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DELAY-INDEPENDENT STABILITY OF RETARDED DYNAMIC SYSTEMS OF MULTIPLE DEGREES OF FREEDOM

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On the basis of generalized Sturm theory, this paper presents a simple, but systematic approach to the delay-independent stability analysis of the linear dynamic systems involving multiple degrees of freedom and possibly two time delays. The approach enables one to complete the stability analysis in a much simpler way than before through the use of a MAPLE routine given in the paper. To demonstrate the approach, the paper gives a detailed analysis, the corresponding sufficient and necessary conditions as well as stable regions in parameter space of concern for the delay-independent stability of a vibrating system with time delays in state feedback, an active-tendon for a tall structure and an active suspension of a quarter car model.

1. INTRODUCTION

Unavoidable time delays frequently appear in the controlled mechanical or structural systems, especially in hydraulic actuators such as those used in the active suspensions of ground vehicles and the active tendons of tall structures. Over the last decade, great attention has been paid to the dynamics of those systems. It has been found that the time delays not only make the systems retarded, but also give the systems a series of unique dynamic features, say, the infinite-dimensional subspace of solutions, the possible chaotic behavior when the systems are characterized by only one unknown variable of single dimension, etc. These features are quite different from those of the dynamic systems described by ordinary differential equations. The time delays also give rise to some tough problems in studying the system dynamics. For example, the characteristic equation of a linear system with time delays is a transcendental equation involving exponential functions, and has an infinite number of roots. Thus, it is usually difficult to carry out stability analyses for linear multiple degrees of freedom (Md.o.f.) systems, though most engineering systems have to be simplified to the models of multiple degrees of freedom. It is one of the open problems, therefore, to develop concise stability criteria for linear Md.o.f. systems with time delays.

The current stability criteria can be classified into two catalogues according to whether the stability depends on time delays or not. In the latter case, the system is

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exponentially stable for arbitrary time delays. From the viewpoint of mathematicians, the stability problem of a linear retarded dynamic system has been solved since some sufficient and necessary conditions are available for the stability analysis when the time delays in the system are given [1–3]. However, these conditions do not show any explicit relationship between the system parameters that the engineers are interested in. Moreover, when those conditions are used, the stability test involves very tedious computations such as solving transcendental equations or computing the spectrum of operators. In general, the analysis of delay-independent stability is relatively simple. Though the conditions for the delay-independent stability are usually too strict from the practical viewpoint, they can offer the engineers useful information in the initial phase of design. For instance, the criterion of delay-independent stability in reference [4] enables one to gain an insight into the dynamics of a class of single-degree-of-freedom systems with one or two time delays in the state feedback. However, for linear Md.o.f. systems with single delay or multiple time delays, even the analysis of the delay-independent stability is not a trivial task [2]. Some examples can be found in reference [2].

The aim of this paper is to develop a systematic approach to the delay-independent stability analysis for linear retarded dynamic systems of multiple degrees of freedom, possibly with two time delays. As usually done in analyzing the stability of a linear dynamic system, the paper focuses on the corresponding characteristic function, which is a quasi-polynomial when the system involves time delays. In the analysis of delay-independent stability, the key is to determine if a corresponding polynomial has real roots. For lower-dimensional systems with given parameters, the delay-independent stability analysis can be completed in a relatively simple way. The classical Sturm criterion can be used to determine whether the corresponding polynomial has real roots. As for the systems with parameters to be determined, the classical theory shows little hope to give a full answer to the stability analysis. However, the generalized Sturm theory developed in reference [5] is especially suitable for this purpose. Hence, the paper begins with some mathematical preliminaries about the classical Sturm theory and the generalized Sturm sequence in section 2, and then presents the systematic approach to the delay-independent stability analysis in section 3. The approach gives the stability criterion characterized by a sufficient and necessary condition. For the stability test based on this approach, one needs to complete only some algebraic computations to check this kind of condition. The approach is not only applicable to the systems with single time delays, but also to those with two time delays for some applications. To illustrate the approach, a detailed delay-independent stability analysis is given in section 4 for a vibrating system with time delays in state feedback, an active tendon of tall structure [6] and an active suspension of a quarter car model [7]. For these systems, analysis shows that it is the damping that makes the delay-independent stability possible. The delay-independent stable regions in the space of interesting parameters are bounded. In the computation of the examples, the computer algebra package MAPLE [8] was used so that the stability analysis became very simple. Finally in section 5, some concluding remarks are made.

2. THEORETICAL BACKGROUND OF THE APPROACH

In the subsequent sections of this paper, one needs to know if a polynomial has real roots (or to determine the exact number of real roots of the polynomial for certain applications). Given a polynomial, the classical Sturm theory gives a full answer to this problem. However, it does not work well if the polynomial involves some unknown parameters. In these cases, the generalized Sturm theory developed in reference [5] shows great advantages. To gain a better understanding of the theory, we will first, in this section, give an outline of the classical Sturm theory, and then present some basic facts about the generalized Sturm theory, following chapter 6 in reference [5] and reference [9].

2.1. THE CLASSICAL STURM CRITERION

Given a real number sequence $l_1, l_2, \ldots, l_n$ $(l_1 l_2 \cdots l_n \neq 0)$, the sign sequence $[s_1, s_2, \ldots, s_n]$ with $s_i = sign(l_1)$, $(i = 1, 2, \ldots, n)$ is called the sign table of the sequence. If $l_j l_{j+1} < 0$, we say that the sign has changed once from l_j to l_{j+1}. For any given real sequence $l_1, l_2, \ldots, l_n$ $(l_1 l_2 \cdots l_n \neq 0)$, we can figure out the number of variation of signs of the sequence. For instance, the number of variation of signs of 1, 3, -2, 1, -4, -10, -1, 4, 2 is 4. If a sequence contains some zeros, then the number of variation of signs is defined as the number of variation of a new sequence obtained by deleting the zeros. For example, the number of variation of signs of real sequence -3, -5, 0, 3, 0, 2, -6, 0 is 2, while the number of variation of signs of 0, -3, 2 is 1.

