$ are the coefficients of Chirstoffel.

Using the components of the displacement in the natural local base associated to the surface, equation (5a) becomes

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left\{\mathbf{a}_\alpha \cdot \frac{\partial \mathbf{U}}{\partial x^\beta} + \mathbf{a}_\beta \cdot \frac{\partial \mathbf{U}}{\partial x^\alpha} - x^3 C^\lambda_\alpha \mathbf{a}_\lambda \cdot \frac{\partial \mathbf{U}}{\partial x^\beta} - x^3 C^\lambda_\beta \mathbf{a}_\lambda \cdot \frac{\partial \mathbf{U}}{\partial x^\alpha}\right\}$$

$$= \frac{1}{2}\left\{\begin{matrix} (U_{\alpha\perp\beta} + U_{\beta\perp\alpha} - 2U_3 C_{\alpha\beta}) \\ - x^3(U_{\lambda\perp\alpha} C^\lambda_\beta + U_{\lambda\perp\beta} C^\lambda_\alpha - 2U_3 C^\lambda_\alpha C_{\lambda\beta}) \end{matrix}\right\}, \tag{5b}$$

$$2\varepsilon_{\alpha 3} = (U_{3,\alpha} + U_{\alpha,3}) + U_\lambda C^\lambda_\alpha - x^3 U_{\lambda,3} C^\lambda_\alpha.$$

In the small displacement theory, the deformation tensor will be the following:

- For the Warping–Kirchhoff theory,

$$\varepsilon_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{u}_\omega) - x^3 K_{\alpha\beta}(\mathbf{u}_\omega) + \phi_n \boldsymbol{\gamma}_{\alpha\beta}(\mathbf{W}^n),$$

$$2\varepsilon_{\alpha 3} = \phi_{n,3} W^n_\alpha + \phi_n W^n_\lambda C^\lambda_\alpha, \tag{6a}$$

- For the Warping–Mindlin theory,

$$\varepsilon_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{u}_\omega) + x^3\gamma_{\alpha\beta}(\boldsymbol{\beta}) - x^3\rho_{\alpha\beta}(\mathbf{u}_\omega) + \phi_n\boldsymbol{\gamma}_{\alpha\beta}(\mathbf{W}^n),$$

$$2\varepsilon_{\alpha 3} = (\beta_\alpha + u_\lambda C^\lambda_\alpha + u_{3,\alpha}) + (\phi_{n,3}W^n_\alpha + \phi_n W^n_\lambda C^\lambda_\alpha), \tag{6b}$$

with

$$\gamma_{\alpha\beta}(\mathbf{V}) = \frac{1}{2}(V_{\beta\perp\alpha} + V_{\alpha\perp\beta}) - V^3 C_{\alpha\beta},$$

$$K_{\alpha\beta}(\mathbf{V}) = \frac{1}{2}\begin{Bmatrix}[V^3_{,\alpha}]_{\perp\beta} + [V^3_{,\beta}]_{\perp\alpha} + [V_\lambda C^\lambda_\alpha]_{\perp\beta} + [V_\lambda C^\lambda_\beta]_{\perp\alpha} \\ + V_{\lambda\perp\alpha}C^\lambda_\beta + V_{\lambda\perp\beta}C^\lambda_\alpha - 2V^3 C^\lambda_\alpha C_{\lambda\beta}\end{Bmatrix},$$

$$\rho_{\alpha\beta}(\mathbf{V}) = \frac{1}{2}(V_{\lambda\perp\alpha}C^\lambda_\beta + V_{\lambda\perp\beta}C^\lambda_\alpha) - V^3 C^\lambda_\alpha C_{\lambda\beta}.$$

γ and K are, respectively, the membranous deformation tensor and the curvature variation tensor. For thin shells, the components of tensor $\rho_{\alpha\beta}$ can be neglected; this is from the Novozhilov–Donnell theory [12,13].

4.2. Stress tensor

The normal stress component and the shear stress components are the following:

- For the Warping–Kirchhoff theory,

$$\sigma_{\alpha\beta} = \frac{E}{1-v^2}\begin{bmatrix}(1-v)\gamma_{\alpha\beta}(\mathbf{u}_\omega) + \phi_n\mathbf{W}^n) + v\mathrm{Tr}\{\gamma_{\alpha\beta}(\mathbf{u}_\omega + \phi_n\mathbf{W}^n)\} \\ -(1-v)x^3K_{\alpha\beta}(\mathbf{u}_\omega) - v\mathrm{Tr}\{x^3K_{\alpha\beta}(\mathbf{u}_\omega)\}\end{bmatrix},$$

$$\sigma_{\alpha 3} = \frac{E}{2(1+v)}[\phi_{n,3}W^n_\alpha + \phi_n W^n_\lambda C^\lambda_\alpha] \tag{7a}$$

- For the Warping–Mindlin theory,

$$\sigma_{\alpha\beta} = \frac{E}{1-v^2}\begin{bmatrix}(1-v)[\gamma_{\alpha\beta}(\mathbf{u}_\omega) + x^3\boldsymbol{\beta} - \phi_n\mathbf{W}^n) - x^3\rho_{\alpha\beta}(\mathbf{u}_\omega)] \\ + v\mathrm{Tr}\{\gamma_{\alpha\beta}(\mathbf{u}_\omega + x^3\boldsymbol{\beta} + \phi_n\mathbf{W}^n) - x^3\rho_{\alpha\beta}(\mathbf{u}_\omega)\}\end{bmatrix},$$

$$\sigma_{\alpha 3} = \frac{E}{2(1+v)}[(\beta_\alpha + u_\lambda C^\lambda_\alpha + u_{3,\alpha}) + (\phi_{n,3}W^n_\alpha + \phi_n W^n_\lambda C^\lambda_\alpha)], \tag{7b}$$

where E is Young's modulus and v is the Poisson ratio.

4.3. STRESS RESULTANT

The stress resultants are defined by

$$[N] = \begin{bmatrix} N^{11} & N^{12} \\ N^{21} & N^{22} \end{bmatrix}, \quad [M] = \begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix}, \quad [P_n] = \begin{bmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{bmatrix},$$

$$\mathbf{T} = \begin{pmatrix} T^1 \\ T^2 \end{pmatrix}; \quad \mathbf{L}_n = \begin{pmatrix} L_n^1 \\ L_n^2 \end{pmatrix}; \quad \mathbf{Q}_n = \begin{pmatrix} Q_n^1 \\ Q_n^2 \end{pmatrix} \tag{8}$$

with

$$\begin{bmatrix} N^{11} & N^{12} & N^{22} \\ M^{11} & M^{12} & M^{22} \\ P_n^{11} & P_n^{12} & P_n^{22} \end{bmatrix} = \int_{\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} 1 \\ x_3 \\ \phi_n \end{bmatrix} [\sigma^{11}\sigma^{12}\sigma^{22}]\mathrm{d}x_3;$$

$$\begin{bmatrix} T^1 T^2 \\ L_n^1 L_n^2 \\ Q_n^1 Q_n^2 \end{bmatrix} = \int_{\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} 1 \\ \phi_n \\ \phi_{n,3} \end{bmatrix} [\sigma^{13}\sigma^{23}]\mathrm{d}x_3.$$