Suppose that $f(x)$ is a real polynomial without repeated roots. A sequence of real polynomials.

$$f_0(x) = f(x), \quad f_1(x), \ \ldots, f_s(x) \tag{1}$$

is called the Sturm sequence of polynomial $f(x)$ if the following conditions hold:

(i) Any two neighboring polynomials in equation (1) have no common roots:
(ii) The last polynomial $f_s(x)$ has no real roots;
(iii) If there exist some k, $(1 \leqslant k \leqslant s-1)$ such that $f_k(\alpha) = 0$, then $f_{k-1}(\alpha) f_{k+1}(\alpha) < 0$;
(iv) If $f(\alpha) = 0$, then there exists a sufficiently small positive number ε such that $f_0(k) f_1(k) < 0$ for $k \in (\alpha - \varepsilon, \alpha)$ and $f_0(k) f_1(k) > 0$ for $k \in (\alpha, \alpha + \varepsilon)$.

Now, we are in a position to state the following classical Sturm theorem.

Sturm criterion. Assume that a real polynomial $f(x)$ has no repeated roots and has p real roots in the interval (α, β), satisfying $f(\alpha) f(\beta) \neq 0$. If the numbers of variation of signs of the sequences

$$f_0(\alpha), \quad f_1(\alpha), \ldots, f_s(\alpha); \qquad f_0(\beta), \quad f_1(\beta), \ldots, f_s(\beta) \tag{2a, b}$$

are $v(\alpha)$ and $v(\beta)$, respectively, then $p = v(\beta) - v(\alpha)$.

Some methods are available to construct the Sturm sequence of a polynomial with no repeated roots. What follows is the most common way. Let $f_0(x) = f(x)$, and $f_1(x) = f'(x)$ be the derivative of $f(x)$. Dividing $f(x)$ one gets the polynomial $f_2(x)$ from $f_0(x) = f_1(x)q_1(x) - f_2(x)$. The other polynomials in the Sturm sequence can be constructed in the same way, namely,

$$f_k(x) = f_{k+1}(x)q_{k+1}(x) - f_{k+2}(x) \quad (0 \leqslant k < s-2) \tag{3}$$

except for the last one by $f_{s-1}(x) = f_s(x)q_s(x)$.

Since only the signs of the Sturm sequence are used in the applications of the Sturm criterion, all the positive factors can be deleted out at each step of constructing a Sturm sequence.

Example. Considering a real polynomial

$$f_0(x) = x^5 + 5x^4 + 5x^3 - 5x^2 - 5x - 7 \tag{4}$$

the Sturm sequence can be constructed as follows. Let

$$f_0(x) = f(x) = x^5 + 5x^4 + 5x^3 - 5x^2 - 5x - 7,$$

$$f_1(x) = f'(x)/5 = x^4 + 4x^3 + 3x^2 - 2x - 1,$$

Dividing $f_0(x)$ by $f_1(x)$, one arrives at $f_0(x) = f_1(x)(x+1) - 2(x^2+1)(x+3)$. Since $2(x^2+1)$ is a positive factor, one can take

$$f_2(x) = x + 3.$$

Dividing $f_1(x)$ by $f_2(x)$, one has $f_1(x) = f_2(x)(x^3 + x^2 - 2) - (-5)$. Hence it follows that $f_3(x) = -5$. The Sturm sequence of $f(x)$ consists of the four polynomials $f_0(x)$, $f_1(x), f_2(x), f_3(x)$.

In order to understand the real roots of $f(x)$, we study the number variation of signs of its Sturm sequence. Some cases are listed in Table 1. By using the Sturm criterion, we see that $f(x)$ has only one real root in $(-\infty, +\infty)$, and the root falls into (1, 2).

If $f(x)$ has repeated roots, the last polynomial $f_s(x)$ in the Sturm sequence is the (non-constant) greatest common divisor of $f_0(x)$ and $f_1(x)$, i.e., $f_s(x) = d(x)g_x(x)$, where $d(x) = g.c.d(f(x), f'(x))$, the leading coefficient is set to be 1, while $g_s(x)$ is the leading coefficient of $f_s(x)$. Let

$$f_k(x) = d(x)g_k(x) \quad (k = 0, 1, 2, \ldots, s) \tag{5}$$

TABLE 1

The sign tables of the Sturm sequence

x	$f_0(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	v
$-\infty$	$-$	$+$	$-$	$-$	2
0	$-$	$-$	$+$	$-$	2
1	$-$	$+$	$+$	$-$	2
2	$+$	$+$	$+$	$-$	1
$+\infty$	$+$	$+$	$+$	$-$	1

then $g_0(x)$ is a real polynomial with no repeated roots, and $g_0(x), g_1(x), \ldots, g_s(x)$ is the Sturm sequence of $g_0(x)$. Thus, the Sturm criterion is still valid for polynomials with repeated roots in the sense that each of the repeated roots is counted only once.

2.2. DISCRIMINATION'S SEQUENCE

In the generalized Sturm theory, the discrimination's sequence takes the same role as the Sturm sequence in the classical Sturm theory. The discrimination's sequence is constructed by using the so-called Bezout matrix. The Bezout matrix is originally created for simplifying the computation of the resultant, which is one of the most important concepts in the theory of polynomial.

Consider two real polynomials of order n:

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n, \tag{6a}$$

$$g(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n \tag{6b}$$

and rewrite them in the form

$$f(x) = f_{i1}(x) x^i + f_{i2}(x), \qquad g(x) = g_{i1}(x) x^i + g_{i2}(x), \tag{7}$$

where

$$f_{i1}(x) = a_0 x^{n-i} + a_1 x^{n-1-i} + \cdots + a_{n-i},$$

$$f_{i2}(x) = a_{n-i+1} x^{i-1} + a_{n-i+2} x^{i-2} + \cdots + \mathrm{a}_n,$$

$$g_{i1}(x) = b_0 x^{n-i} + b_1 x^{n-1-i} + \cdots + b_{n-i},$$

$$g_{i2}(x) = b_{n-i+1} x^{i-1} + b_{n-i+2} x^{i-2} + \cdots + \mathrm{b}_n.$$

Let

$$p_i(x) = \begin{vmatrix} f_{i1}(x) & f_{i2}(x) \\ g_{i1}(x) & g_{i2}(x) \end{vmatrix} \equiv \sum_{j=1}^{n} d_{ij} x^{n-j} \tag{8}$$

then the Bezout matrix is defined as the coefficient matrix of n polynomials $p_{n-i+1}(x)$, $(i = 1, 2, \ldots, n)$ in the form

$$\begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix}. \tag{9}$$

Now, let $f(x)$ be a real polynomial of order n and regard $f'(x)$ as a polynomial of order n with zero leading coefficient, one defines the Bezout matrix of $f(x)$ and $f'(x)$ as the discriminant matrix of $f(x)$ and denotes it by *discr* (f), which can be proved to take the following explicit form:

$$discr(f) = (c_{n-i,\,j-1}), \quad i, j = 1, 2, \ldots, n,$$

$$c_{ij} = (n - max(i, j))a_i a_j - \sum_{p=0}^{\min(i,j)-1} (i + j - 2p) a_p a_{i+j-p}, \tag{10}$$

where $a_k \equiv 0$ if $k < 0$ or $k > n$.

The discriminant sequence of $f(x)$ is defined as the principal sub-determinant sequence taken in order, and denoted by

$$D_1(f), D_2(f), \ldots, D_n(f) \tag{11}$$

respectively. This sequence can be simply obtained by using a short MAPLE routine *discr* shown in Figure 1. The command *discr* $(f(x), x)$ on the MAPLE platform gives sequence (11).

The repeated roots of a polynomial $f(x)$ are determined exactly by the greatest common divisor of $f(x)$ and $f'(x)$. One defines the sequence of sub-resultants of $f(x)$ and $f'(x)$ as the sequence of multiple order factors of $f(x)$, and denotes it by $\Delta_0(f)$, $\Delta_1(f), \ldots, \Delta_{n-1}(f)$.

2.3. A MODIFIED SIGN TABLE

To express the generalized Sturm theory in a compact form, some agreements are needed. Given a real number sequence $l_1, l_2, \ldots, l_n$ $(l_1 \neq 0)$, the sign sequence $[s_1, s_2, \ldots, s_n]$, with $s_i = sign(l_1)$, $(i = 1, 2, \ldots, n]$ is called the sign table of the

```
> discr: = proc(poly,var)
>     local f, g, tt, d, bz, i, ar, j, mm, dd:
>         f = expand(poly): d: = degree (f, var):
>         g:=tt var^d+diff(f, var):
>         with(linalg):
>         bz:=subs(tt=0, bezout(f,g,var)): ar:=[ ]:
>         for i to d do
>         ar:=[op(ar), row(bz,d+1-i..d+1-i)] od:
>         mm:=matrix(ar): dd:= [ ]:
>         for j to d do
>         dd:=[op(dd),det(submatrix(mm,1..j,1..j))] od:
>         dd:=map(primpart,dd)
> end:
```

Figure 1. The Maple routine of *discr*.

original sequence. The modified sign table $[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]$ can be written out by following the rules as below:

(1) For any segment $[s_i, s_{i+1}, s_{i+2}, \ldots, s_{i+j}]$ with $s_i \neq 0$, $s_{i+1} = s_{i+2} \cdots = s_{i+j-1} = 0$ and $s_{i+j} \neq 0$ of a given sign table, $[s_{i+1}, {}_{i+2}, \ldots, s_{i+j-1}]$ is replaced by $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \ldots]$.
(2) All the other terms in the table are kept unchanged.

For example, the sign table $[1, -1, 0,0, 0, 0, 1, 0, 0, 0, -1, 0]$ will be modified by $[1, -1, 1, 1, -1, -1, 1, -1, -1, 1, -1, 0]$ according to these two rules.

2.4. NUMBER OF REAL ROOTS OF A POLYNOMIAL VIA DISCRIMINANT SEQUENCE

Based on the discriminant sequence and its modified sign table, we state the following theorem that plays an important role in the stability analysis.