The stress resultants in terms of displacements are given by the following:

- For the Warping–Kirchhoff theory,

$$\mathbf{N} = D_2[(1-\nu)\boldsymbol{\gamma}(\mathbf{u}_\omega) + \nu \mathrm{Tr}\{\boldsymbol{\gamma}(\mathbf{u}_\omega)\} \cdot \mathbf{1}], \tag{9a}$$

$$\mathbf{M} = -D_1[(1-\nu)(\mathbf{K}(\mathbf{u}_\omega)) + \nu \mathrm{Tr}\{\mathbf{K}(\mathbf{u}_\omega)\} \cdot \mathbf{1}], \tag{10a}$$

$$\mathbf{P}_n = D_2[(1-\nu)\boldsymbol{\gamma}(\Sigma_n \mathbf{W}^n) + \nu \mathrm{Tr}(\boldsymbol{\gamma}(\Sigma_n \mathbf{W}^n)) \cdot \mathbf{1}], \tag{11a}$$

$$\mathbf{T} = D_3 \mathbf{W}^n \Theta_n, \tag{12a}$$

$$\mathbf{L}_n = D_3[\Sigma_n \mathbf{C} \cdot \mathbf{W}^n], \tag{13a}$$

$$\mathbf{Q}_n = D_3[\Xi_n \mathbf{W}^n], \tag{14a}$$

- For the Warping–Mindlin theory,

$$\mathbf{N} = D_2[(1-\nu)\boldsymbol{\gamma}(\mathbf{u}_\omega) + \nu \mathrm{Tr}\{\boldsymbol{\gamma}(\mathbf{u}_\omega)\} \cdot \mathbf{1}], \tag{9b}$$

$$\mathbf{M} = D_1[(1-\nu)(\boldsymbol{\gamma}(\boldsymbol{\beta}) - \rho(\mathbf{u}_\omega)) + \nu \mathrm{Tr}(\boldsymbol{\gamma}(\boldsymbol{\beta}) - \rho(\mathbf{u}_\omega)) \cdot \mathbf{1}], \tag{10b}$$

$$\mathbf{P}_n = D_2[(1-\nu)\boldsymbol{\gamma}(\Sigma_n \mathbf{W}^n) + \nu \mathrm{Tr}(\boldsymbol{\gamma}(\Sigma_n \mathbf{W}^n)) \cdot \mathbf{1}], \tag{11b}$$

$$\mathbf{T} = D_3[\boldsymbol{\beta} + \nabla \mathbf{u}_3 + \mathbf{C} \cdot \mathbf{u}_\omega + \mathbf{W}^n \Theta_n], \tag{12b}$$

$$\mathbf{L}_n = D_3[\Sigma_n \mathbf{C} \cdot \mathbf{W}^n], \tag{13b}$$

$$\mathbf{Q}_n = D_3[\Theta_n(\boldsymbol{\beta} + \mathbf{C} \cdot \mathbf{u}_\omega + \nabla \mathbf{u}_3) + \mathbf{W}^n \Xi_n], \tag{14b}$$

where **1** is the unit tensor and (D_1, D_2, D_3) are the stiffness coefficients defined by

$$D_1 = \frac{Eh^3}{12(1-\nu^2)}, \quad D_2 = \frac{Eh}{(1-\nu^2)}, \quad D_3 = \frac{Eh}{2(1+\nu)}. \tag{15}$$

The constants used in equations (9)–(14) are defined by

$$\Theta_n = \frac{1}{h}\int_{\frac{h}{2}}^{\frac{h}{2}} (\phi_n)_{,3}\,\mathrm{d}x^3, \quad \Sigma_n = \frac{1}{h}\int_{\frac{h}{2}}^{\frac{h}{2}} (\phi_n)^2\,\mathrm{d}x^3, \quad \Xi_n = \frac{1}{h}\int_{\frac{h}{2}}^{\frac{h}{2}} (\phi_n)_{,3}\cdot(\phi_n)_{,3}\,\mathrm{d}x^3. \tag{16}$$

5. EQUILIBRIUM EQUATIONS

The principle of the virtual work is used to derive the governing equilibrium equations (for more details see Appendices B and C):

- For the Warping–Kirchhoff theory,

$$\begin{aligned}
&N^{\alpha\beta}_{\perp\beta} - 2M^{\lambda\beta}_{\perp\beta}C^{\alpha}_{\lambda} - M^{\lambda\beta}C^{\alpha}_{\lambda\perp\beta} + F^{\alpha} = \rho h\ddot{u}^{\alpha},\\
&N^{\alpha\beta}C_{\alpha\beta} + (M^{\alpha\beta}_{\perp\beta})_{\perp\alpha} - M^{\alpha\beta}C^{\lambda}_{\alpha}C_{\lambda\beta} + F^{3} + m^{\alpha}_{\perp\alpha} = \rho h\ddot{u}_3,\\
&(P^{\alpha\beta}_{n})_{\perp\beta} - Q^{\alpha}_{n} - C^{\alpha}_{\lambda}L^{\lambda}_{n} + F^{\alpha}_{n} = \rho h\Sigma_n(\ddot{W}^{\alpha})^n,
\end{aligned} \tag{17a}$$

or

$$\begin{aligned}
&\operatorname{div}\mathbf{N} - 2\mathbf{C}\operatorname{div}\mathbf{M} - \nabla\mathbf{C}:\mathbf{M} + \mathbf{F}_{\omega} = \rho h\,\pi(\ddot{\mathbf{u}}_{\omega}),\\
&\mathbf{N}:\mathbf{C} + \operatorname{div}(\operatorname{div}\mathbf{M}) - (\mathbf{C}\cdot\mathbf{M}):\mathbf{C} + F^3 + \operatorname{div}\mathbf{m}_{\omega} = \rho h\,\ddot{u}_3,\\
&\operatorname{div}\mathbf{P}_n - \mathbf{Q}_n - \mathbf{C}\cdot\mathbf{L}_n + \mathbf{F}_n = \rho h\Sigma_n\ddot{\mathbf{W}}^n,
\end{aligned} \tag{18a}$$

- For the Warping–Mindlin theory,

$$\begin{aligned}
&N^{\alpha\beta}_{\perp\beta} - (M^{\lambda\beta}C^{\alpha}_{\lambda})_{\perp\beta} - C^{\alpha}_{\lambda}T^{\lambda} + F^{\alpha} = \rho h\,\ddot{u}^{\alpha},\\
&M^{\alpha\beta}_{\perp\beta} - T^{\alpha} + m^{\alpha} = \frac{\rho h^3}{12}\ddot{\beta}^{\alpha},\\
&N^{\alpha\beta}C_{\alpha\beta} + T^{\alpha}_{\perp\alpha} - M^{\alpha\beta}C^{\lambda}_{\alpha}C_{\lambda\beta} + F^3 = \rho h\,\ddot{u}_3,\\
&(P^{\alpha\beta}_{n})_{\perp\beta} - Q^{\alpha}_{n} - C^{\alpha}_{\lambda}L^{\lambda}_{n} + F^{\alpha}_{n} = \rho h\,\Sigma_n(\ddot{W}^{\alpha})^n
\end{aligned} \tag{17b}$$