Theorem 1. *Let $f(x)$ be a polynomial of order n and $D_1(f), D_2(f), \ldots, D_n(f)$ be the corresponding discriminant sequence, and $\sigma_i = D_{qi}$ $(i = 1, 2, \ldots, k)$ is the ith non-zero term of the discriminant sequence, $\sigma_0 = 1$. Let $q_0 = 0$, $s_i = q_{i+1} - q_i - 1$, $(i = 0, 1, 2, \ldots, k-1)$. Assume that the number of variation of signs in the modified sign table is v. If l is the integer satisfying $D_l(f) \neq 0$, $D_m(f) = 0$ $(m > l)$, then we have*

(1) *The number of distinct pairs of conjugate complex roots of $f(x)$* is v.
(2) *The number of distinct real roots of $f(x)$ is $l - 2v$, which satisfies* 1

$$l - 2v = \sum_{i=0,\ s_i\ are\ even}^{k-1} (-1)^{s_i/2}\, sign\left(\frac{\sigma_{i+1}}{\sigma_i}\right). \tag{12}$$

(3) *α is a root of multiplicity of p of $f(x)$ if and only if it is a root of multiplicity $p-1$ of $\Delta_{n-l}(f)$.*
(4) *Except for some positive factors, $D_1(f)$, $D_2(f), \ldots, D_l(f)$ is the discriminant sequence of polynomial $f/g.c.d.(f,f')$, which has no repeated roots.*

Example 1. Let $f(x) = x^{18} - x^{16} + 2x^{15} - x^{14} - x^5 + x^4 + x^3 - 3x^2 + 3x - 1$. Then, the sign table of its discriminant sequence is [1, 1, −1, −1, −1, 0, 0, 0, −1, 1, 1, −1, −1, 1, −1, −1, 0, 0]. Thus, the modified sign table is [1, 1, −1, −1, −1, 1, 1, −1, −1, 1, 1, −1, −1, 1, −1, 1, −1, 0, 0] and the number of variation of signs is 7. As a result, $f(x)$ has two distinct real roots and seven distinct pairs of conjugate complex roots. As $g.c.d.(f,f') = x^2 - x + 1$, the remaining two roots of $f(x)$ are a pair of conjugate complex roots repeated to one of the above seven pairs.

Example 2. To ensure that a polynomial of order 6 with positive leading coefficient has no real roots, one needs to make sure that one of the following 14 cases, listed in Appendix A (Table 2) of the modified sign table is held true.

The above theorem gives us full information about the numbers of real roots and complex roots of a polynomial. However, it is the case $l = 2v$ for some $v = 1, 2, \ldots,$ which serves the purpose of delay-independent stability analysis.

3. STABILITY ANALYSIS OF RETARDED DYNAMIC SYSTEMS

Now, consider the delay-independent stability analysis of a linear dynamic system with possibly two time delays. The characteristic equation of the system is in the form

$$D(\lambda, \tau_1, \tau_2) \equiv P(\lambda) + Q_1(\lambda)\exp(-\lambda\tau_1) + Q_2(\lambda)\exp(-\lambda\tau_2) = 0, \tag{13}$$

where $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$ are the time delays, $P(\lambda)$, $Q_1(\lambda)$ and $Q_2(\lambda)$ are three polynomials of real coefficients under the conditions $\deg(P) = n > \deg(Q_i)$, $(i = 1, 2)$. As is well known, the system of concern is delay-independent stable if and only if each of its characteristic roots has negative real part for all given $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$. Moreover, we have the following theorem to ensure delay-independent stability.

Theorem 2. *The linear delayed dynamic system with equation* (13) *as the characteristic equation is delay-independent stable if and only if the following two conditions are held true*:

(i) *The polynomial $P(\lambda) + Q_1(\lambda) + Q_2(\lambda)$, corresponding to the case $\tau_1 = \tau_2 = 0$, is Hurwitz stable, and*
(ii) *$D(\mathrm{i}\omega, \tau_1, \tau_2) = 0$ has no real roots ω for all $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$.*

To demonstrate the theorem, let us first consider a one-dimensional system

$$\dot{x} = ax + bx(t-\tau), \quad a, b \in R. \tag{14}$$

The characteristic equation of the system reads

$$D(\lambda) \equiv \lambda + a + b\exp(-\lambda\tau) = 0. \tag{15}$$

When $\tau = 0$, the characteristic function $\lambda + a + b$ is Hurwitz stable if and only if $a + b < 0$. In order that $D(\mathrm{i}\omega) = 0$ has no real roots for all $\tau \geqslant 0$, it is sufficient and necessary to ensure that $|\mathrm{i}\omega + a| = |b|$ or $\omega^2 = b^2 - a^2$ has no real roots, that is, $b^2 - a^2 < 0$. Thus, the system is delay-independent stable if and only if $a + b < 0$ and $b^2 - a^2 < 0$. Here, system (14) with $a = 0$ cannot be delay-independent stable.

In general, the well-known Routh–Hurwitz criterion is available to make sure that the system is asymptotically stable [namely, $P(\lambda) + Q_1(\lambda) + Q_2(\lambda)$ is Hurwitz stable] when $\tau_1 = \tau_2 = 0$. With these stability conditions as a basic assumption, the system is delay-independent stable if and only if equation (13) has no pure imaginary roots $\pm \mathrm{i}w$ for all $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$. Let

$$P^R(w) = \mathrm{Re}(P(\mathrm{i}w)) \qquad P^I(w) = \mathrm{Im}(P(\mathrm{i}w)),$$

$$Q_j^R(w) = \mathrm{Re}(Q_j(\mathrm{i}w), \qquad Q_j^I(w) = \mathrm{Im}(Q_j(\mathrm{i}w)) \qquad (j = 1, 2). \tag{16}$$

Since $P^R(w), Q_j^R(w), (j = 1, 2)$ are even functions and $P^I(w), Q_j^I(w), (j = 1, 2)$ are odd functions, equation (13) has no pure imaginary roots $\pm \mathrm{i}w$ for all $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$ if and only if

$$\begin{aligned}(P^R(w))^2 &+ (P^I(w))^2 - ((Q_1^R(w))^2 + (Q_1^I(w))^2 + (Q_2^R(w))^2 + (Q_2^I(w))^2)\\ &- (-2Q_1^I(w)Q_2^R(w) + 2Q_1^R(w)Q_2^I(w))\sin(w(\tau_2 - \tau_1))\\ &- (2Q_1^I(w)Q_2^I(w) + 2Q_1^R(w)Q_2^R(w))\cos(w(\tau_2 - \tau_1)) = 0\end{aligned} \tag{17}$$

has no non-negative root w for all $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$.

In the case of $\tau_1 = \tau_2$, the left-hand side of equation (16) can simply be rewritten as

$$F(w) \equiv w^{2n} + b_1 w^{2(n-1)} + b_2 w^{2(n-2)} + \cdots + b_{n-1}w^2 + b_n. \tag{18}$$

If $\tau_1 \neq \tau_2$, an even function $G(w)$ in w is defined as follows:

$$\begin{aligned}G(w) = (P^R(w))^2 &+ (P^I(w))^2 - ((Q_1^R(w))^2 + (Q_1^I(w))^2 + (Q_2^R(w))^2 + (Q_2^I(w))^2)\\ &- \sqrt{(-2Q_1^I(w)Q_2^R(w) + 2Q_1^R(w)Q_2^I(w))^2 + (2Q_1^I(w)Q_2^I(w) + 2Q_1^R(w)Q_2^R(w))^2}.\end{aligned} \tag{19}$$

It can be shown that equation (17) has no non-negative roots for all $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$ if and only if $G(w)$ has no non-negative roots, or $G(w^2)$ has no real roots. In general, $G(w)$ is not a polynomial. However, for certain applications, the system with linear, delayed state feedback fall into this category. For these systems, the analysis of delay-independent stability can be converted to the study on the real roots of a polynomial.

According to (ii) of Theorem 1, one can find that $F(w)$ (or $G(w^2)$) has no real roots if and only if the modified sign table subjects to $l = 2v$ for $v = 1, 2, 3, 4, \ldots, n$ (or $2n$), where l and v are defined in Theorem 1.

Based on the above analysis, an approach to the delay-independent stability can be summarized as below.

Algorithm

(1) Find out the characteristic equation (13) and the corresponding function $F(w)$ (*or* $G(w^2)$).
(2) Determine the discriminant sequence (11) by using the MAPLE routine *discr*.
(3) Justify the stability as follows. The system is delay-independent stable if and only if $l = 2v$ for some $v = 1, 2, 3, 4, \ldots, n$ (or $2n$), and the polynomial $P(\lambda) + Q_1(\lambda) + Q_2(\lambda)$ is Hurwitz stable.

Remark. If the terms in the discriminant sequence are factorized, the computation in stability test will be greatly reduced.

In practice, especially in the design phase of a controlled system, one usually wants to know the delay-independent stable region in a parameter space. Once the discriminant sequence is obtained and each term is factorized, one can set each factor equal to zero and draw its graph. These graphs divide the parameter space into several sub-regions. Each of them can be easily determined to be or not to be delay-independent stable by checking the corresponding sign tables. This will be discussed in detail later.

4. APPLICATIONS

This section gives three example to illustrate the presented approach. The first example is about the stability analysis of a vibrating system with time delays in state feedback. It shows that the approach is not only applicable to the systems with single time delay, but also available for some systems with multiple time delays. The second example deals with the delay-independent stability of an active-tendon for tall structure, which is a three-dimensional system. The final example is on the stability analysis of an active suspension of a quarter car model with 2 d.o.f.

4.1. STABILITY OF A VIBRATING SYSTEM WITH TIME DELAYS IN STATE FEEDBACK

Consider a vibrating system of single d.o.f. with two time delays in the paths of displacement feedback and velocity feedback, respectively. The equation of motion of the system yields

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = s_1 x(t - \tau_1) + s_2 \dot{x}(t - \tau_2). \tag{20}$$

The corresponding characteristic equation reads

$$\lambda^2 + a\lambda + b - s_1 \exp(-\lambda\tau_1) - s_2\lambda \exp(-\lambda\tau_2) = 0. \tag{21}$$

4.1.1. *General results*

Assume that $a > s_2$ and $b > s_1$. These inequalities are the sufficient and necessary conditions under which the system is exponentially stable when $\tau_1 = \tau_2 = 0$. In order that the system is delay-independent stable, it is sufficient and necessary to make sure that equation (13) has no pure imaginary roots $\pm iw$ for all $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$.

In the case of single time delay, i.e., $\tau_1 = \tau_2$, it is easy to find out the discriminant sequence of the system as follows:

$$\begin{aligned} &1,\ -(a^2 - s_2^2 - 2b),\ (a^2 - s_2^2 - 2b)(4(b^2 - s_1^2) - (a^2 - s_2^2 - 2b)^2),\\ &(4(b^2 - s_1^2) - (a^2 - s_2^2 - 2b)^2)^2(b^2 - s_1^2). \end{aligned} \tag{22}$$

Therefore, the system is delay-independent stable if and only if the *modified* sign tables of the discriminant sequence (22) is in one of the following three cases: [1, −1, 0, 0], [1, −1, −1, 1] and [1, 1, −1, 1].