or

$$\begin{aligned}
&\operatorname{div}\mathbf{N} - \operatorname{div}(\mathbf{C}\cdot\mathbf{M}) - \mathbf{C}\cdot\mathbf{T} + \mathbf{F}_{\omega} = \rho h\,\pi(\ddot{\mathbf{u}}_{\omega}),\\
&\operatorname{div}\mathbf{M} - \mathbf{T} + \mathbf{m}_{\omega} = \frac{\rho h^3}{12}\ddot{\boldsymbol{\beta}},\\
&\mathbf{N}:\mathbf{C} + \operatorname{div}(\mathbf{T}) - (\mathbf{C}\cdot\mathbf{M}):\mathbf{C} + F^3 = \rho h\,\ddot{u}^3,\\
&\operatorname{div}\mathbf{P}_n - \mathbf{Q}_n - \mathbf{C}\cdot\mathbf{L}_n + \mathbf{F}_n = \rho h\,\Sigma_n\ddot{\mathbf{W}}^n,
\end{aligned} \tag{18b}$$

where ρ is the mass density, h is the thickness of the shell, $\pi(\ddot{\mathbf{u}}_\omega)$ is the projection on surface of the surface's acceleration $\ddot{\mathbf{u}}_\omega$, $\mathbf{F}_\omega$ is the in-surface force vector, $\mathbf{m}_\omega$ is the in-surface moment vector and $\mathbf{F}^n$ is the projection of the in-surface vector force on the nth transverse normal mode.

For a shell loaded by a surface density force applied on $\partial\omega\, x\;]-h/2, h/2[$ ($\partial\omega$ is the boundary of the surface ω), boundary conditions are the following:

- For the Warping–Kirchhoff theory:

$$\begin{aligned} &N^{\alpha\beta}v_\beta + 2M^{\lambda\beta}C^\alpha_\lambda v_\beta + F^\alpha_S = 0,\\ &(M^{\alpha\beta}_{\perp\beta})v_\alpha - (M^{\alpha\beta}v_\beta)_{\perp\alpha} + F^3_S = 0,\\ &P^{\alpha\beta}_n v_\beta + (F^\alpha_S)_n = 0, \end{aligned} \tag{19a}$$

or

$$\begin{aligned} &\mathbf{N}\cdot\mathbf{v} + 2\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{v} + (\mathbf{F}_\omega)_s = \mathbf{0},\\ &(\operatorname{div}\mathbf{M})\cdot\mathbf{v} - \operatorname{div}(\mathbf{M}\cdot\mathbf{v}) + F^3_S = 0,\\ &(\mathbf{P})_n\cdot\mathbf{v} + (\mathbf{F}_s)_n = \mathbf{0}, \end{aligned} \tag{20a}$$

- For the Warping–Mindlin theory:

$$\begin{aligned} &N^{\alpha\beta}v_\beta + M^{\lambda\beta}v_\lambda C^\alpha_\beta + F^\alpha_S = 0,\\ &M^{\alpha\beta}v_\beta + m^\alpha_S = 0,\\ &T^\alpha v_\alpha + F^3_S = 0,\\ &P^{\alpha\beta}_n v_\beta + (F^\alpha_S)_n = 0, \end{aligned} \tag{19b}$$

$$\begin{aligned} &\mathbf{N}\cdot\mathbf{v} + \mathbf{C}\cdot\mathbf{M}\cdot\mathbf{v} + (\mathbf{F}_\omega)_s = \mathbf{0},\\ &\mathbf{M}\cdot\mathbf{v} + (\mathbf{m}_\omega)_s = \mathbf{0},\\ &\mathbf{T}\cdot\mathbf{v} + F^3_S = 0,\\ &(\mathbf{P})_n\cdot\mathbf{v} + (\mathbf{F}_s)_n = \mathbf{0}, \end{aligned} \tag{20b}$$

where $(\mathbf{F}_\omega)_s$ is the in-surface boundary force vector, $(\mathbf{m}_\omega)_s$ is the in-surface boundary moment vector and $(\mathbf{F}_s)_n$ is the projection of the in-surface boundary force vector on the nth transverse normal mode.

6. EVALUATION OF THE PRESENT THEORY

6.1. APPLICATION 1

Consider an infinite cylindrical shell with internal radius R_1 and external radius R_2 (see Figure 2). The cylindrical shell is submitted to internal shear forces F/R_1 and external shear forces F/R_2 (F is a linear density force).

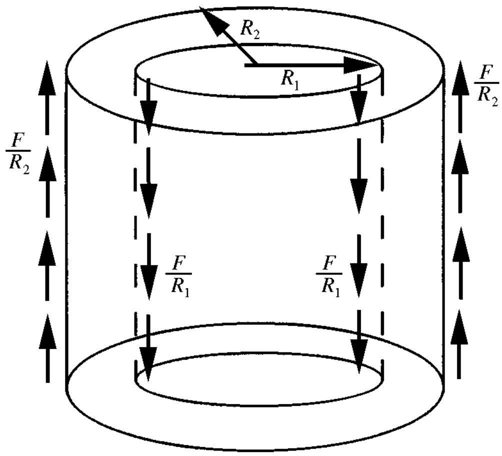

Figure 2. Cylindrical shell.

The elastic solution of the shear stress as a function of the radius r is

$$\sigma_{rz} = \frac{F}{r}. \tag{21}$$

Note: σ_{rz} is zero for the Kirchhoff theory.

Using the Reissner–Mindlin model, the shear stress is written as

$$\sigma_{rz} = -\frac{F}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{22}$$

By considering the present theory, the shear stress is written as follows:

$$\sigma_{rz} = \frac{D_3}{h}\left[\beta_1 + \frac{\partial \phi_n}{\partial x^3} W_1^n\right], \tag{23}$$

with

$$W_1^n = \frac{F}{D_3(\Xi_n - \Theta_n^2)}\left[\frac{\phi_n(-h/2)}{R_2} - \frac{\phi_n(h/2)}{R_1} - \Theta_n \frac{h}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]$$

$$\beta_1 = -\Theta_n W_1^n - \frac{Fh}{2D_3}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

Shear stress calculated according to the present theory (23) and the Reissner–Mindlin theory (22) are compared, for two cases, with the elastic exact solution (21) in Figure 3. For this application, modelling the non-linear distribution of stress is necessary. As we can see, and using only two warping functions, close agreement of the warping model results with the exact solutions is obtained.

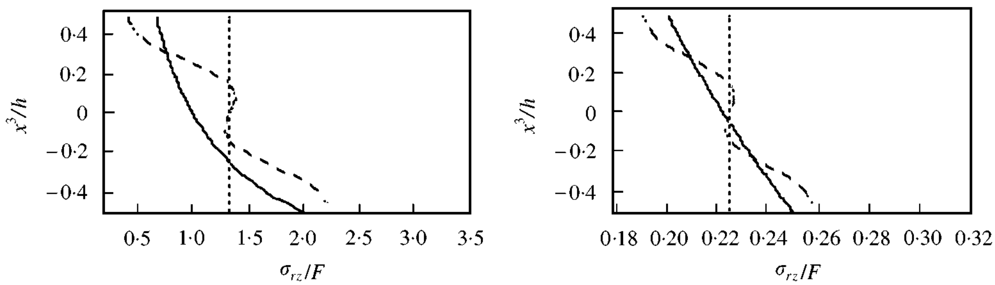

Figure 3. Shear stress distribution for a cylindrical shell (a) ($R_1 = 0{\cdot}5$ m; $R_2 = 1{\cdot}5$ m; $h = 1$ m); (b) ($R_1 = 4$ m; $R_2 = 5$ m; $h = 1$ m). --- Present theory; —— exact solution; - - - Mindlin solution; Kirchhoff: $\sigma_{rz} = 0$.