In practice, it is required that the stability condition should be in terms of the *original* sign tables of the discriminant sequence. As [1, −1, −1, 1] may be the modified sign table of [1, 0, 0, 1], the second and the third term in [1, −1, −1, 1] should be understood to be non-positive when it is regarded as an *original* sign Table 1. Thus, the stability conditions can be expressed in a more explicit form. That is, the system with equal time delays $\tau_1 = \tau_2$ is delay-independent stable if and only if the sign table of $[A,\ 4B - A^2,\ B] \equiv [a^2 - s_2^2 - 2b,\ 4(b^2 - s_1^2) - (a^2 - s_2^2 - 2b)^2,\ b^2 - s_1^2]$ takes one of the following four cases: [1, 0, 1], [1, 1, 1], [1, −1, 1], [−1, 1, 1]. These conditions can be simplified to

$$A > 0,\ B > 0, \quad \text{or} \quad A \leqslant 0,\ A^2 - 4B < 0 \tag{23}$$

which are identical to those derived in reference [2] by using another method.

It is easy to know from the above conditions that the delay-independent stable region in the parameter space (s_1, s_2) is symmetrical and bounded in the region $D = \{(s_1, s_2) : |s_1| < |b|,\ |s_2| < |a|\}$. If $ab = 0$, the vibrating system with retarded feedback cannot be delay-independent stable.

When two distinct time delays $\tau_1 \neq \tau_2$ appear in the feedback paths, the system is delay-independent stable if and only if the following equation

$$w^4 + (a^2 - s_2^2 - 2b)w^2 + b^2 - s_1^2 - 2s_1 s_2 w \sin(w\tau_2 - w\tau_1) = 0 \tag{24}$$

has no real roots [4], or in turn, if and only if

$$w^8 + (a^2 - s_2^2 - 2b)w^4 + b^2 - s_1^2 - 2|s_1 s_2|w^2 \equiv w^8 + pw^4 + qw^2 + r = 0 \tag{25}$$

has no real roots. Here, the left-hand side of equation (25) is the function $G(w^2)$, and in fact, the coefficients p, and r are the same as A, and B respectively. The conditions in equation (23) are certainly true in this case.

The discriminant sequence of $w^8 + pw^4 + qw^2 + r$ is in the form

$$1,\quad d_0,\quad d_0 d_1,\quad d_1 d_2,\quad d_2 d_3,\quad d_3 d_4,\quad d_4 d_5,\quad d_5^2 d_6, \tag{26}$$

where

$$\begin{aligned} &d_0 = 0, \quad d_1 = -p, \quad d_2 = -p^2, \quad d_3 = -(2p^3 - 8pr + 9q^2),\\ &d_4 = q(-48pr + 27q^2 + 4p^3),\\ &d_5 = -(27q^4 + 4p^3q^2 - 144pq^2r - 16p^4r + 128p^2r^2 - 256r^3),\\ &d_6 = r > 0. \end{aligned}$$

As the second and the third term in the discriminant sequence vanish, the first three terms of the modified sign tables must be 1, -1, -1. Thus, the system is delay-independent stable if and only if its *modified* sign table of the discriminant sequence is in one of the 13 cases listed in Appendix A. (Table 3).

4.1.2. *Case study*

Now, two examples are given to demonstrate how to get the delay-independent stable region in the (s_1, s_2) plane.

Example 1. Consider the case of $a = 0{\cdot}5$, and $b = 1$. Since the expressions of d_i are even functions with respect to s_1 and s_2, we give the d_i's for positive s_1 and s_2 only.

$$\begin{aligned} &d_0 = 0, \qquad d_1 = 1{\cdot}75 + s_2^2 > 0, \qquad d_2 = -(1{\cdot}75 + s_2^2)^2 < 0,\\ &d_3 = 2s_2^6 + 1{\cdot}05\times 10 s_2^4 + (-28s_1^2 + 1{\cdot}0375\times 10)s_2^2 - 3{\cdot}2813 + 14{\cdot}00 s_1^2,\\ &d_4 = s_1 s_2[8s_2^6 + 42s_2^4 + (-2{\cdot}25\times 10 - 120s_1^4)s_2^2 + (-1{\cdot}2513\times 10^2 + 168s_1^2)],\\ &d_5 = 16s_2^8 + (112 - 28s_1^2)s_2^6 + (166 - 467s_1^2 + 16s_1^4)s_2^4\\ &\qquad + (-105 - 3{\cdot}6925\times 10^2 s_1^2 + 560s_1^4)s_2^2 + 1{\cdot}4063\times 10\\ &\qquad - 1{\cdot}3406\times 10^2 s_1^2 + 376s_1^4 - 256s_2^6,\\ &d_6 = 1 - s_1^2 > 0. \end{aligned} \tag{28}$$

The curve determined by $d_3 = 0$ is ellipse-like, as shown in Figure 2. The inside of the "ellipse" is governed by $d_3 < 0$, while the outside by $d_3 > 0$. The graph of $d_4 = 0$ is hyperbola-like, the corresponding sub-region containing the origin is featured by $d_4 < 0$, the other parts by $d_4 > 0$. The graph of $d_5 = 0$ consists of four line-like

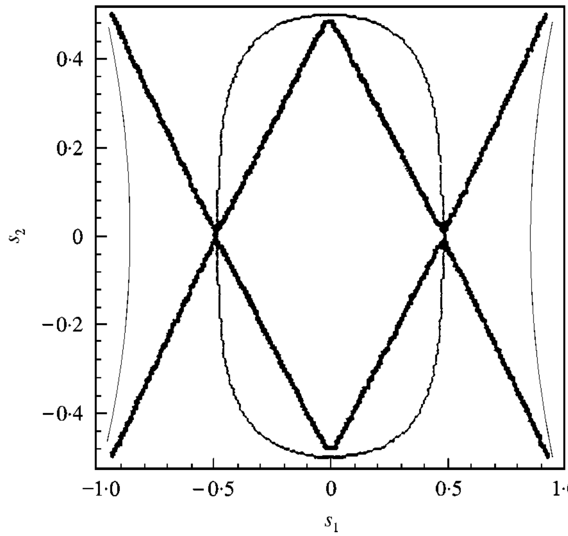

Figure 2. The rhombus-like region is the delay-independent stable region in (s_1, s_2) plane for the vibrating system with two time delays in state feedback when $a = 0{\cdot}5$, and $b = 1$.