6.2. APPLICATION 2

Consider a cylindrical shell (with a radius R, a height H and a thickness h) submitted to a linear density of forces applied on the sides defined by $x^1 = -H/2$ and $x^1 = H/2$.

The two cases of in-surface loads are defined by (see also Figure 4);

Case 1:

$$\mathbf{f}_s = q\delta(x^3)\mathbf{a}_1 \quad \text{on } x^1 = -\frac{H}{2},$$

$$\mathbf{f}_s = -q\delta(x^3)\mathbf{a}_1 \quad \text{on } x^1 = \frac{H}{2},$$

Case 2:

$$\mathbf{f}_s = \frac{q}{2}\delta(x^3 - h/4)\mathbf{a}_1 \quad \text{on } x^1 = -\frac{H}{2},$$

$$\mathbf{f}_s = \frac{q}{2}\delta(x^3 + h/4)\mathbf{a}_1 \quad \text{on } x^1 = -\frac{H}{2},$$

$$\mathbf{f}_s = -\frac{q}{2}\delta(x_3^3 - h/4)\mathbf{a}_1 \quad \text{on } x^1 = \frac{H}{2},$$

$$\mathbf{f}_s = -\frac{q}{2}\delta(x^3 + h/4)\mathbf{a}_1 \quad \text{on } x^1 = \frac{H}{2}.$$

$\mathbf{a}_1$ is the first tangent vector of the surface (associated to the parameter x^1), δ is the Dirac's function and q is a constant.

6.2.1. *Theory of Reissner–Mindlin*

The Reissner–Mindlin solution is

$$N_{11} = -q, \quad N_{12} = 0, \quad N_{22} = -\nu R^2 N_{11}, \quad u_1 = -\frac{q}{D_2}x^1.$$

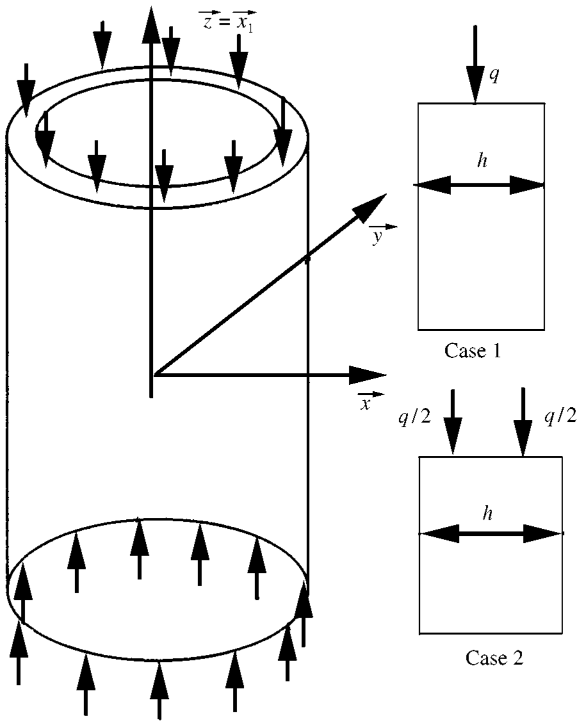

Figure 4. Cylindrical shell submitted to two cases of in-surface loads.

6.2.2. *Theory of Lo*

The displacement field is

$$\mathbf{U} = \begin{cases} u_1(x^1) + W_1(x^1)(x^3)^2 \\ 0 \\ 0 \end{cases} \qquad (\mathbf{a}^1, \mathbf{a}^2, \mathbf{n}).$$

The stress resultants are defined by

$$N = D_2[(1-\nu)\gamma(\mathbf{u}_\omega) + \nu\mathrm{Tr}\{\gamma(\mathbf{u}_\omega)\}\cdot\mathbf{1}] + D_1[(1-\nu)\gamma(\mathbf{W}) + \nu\mathrm{Tr}\{\gamma(\mathbf{W})\}\cdot\mathbf{1}],$$

$$P^1 = D_1[(1-\nu)\gamma(\mathbf{u}_\omega) + \nu\mathrm{Tr}(\gamma(\mathbf{u}_\omega))\cdot\mathbf{1}] + D_4[(1-\nu)\gamma(\mathbf{W}) + \nu\mathrm{Tr}(\gamma(\mathbf{W}))\cdot\mathbf{1}],$$

$$\mathbf{Q}^1 = 2D_1\mathbf{W},$$

where

$$\mathbf{u}_\omega = \begin{vmatrix} u_1 \\ 0(\mathbf{a}^1, \mathbf{a}^2) \end{vmatrix}; \quad \mathbf{W} = \begin{vmatrix} W_1 \\ 0(\mathbf{a}^1, \mathbf{a}^2) \end{vmatrix}.$$

(D_1, D_2, D_3) are defined by equation (15) and D_4 is defined by

$$D_4 = \frac{Eh^5}{80(1-\nu^2)}.$$

Lo's displacement is given by

$$u_1 = -\frac{\lambda_1(F_s^1 + (D_1/D_2)q)\sinh(\lambda_1 x^1)}{2D_2\cosh\left(\lambda_1\frac{H}{2}\right)} - \frac{q}{D_2}x^1, \quad W_1 = \frac{\lambda_1(F_s^1 + (D_1/D_2)q)\sinh(\lambda_1 x^1)}{2D_1\cosh\left(\lambda_1\frac{H}{2}\right)}$$

with

$$(\lambda_1)^2 = \frac{2D_1}{D_4 - (D_1)^2/D_2}.$$

In the case 1, $F_s^1 = 0$ and in the case 2, $F_s^1 = \frac{h^2}{8}q$.

The stress resultants are

$$N_{11} = -\frac{q}{D_2}, \quad N_{12} = 0, \quad N_{22} = \nu R^2 N_{11},$$

$$P_{11}^1 = \frac{(F_s^1 + (D_1/D_2)q)\cosh(\lambda_1 x^1)}{\cosh\left(\lambda_1\frac{H}{2}\right)} - \frac{D_1}{D_2}q, \quad P_{12}^1 = 0, \quad P_{22}^1 = \nu R^2 P_{11}^1.$$

6.2.3. *Warping theory*

The displacement field is

$$\mathbf{U} = \left| \begin{array}{ll} u_1(x^1) + \sum_n W_1^n(x^1)\phi_n(x^3) & \\ 0 & \\ 0 & (\mathbf{a}^1, \mathbf{a}^2, \mathbf{n}). \end{array} \right.$$

For the warping theory, the displacement solution is

$$u_1 = -\frac{q}{D_2}x^1, \quad W_1^n = \frac{\sinh(\lambda^n x^1)}{\Sigma_n D_2 \lambda_2^n \cosh\left(\lambda^n\frac{H}{2}\right)} F_s^n \quad \text{with } (\lambda^n)^2 = \frac{D_3}{D_2}\frac{\Sigma_n}{\Xi_n}.$$

Σ_n and Ξ_n are defined by equation (16).