curves, the w-like region and the m-like region are determined by $d_5 < 0$, and the other parts by $d_5 > 0$. Following the present approach, one can see that only the combination (s_1, s_2) in the rhombus-like region that makes the vibrating system delay-independent stable. The stable region is characterized by $d_5 > 0$ and $|s_1| < 1$.

Example 2. When $a = 1{\cdot}5$, and $b = 1$, one has

$$
\begin{aligned}
&d_0 = 0, \qquad d_1 = 0{\cdot}25 + s_2^2 > 0, \qquad d_2 = -(0{\cdot}25 + s_2^2)^2 < 0,\\
&d_3 = 2s_2^6 - 1{\cdot}50 s_2^4 + (-7.625 - 28 s_1^2) s_2^2 + 1{\cdot}9688 - 2 s_1^2,\\
&d_4 = s_1 s_2 (8 s_2^6 - 6 s_2^4 + (-94{\cdot}5 - 120 s_1^2) s_2^2 + (2{\cdot}3875 \times 10 - 24 s_1^2),\\
&d_5 = 16 s_2^8 + (-16 + 4 s_1^2) s_2^6 + (-122 - 323 s_1^2 + 16 s_1^4) s_2^4 + (63 + 1{\cdot}675\\
&\qquad \times 10 s_1^2 - 80 s_1^4) s_2^2 + 2.4806 \times 10^2 - 7{\cdot}5206 \times 10^2 s_1^2 + 760 s_1^4 - 256 s_1^6\\
&d_6 = 1 - s_1^2 > 0.
\end{aligned} \tag{29}
$$

As done in the above example, the delay-independent stable region in the (s_1, s_2) plane can be determined easily by checking the sign tables of the discriminant sequence as shown in Figure 3.

It should be pointed out that this problem has been studied by a different method in reference [4]. However, from the viewpoint of computation in the stability test, the present approach is simpler.

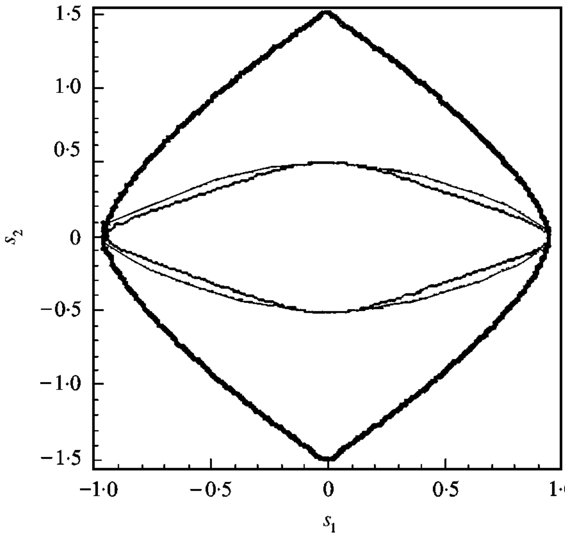

Figure 3. The delay-independent stable region, the biggest rhombus-like region, in (s_1, s_2) plane for the vibrating system with two time delays in stable feedback when $a = 1{\cdot}5$, and $b = 1$.

4.2. STABILITY OF AN ACTIVE-TENDON FOR TALL STRUCTURE

Following reference [6], one has the dimensionless equation of motion of the active-tendon of tall structure

$$\begin{aligned} \dot{X}(t) &= V(t) \\ \dot{V}(t) &= -2\xi V(t) - X(t) + e(t) - Z(t), \\ \dot{Z}(t) &= \alpha\gamma X(t-\tau) + \alpha\delta V(t-\tau) - \alpha Z(t), \end{aligned} \tag{30}$$

where $\tau > 0$ is the time delay, $e(t)$ is the external force, α is a positive number, ξ is the damping ratio, γ and δ are the displacement and velocity control gains respectively. The characteristic equation of the system is the quasi-polynomial

$$\lambda^3 + (\alpha + 2\xi)\lambda^2 + (1 + 2\alpha\xi)\lambda + \alpha + \alpha(\delta\lambda + \gamma)\exp(-\lambda\tau) = 0. \tag{31}$$

4.2.1. *General results*

In the case of $\tau = 0$, the system is stable if and only if

$$\alpha + 2\xi > 0, \qquad 1 + \gamma > 0, \qquad (\alpha + 2\xi)(\alpha\delta + 2\alpha\xi + 1) > \alpha(1 + \gamma). \tag{32}$$

The first inequality in equation (22) is a trivial case due to the positiveness of α and ξ. When $\tau > 0$, it is easy to find that the function $F(w)$ in equation (9) takes the form

$$\begin{aligned} F(w) &\equiv w^6 + (\alpha^2 + 4\xi^2 - 2)w^4 + (1 - 2\alpha^2 - \alpha^2\delta^2 + 4\alpha^2\xi^2)w^2 + \alpha^2(1 - \gamma^2) \\ &\equiv w^6 + b_1 w^4 + b_2 w^2 + b_3. \end{aligned} \tag{33}$$