The stress resultants in terms of displacement are

$$\mathbf{N} = D_2[(1-\nu)\boldsymbol{\gamma}(\mathbf{u}_\omega) + \nu \mathrm{Tr}\{\boldsymbol{\gamma}(\mathbf{u}_\omega)\}\cdot \mathbf{1}],$$

$$\mathbf{P}_n = D_2\Sigma_1[(1-\nu)\boldsymbol{\gamma}(\mathbf{W}^n) + \nu \mathrm{Tr}(\boldsymbol{\gamma}(\mathbf{W}^n))\cdot \mathbf{1}],$$

$$\mathbf{Q}_n = D_3\Xi_1\mathbf{W}^n.$$

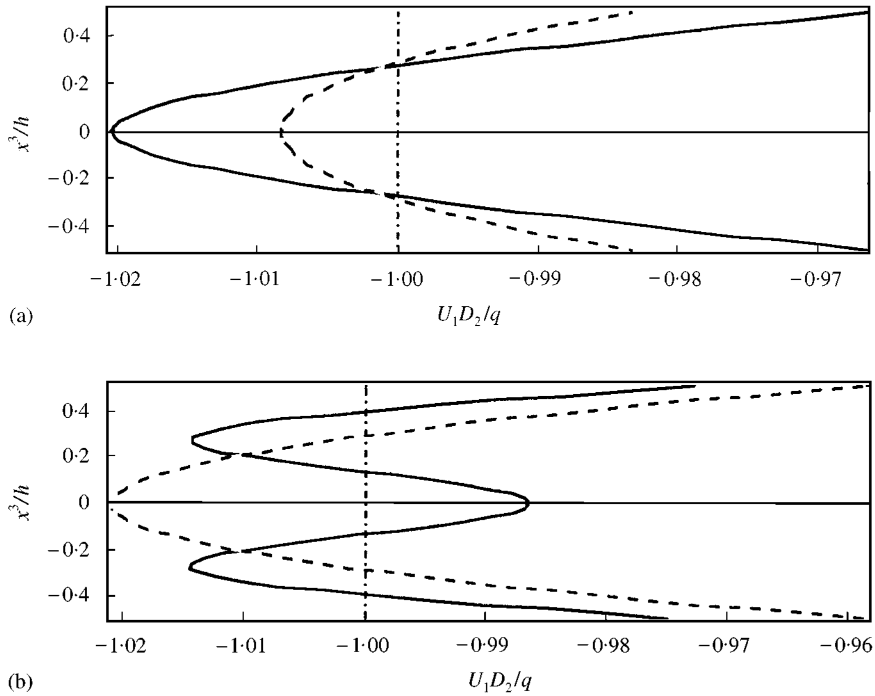

Figure 5. Evolution of the in-surface displacement as a function of the normal co-ordinate. (a) Case 1, .-.–.- Reissner–Mindlin; —— Warping: case 1; – – – Lo: case 1; (b) case 2, ·-·-· Reissner–Mindlin; —— Warping: case 2; – – – Lo: case 2.

The warping stress resultants are

$$N_{11} = -\frac{q}{D_2}, \quad N_{12} = 0, \quad N_{22} = \nu R^2 N_{11},$$

$$P^n_{11} = \frac{\cosh(\lambda^n x^1)}{\cosh\left(\lambda^n \dfrac{H}{2}\right)} (F^n_s), \quad P^n_{12} = 0, \quad P^n_{22} = \nu R^2 P^1_{11}.$$

Case 1: $(F^n_s) = -q\phi_n(0)$,

Case 2: $(F^n_s) = -\frac{q}{2}\left[\phi_n\left(\frac{h}{4}\right) + \phi_n\left(-\frac{h}{4}\right)\right]$.

The evolution of the first in-surface displacement as a function of the normal co-ordinate is given in Figure 5.

Lo's theory does not have enough functions to distinguish the two cases of loads. For the warping theory, mode 1 is used for case 1 and the two first modes are used for case 2. The development of Lo's solution on the function $(x^3)^2$ constrains the middle of the fibre not to move; there is no physical reason to have that kind of deformation.

7. DISCUSSION–CONCLUSION

- For the cylindrical shell problems presented here, the classical Reissner–Mindlin solution cannot reproduce the non-uniformity of the exact solution which deviates from the solution. The present theory can reproduce the non-uniform shear and displacement distribution.
- Due to the high order of the terms included in this theory, it is, however, inconvenient to solve classical problems. The examples of infinite plates and circular plates presented in Hassis 1998 and other examples presented here would be helpful for providing guidelines by which one can ascertain when it is necessary to use a high order theory and when a lower order theory will suffice.
- This high order theory can be extended to laminated plates and shells. It is known that distribution of in-plane displacements across the thickness may be strongly non-linear [14].
- After the application of the warping theory for plates [1, 2] and this work for shells, it can be stated that for problems which involve rapidly fluctuating loads with the characteristic length of the order of the thickness, or high frequencies analysis of plates and shells, a high order theory is required to give meaningful results. For the other problems, a low simple theory is sufficient.

REFERENCES

1. H. Hassis 1998 *European Journal of Mechanics* **17**(5), 843–853. A warping theory of plate deformation.
2. K. H. Lo, R. M. Christensen and E. M. Wu 1977 *Journal of Applied Mechanics* **44**, 663–676. A high order theory of plate deformation. Part 1: homogeneous plates.
3. E. Reissner 1944 *Journal of Mathematics and Physics* **23**, 184–191. On the theory of bending of elastic plates.
4. E. Reissner 1945 *Journal of Applied Mechanics* **12**, 69–77. The effects of transverse shear deformation on bending of elastic plates.
5. P. M. Naghdi 1957 *Journal of Applied Mechanics* **44**, 663–676. *Quarterly of Applied Mathematics* **14**, 369–380. On the theory of thin elastic shells.
6. F. Essenburg 1975 *Journal of Applied Mechanics* **42**(1), 127–132. On the significance of the inclusion of the effects of transverse normal strain in problems involving beams with surface constrains.
7. R. B. Nelson and D. R. Lorch 1974 *Journal of Applied Mechanics* **41**. *Transactions of the ASME* **96** *Series E*, *Mar* 1974 177–183. A Raffined theory of laminated orthotropic plates.
8. E. Reissner 1975 *International Journal of Solids and Structures* **11,** 569–573. On transverse bending of plates including the effects of transverse shear deformation.
9. E. Reissner 1985 *Applied Mechanics Review* **38**, 1453–1464. Reflections on the theory of elastic plates.
10. A. Bhimaraddi 1984 *International Solids and Structures*, **20** 623–630. A higher order theory for free vibration analysis of circular cylindrical shells.
11. R. J. Gibert 1988 *EYROLLES 1988*. Vibration des structures.
12. Ph. Destuynder 1986 *Masson 1986*. Une théorie asymptotique des plaques minces en élasticité linéaire.
13. Ph. Destuynder 1990 *Masson 1990*. Modélisation des coques minces élastiques.
14. K. H. Lo, R. M. Christensen and E. M. Wu 1977 *Journal of Applied Mechanics* **44**, 663–676. A high order theory of plate deformation. Part 2: laminated plates.

APPENDIX A

The transverse normal modes for a free-free beam are

$$\phi_n = \cos\left(\frac{\alpha_n x^3}{h}\right) + \cosh\left(\frac{\alpha_n x^3}{h}\right) - R_n\left[\sin\left(\frac{\alpha_n x^3}{h}\right) + \sinh\left(\frac{\alpha_3 x^3}{h}\right)\right].$$

The coefficients α_n and R_n take the following values:

$$R_1 = 0{\cdot}9825; \quad R_2 = 1{\cdot}0008; \quad R_3 = 1{\cdot}0000; \quad R_4 = 1{\cdot}0000.$$

$$\alpha_1 = 4{\cdot}730; \quad \alpha_2 = 7{\cdot}853; \quad \alpha_3 = 10{\cdot}996; \quad \alpha_4 = 14{\cdot}137.$$

For a free-free beam, the longitudinal modes are written as follows:

$$\Phi_k = \cos\left(k\pi\left(\frac{x^3}{h} + \frac{1}{2}\right)\right).$$

APPENDIX B

For the Warping–Kirchhoff model, the exterior virtual work is (for a unit density forces $\mathbf{f}_v$ and the surface density forces $\mathbf{f}_s$):

$$W_e = \int_\omega \int_{\frac{h}{2}}^{\frac{h}{2}} [f_v^\alpha u_\alpha^* - f_v^\alpha x^3(u_{3,\alpha}^* + C_\alpha^\lambda u_\lambda^*) + f_v^\alpha \phi_n W_\alpha^{n^*} + f_v^3 u_3^*]\,\mathrm{d}x^3\,\mathrm{d}\omega$$

$$+ \int_{\partial\omega} \int_{\frac{h}{2}}^{\frac{h}{2}} [f_s^\alpha u_\alpha^* - f_s^\alpha x^3(u_{3,\alpha}^* + C_\alpha^\lambda u_\lambda^*) + f_s^\alpha \phi_n W_\alpha^{n^*} + f_s^3 u_3^*]\,\mathrm{d}x^3\,\mathrm{d}\Gamma$$

$$= \int_\omega [F^\alpha u_\alpha^* - m^\alpha(u_{3,\alpha}^* + C_\alpha^\lambda u_\lambda^*) + F_n^\alpha W_\alpha^{n^*} + F^3 u_3^*]\,\mathrm{d}\omega$$

$$+ \int_{\partial\omega} [F_s^\alpha u_\alpha^* - m_s^\alpha(u_{3,\alpha}^* + C_\alpha^\lambda u_\lambda^*) + (F_s^\alpha)^n W_\alpha^{n^*} + F_s^3 u_3^*]\,\mathrm{d}\Gamma$$

$$= \int_\omega [F^\alpha u_\alpha^* + m_{\perp\alpha}^\alpha u_3^* - m^\alpha C_\alpha^\lambda u_\lambda^* + F_n^\alpha W_\alpha^{n^*} + F^3 u_3^*]\,\mathrm{d}\omega$$

$$+ \int_{\partial\omega} [F_s^\alpha u_\alpha^* + m_{s\perp\alpha}^\alpha u_3^* - m_s^\alpha C_\alpha^\lambda u_\lambda^* + (F_s^\alpha)^n W_\alpha^{n^*} + F_s^3 u_3^*]\,\mathrm{d}\Gamma,$$

where

$$F^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_v^{\alpha}\,\mathrm{d}x^3, \quad m^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x^3 f_v^{\alpha}\,\mathrm{d}x^3, \quad F_n^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \phi_n f_v^{\alpha}\,\mathrm{d}x^3,$$

$$F_s^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_s^{\alpha}\,\mathrm{d}x^3, \quad m_s^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x^3 f_s^{\alpha}\,\mathrm{d}x^3, \quad (F_s^{\alpha})^n = \int_{-\frac{h}{2}}^{\frac{h}{2}} \phi_n f_s^{\alpha}\,\mathrm{d}x^3.$$

For the Warping–Kirchhoff model, the inertial virtual work is (the second order terms are neglected):

$$W_j = -\int_{\omega}\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\ddot{u}^{\alpha}u_{\alpha}^{*} + \ddot{u}^{3}u_{3}^{*} + \boldsymbol{\phi}_{\mathbf{n}}^{2}(\ddot{\mathbf{W}}^{\boldsymbol{\alpha}})^{\mathbf{n}}\mathbf{W}_{\boldsymbol{\alpha}}^{\mathbf{n}^{*}})\,\mathrm{d}x^3\,\mathrm{d}\omega$$

$$= -\int_{\omega} \rho h(\ddot{u}^{\alpha}u_{\alpha}^{*} + \ddot{u}^{3}u_{3}^{*}) + \rho h\boldsymbol{\Sigma}_{\mathbf{n}}(\ddot{\mathbf{W}}^{\boldsymbol{\alpha}})^{\mathbf{n}}\mathbf{W}_{\boldsymbol{\alpha}}^{\mathbf{n}^{*}}\,\mathrm{d}\omega.$$

For the Warping–Kirchhoff model, the interior virtual work is

$$W_i = -\int_{\omega}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha\beta}[\gamma_{\alpha\beta}(\mathbf{u}_{\omega}^{*}) - x^3K_{\alpha\beta}(\mathbf{u}_{\omega}^{*}) + \phi_n\gamma_{\alpha\beta}(\mathbf{W}^{n^{*}})]\,\mathrm{d}x^3\,\mathrm{d}\omega$$

$$-\int_{\omega}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha 3}[\phi_{n,3}W_{\alpha}^{n^{*}} + \phi_n C_{\alpha}^{\lambda}W_{\lambda}^{n^{*}}]\,\mathrm{d}x^3\,\mathrm{d}\omega$$

using the stress resultant defined by equation (8), the interior virtual work becomes:

$$W_i = -\int_{\omega} [N^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}_{\omega}^{*}) - M^{\alpha\beta}K_{\alpha\beta}(\mathbf{u}_{\omega}^{*}) + P_n^{\alpha\beta}\boldsymbol{\gamma}_{\boldsymbol{\alpha\beta}}(\mathbf{W}^{\mathbf{n}^{*}})]\,\mathrm{d}\omega$$

$$-\int_{\omega} [Q_n^{\alpha}W_{\alpha}^{n^{*}} + L_n^{\alpha}C_{\alpha}^{\lambda}W_{\lambda}^{n^{*}}]\,\mathrm{d}\omega$$

using relation (6) and after integration by parts, it becomes

$$W_i = \int_{\omega} N_{\perp\beta}^{\alpha\beta}u_{\alpha}^{*}\,\mathrm{d}\omega - \int_{\partial\omega} N^{\alpha\beta}u_{\alpha}^{*}v_{\beta}\,\mathrm{d}\Gamma + \int_{\omega} N^{\alpha\beta}C_{\alpha\beta}u_3^{*}\,\mathrm{d}\omega + \int_{\omega} M_{\perp\alpha\beta}^{\alpha\beta}u_3^{*}\,\mathrm{d}\omega$$

$$-\int_{\partial\omega} M_{\perp\beta}^{\alpha\beta}v_{\alpha}u_3^{*}\,\mathrm{d}\Gamma - \int_{\partial\omega} (M^{\alpha\beta}v_{\beta})_{\perp\alpha}u_{\alpha}^{*}\,\mathrm{d}\Gamma - \int_{\omega} M_{\perp\beta}^{\alpha\beta}C_{\alpha}^{\lambda}u_{\lambda}^{*}\,\mathrm{d}\omega + \int_{\partial\omega} M^{\alpha\beta}C_{\alpha}^{\lambda}v_{\beta}u_{\lambda}^{*}\,\mathrm{d}\Gamma$$

$$- \int_{\omega} (M^{\alpha\beta} C_{\beta}^{\lambda})_{\perp\alpha} u_{\lambda}^{*} \, \mathrm{d}\omega + \int_{\partial\omega} M^{\alpha\beta} C_{\beta}^{\lambda} v_{\alpha} u_{\lambda}^{*} \, \mathrm{d}\Gamma - \int_{\omega} M^{\alpha\beta} C_{\alpha}^{\lambda} C_{\lambda\beta} u_{3}^{*} \, \mathrm{d}\omega$$

$$+ \int_{\omega} P_{n\perp\beta}^{\alpha\beta} W_{\alpha}^{n^*} \, \mathrm{d}\omega - \int_{\partial\omega} P_{n}^{\alpha\beta} W_{\alpha}^{n^*} v_{\beta} \, \mathrm{d}\Gamma - \int_{\omega} Q_{n}^{\alpha} W_{\alpha}^{n^*} \, \mathrm{d}\omega - \int_{\omega} L_{n}^{\alpha} C_{\alpha}^{\lambda} W_{\lambda}^{n^*} \, \mathrm{d}\omega$$

also

$$W_i = \int_{\omega} [N_{\perp\beta}^{\alpha\beta} - M_{\perp\beta}^{\lambda\beta} C_{\lambda}^{\alpha} - (M^{\lambda\beta} C_{\lambda}^{\alpha})_{\perp\beta}] u_{\alpha}^{*} \, \mathrm{d}\omega$$

$$+ \int_{\omega} [N^{\alpha\beta} C_{\alpha\beta} + M_{\perp\alpha\beta}^{\alpha\beta} - (M^{\alpha\beta} C_{\alpha}^{\lambda} C_{\lambda\beta})] u_{3}^{*} \, \mathrm{d}\omega + \int_{\omega} [P_{n\perp\beta}^{\alpha\beta} - Q_{n}^{\alpha} - L_{n}^{\lambda} C_{\lambda}^{\alpha}] W_{\alpha}^{n^*} \, \mathrm{d}\omega$$

$$+ \int_{\partial\omega} [-N^{\alpha\beta} v_{\beta} + 2M^{\lambda\beta} C_{\lambda}^{\alpha} v_{\beta}] u_{\alpha}^{*} \, \mathrm{d}\Gamma$$

$$- \int_{\partial\omega} [M_{\perp\beta}^{\alpha\beta} v_{\alpha} + (M^{\alpha\beta} v_{\beta})_{\perp\alpha}] u_{3}^{*} \, \mathrm{d}\Gamma - \int_{\partial\omega} P_{n}^{\alpha\beta} v_{\beta} W_{\alpha}^{*} \, \mathrm{d}\Gamma.$$

The application of the virtual work leads to the equations 17a (or 18a) and 19a (or 20a).

APPENDIX C

For the Warping–Kirchhoff model, the exterior virtual work is (for the densities $\mathbf{f}_v$ and $\mathbf{f}_s$):

$$W_e = \int_{\omega} \int_{-\frac{h}{2}}^{\frac{h}{2}} [f_v^{\alpha} u_{\alpha}^{*} + f_v^{\alpha} x^3 \beta_{\alpha}^{*} + f_v^{\alpha} \phi_n W_{\alpha}^{n^*} + f_v^{3} u_{3}^{*}] \, \mathrm{d}x^3 \, \mathrm{d}\omega$$

$$+ \int_{\partial\omega} \int_{-\frac{h}{2}}^{\frac{h}{2}} [f_s^{\alpha} u_{\alpha}^{*} + f_s^{\alpha} x^3 \beta_{\alpha}^{*} + f_s^{\alpha} \phi_n W_{\alpha}^{n^*} + f_s^{3} u_{3}^{*}] \, \mathrm{d}x^3 \, \mathrm{d}\Gamma$$

$$= \int_{\omega} [F^{\alpha} u_{\alpha}^{*} + m^{\alpha} \beta_{\alpha}^{*} + F_{n}^{\alpha} W_{\alpha}^{n^*} + F^{3} u_{3}^{*}] \, \mathrm{d}\omega$$

$$+ \int_{\partial\omega} [F_s^{\alpha} u_{\alpha}^{*} + m_s^{\alpha} \beta_{\alpha}^{*} + (F_s^{\alpha})^{n} W_{\alpha}^{n^*} + F_s^{3} u_{3}^{*}] \, \mathrm{d}\Gamma.$$

For the Warping–Mindlin model, the inertial virtual work is

$$W_j = -\int_{\omega}\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\ddot{u}^{\alpha} + x^3\ddot{\beta}^{\alpha} + \ddot{\mathbf{W}}^{\alpha\mathbf{n}}\boldsymbol{\phi}_{\mathbf{n}})(u^*_{\alpha} + x^3\beta^*_{\alpha} + \mathbf{W}^{\mathbf{n}}_{\alpha}\boldsymbol{\phi}^*_{\mathbf{n}})\,\mathrm{d}x^3\,\mathrm{d}\omega$$

$$= -\int_{\omega}^{\omega} \rho h(\ddot{u}^{\alpha}u^*_{\alpha} + \ddot{u}^3 u^*_3) + \rho\frac{h^3}{12}\ddot{\beta}^{\alpha}\beta^*_{\alpha} + \rho h\boldsymbol{\Sigma}_{\mathbf{n}}(\ddot{\mathbf{W}}^{\alpha})^{\mathbf{n}}\mathbf{W}^{\mathbf{n}^*}_{\boldsymbol{\alpha}}\,\mathrm{d}\omega.$$

For the Warping–Mindlin model, the interior virtual work is

$$W_i = -\int_{\omega}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha\beta}[\gamma_{\alpha\beta}(\mathbf{u}^*_{\omega}) + x^3\gamma_{\alpha\beta}(\boldsymbol{\beta}^*) - x^3\rho_{\alpha\beta}(\mathbf{u}^*_{\omega}) + \phi_n\boldsymbol{\gamma}_{\boldsymbol{\alpha\beta}}(\mathbf{W}^{\mathbf{n}^*})]\,\mathrm{d}x^3\,\mathrm{d}\omega$$

$$-\int_{\omega}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha 3}[(\beta^*_{\alpha} + u^*_{\lambda}C^{\lambda}_{\alpha} + u^*_{3,\alpha}) + (\phi_{n,3}W^{n^*}_{\alpha} + \phi_n W^{n^*}_{\lambda} f^{\lambda}_{\alpha}]\,\mathrm{d}x^3\,\mathrm{d}\omega$$

using the stress resultant defined by equation (8), the interior virtual work becomes

$$W_i = -\int_{\omega} [N^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}^*_{\omega}) + M^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\beta}^*) - M^{\alpha\beta}\rho_{\alpha\beta}(\mathbf{u}^*_{\omega}) + P^{\alpha\beta}_n\boldsymbol{\gamma}_{\boldsymbol{\alpha\beta}}(\mathbf{W}^{\mathbf{n}^*})]\,\mathrm{d}\omega$$

$$-\int_{\omega} [T^{\alpha}(\beta^*_{\alpha} + u^*_{\lambda}C^{\lambda}_{\alpha} + u^*_{3,\alpha}) + Q^{\alpha}_n W^{n^*}_{\alpha} + L^{\alpha}_n W^{n^*}_{\lambda}C^{\lambda}_{\alpha}]\,\mathrm{d}\omega,$$

using relation (6) and after integration by parts, it becomes

$$W_i = \int_{\omega} N^{\alpha\beta}_{\perp\beta}u^*_{\alpha}\,\mathrm{d}\omega - \int_{\partial\omega} N^{\alpha\beta}u^*_{\alpha}v_{\beta}\,\mathrm{d}\Gamma + \int_{\omega} N^{\alpha\beta}C_{\alpha\beta}u^*_3\,\mathrm{d}\omega + \int_{\omega} M^{\alpha\beta}_{\perp\alpha\beta}\,\beta^*_{\alpha}\,\mathrm{d}\omega$$

$$-\int_{\partial\omega} M^{\alpha\beta}\beta^*_{\alpha}v_{\beta}\,\mathrm{d}\Gamma + \int_{\omega} P^{\alpha\beta}_{n\perp\beta}W^{n^*}_{\alpha}\,\mathrm{d}\omega - \int_{\partial\omega} P^{\alpha\beta}_n W^{n^*}_{\alpha}v_{\beta}\,\mathrm{d}\Gamma - \int_{\omega} (M^{\alpha\beta}C^{\lambda}_{\beta})_{\perp\alpha}u^*_{\lambda}\,\mathrm{d}\omega$$

$$+\int_{\partial\omega} (M^{\alpha\beta}C^{\lambda}_{\beta})u^*_{\lambda}v_{\alpha}\,\mathrm{d}\Gamma - \int_{\partial\omega} (M^{\alpha\beta}C^{\lambda}_{\alpha}C_{\lambda\beta})u^*_3\,\mathrm{d}\omega - \int_{\omega} T^{\alpha}\beta^*_{\alpha}\,\mathrm{d}\omega - \int_{\omega} T^{\alpha}C^{\lambda}_{\alpha}u^*_{\lambda}\,\mathrm{d}\omega$$

$$+\int_{\omega} T^{\alpha}_{\perp\alpha}u^*_3\,\mathrm{d}\omega - \int_{\partial\omega} T^{\alpha}v_{\alpha}u^*_3\,\mathrm{d}\Gamma - \int_{\omega} Q^{\alpha}_n W^{n^*}_{\alpha}\,\mathrm{d}\omega - \int_{\omega} L^{\alpha}_n C^{\lambda}_{\alpha}W^{n^*}_{\lambda}\,\mathrm{d}\omega$$

also

$$W_i = \int_\omega [N^{\alpha\beta}_{\perp\beta} - (M^{\lambda\beta}C^\alpha_\lambda)_{\perp\beta} - T^\lambda C^\alpha_\lambda] u^*_\alpha \, d\omega + \int_\omega (M^{\alpha\beta}_{\perp\beta} - T^\alpha)] \beta^*_\alpha \, d\omega$$

$$+ \int_\omega [N^{\alpha\beta}C_{\alpha\beta} + T^\alpha_{\perp\alpha} - (M^{\alpha\beta}C^\lambda_\alpha C_{\lambda\beta})] u^*_3 \, d\omega + \int_\omega [P^{\alpha\beta}_{n\perp\beta} - Q^\alpha_n - L^\lambda_n C^\alpha_\lambda] W^{n*}_\alpha \, d\omega$$

$$+ \int_{\partial\omega} [- N^{\alpha\beta} v_\beta + M^{\lambda\beta} C^\alpha_\lambda v_\lambda] u^*_\alpha \, d\Gamma$$

$$- \int_{\partial\omega} M^{\alpha\beta} v_\beta \beta^*_\alpha \, d\Gamma - \int_\omega T^\alpha v_\alpha u^*_3 \, d\Gamma - \int_{\partial\omega} P^{\alpha\beta}_n v_\beta W^{n*}_\alpha \, d\Gamma.$$

The application of the virtual work leads to the equations (17b) or (18b) and (19b) or (20b).

APPENDIX D: NOMENCLATURE

$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$	local natural base
$a_{\alpha\beta}$	covariant metric coefficients
$C_{\alpha\beta}$	coefficients of the curvature tensor
(D_1, D_2, D_3)	stiffness coefficients (equation (15))
E	Young's modulus
$\mathbf{F}_\omega$	the in-surface forces vector
$\mathbf{F}^n$	projection of the in-surface vector forces on the nth transverse normal mode
$(\mathbf{F}_\omega)_s$	the in-surface boundary forces
$(\mathbf{F}_s)^n$	projection of the in-surface boundary forces on the nth mode
h	thickness of the shell
K	curvature variation tensor
$\mathbf{L}_n$	first warping-shear resultant
m	a point of the surface
$\mathbf{m}_\omega$	the in-surface moments vector
$(\mathbf{m}_\omega)_s$	the in-surface boundary moments
M	a point of the shell
$\mathbf{M}$	flexural moment tensor
$\mathbf{N}$	normal resultant tensor
$\mathbf{P}$	warping resultant tensor
$\mathbf{Q}_n$	second warping-shear resultant
r	radius of the cylindrical surface
$\mathbf{T}$	shear stress resultant
(u_λ, u_3)	displacement of the surface written in the local surface co-ordinate
$\ddot{\mathbf{u}}_\omega$	the surface's acceleration
$U_{\alpha\perp\beta}$	covariant derivate
(U_λ, U_3)	displacement of a point of the shell written in the surface co-ordinate
(W^n_1, W^n_2, W^k_3)	participation of the deformation modes to warping
(x^1, x^2)	surface co-ordinates
x^3	normal co-ordinate
β_λ	rotation of the normal fibre

γ	membranous deformation tensor
δ	Dirac's function
$\delta\omega$	boundary of the surface ω
ε	strain tensor
Θ_n, Σ_n, Ξ_n	warping coefficients
ν	the Poisson's ratio
$\pi(\ddot{\mathbf{u}}_\omega)$	projection on surface of the surface's acceleration $\ddot{\mathbf{u}}_\omega$
ρ	mass density
σ	stress tensor
$\Gamma^\lambda_{\alpha\beta}$	coefficients of Christoffel
$\{\phi_n\}$	the nth transverse mode
(ϕ_n, Φ_k)	warping co-ordinates
$\{\Phi_k\}$	the kth longitudinal mode
ω	surface of the shell
$\mathbf{1}$	the unit tensor