$F(w)$ has no real roots only for $b_3 > 0$ (namely $|\gamma| < 1$). By using the MAPLE routine *discr*, one has the discriminant sequence

$$1, \quad -b_1, \quad 3b_1b_2 - b_1^3, \quad 7b_1^2b_2^2 - 9b_1b_2b_3 + b_1^4b_2 + 3b_1^3b_3 + 12b_2^3,$$

$$-8b_1^2b_2^4 + 37b_1^3b_2^2b_3 - 84b_1b_2^3b_3 + 27b_1^2b_2b_3^2 + b_1^4b_2^3 - 4b_1^5b_2b_3$$
$$-12b_1^4b_3^2 - 81b_1b_3^3 + 16b_2^5 + 108b_2^2b_3^2,$$

$$(-16b_1^6b_3^2 + 8b_1^5b_2^2b_3 - b_1^4b_2^4 + 144b_1^4b_2b_3^2 - 216b_1^3b_3^3 - 68b_1^3b_2^3b_3 + 8b_1^2b_2^5$$
$$-270b_1^2b_2^2b_3^2 + 144b_1b_2^4b_3 + 972b_1b_2b_3^3 - 729b_3^4 - 16b_2^6 - 216b_2^3b_3^2)b_3 \quad (34)$$

To reduce the computation in the stability test, one can factorize the above six terms and get

$$1, \quad d_0, \quad d_0d_1, \quad d_1d_2, \quad d_2d_3, \quad d_3^2d_4, \quad (35)$$

where

$$d_0 = -b_1, \qquad d_1 = b_1^2 - 3b_2, \qquad d_2 = b_1^2b_2 + 3b_1b_3 - 4b_2^2,$$
$$d_3 = -(4b_1^3b_3 - b_1^2b_2^2 - 18b_1b_2b_3 + 4b_2^3 + 27b_3^2) \qquad d_4 = -b_3 < 0. \quad (36)$$

This indicates that the sign tables of equation (34) can be obtained by computing the signs of d_i ($i = 0, 1, 2, 3, 4$), instead of computing the terms in equation (34) directly.

According to the analysis in section 2, the system is delay-independent stable if and only if (i) equation (32) holds and (ii) one of the 14 cases of the *modified* sign tables listed in Appendix A (Table 2) holds true.

4.2.2. *Case study*

Now, let $\alpha = 2$. Then we have

$$d_0 = -2 - 4\xi^2 < 0, \qquad d_1 = 25 + 12\delta^2 - 32\xi^2 + 16\xi^4,$$
$$d_2 = -64\delta^4 + (-64\xi^4 + 448\xi^2 - 240)\delta^2 - 200$$
$$+ 986\xi^2 - 880\xi^4 - 24\gamma^2 - 48\gamma^2\xi^2 + 256\xi^6,$$
$$d_3 = 256\delta^6 + (256\xi^4 - 2816\xi^2 + 1408)\delta^4$$
$$+ ((1152\xi^2 + 576)\gamma^2 - 2048\xi^6 + 11136\xi^4 - 11520\xi^2 + 2000)\delta^2$$
$$+ 4096\xi^8 - 16896\xi^6 + 22800\xi^4 - 10000\xi^2$$
$$+ (1024\xi^6 - 3072\xi^4 + 480\xi^2 + 2000)\gamma^2 - 432\gamma^4,$$
$$d_4 = 4\gamma^2 - 4 < 0. \quad (37)$$

It follows that the number of variation of signs $[1, -1, -1, -1, -1, -1]$ of the discriminant sequence is 1 if the damping ratio $\xi = 0$. This means that the system is not delay-independent stable, since $F(w)$ has $4(= 6 - 2\times 1)$ real roots. Thus, it is also the damping that makes the delay-independent stability possible.

For the damped systems, say, $\xi = 0{\cdot}02$ taken from reference [6], one has

$$b_1 = 2{\cdot}0016, \qquad b_2 = -6{\cdot}9936 - 4\delta^2, \qquad b_3 = 4(1-\gamma^2) \in (0, 4]$$

and

$$\begin{aligned} d_0 &= -2{\cdot}0016 < 0, \qquad d_1 = 2{\cdot}4987\times 10 + 12\delta^2 > 0, \\ d_2 &= -64\delta^4 - 2{\cdot}3982\times 10^2\delta^2 - 2{\cdot}4019\times 10\gamma^2 - 1{\cdot}9964\times 10^2 < 0, \\ d_3 &= 256\delta^6 + 1{\cdot}4069\times 10^3\delta^4 + (5{\cdot}7646\times 10^2\gamma^2 + 1{\cdot}9954\times 10^3)\delta^2 \\ &\quad + 2{\cdot}0002\times 10^3\gamma^2 - 432\gamma^4 - 3{\cdot}9964, \\ d_4 &= -b_3 < 0. \end{aligned} \tag{38}$$

If $d_3 > 0$ (or $d_3 = 0$), the sign table of the discriminant sequence (34) is $[1, -1, -1, -1, -1]$ or $[1, -1, -1, -1, 0, 0]$), whose number of variation of signs is 1, the function $F(w)$ has two real roots $(6 - 2\times 1)$, and consequently, the system is not delay-independent stable. Thus, the system with $\alpha = 2$, $\xi = 0{\cdot}02$ is delay-independent stable if and only if the small gains of retarded state feedback control yield

(i) $d_3 < 0$ and

(ii) $-1 < \gamma$, $(2 + 2\times 0{\cdot}02)(2\delta + 2\times 2\times 0{\cdot}02 + 1) > 2(1+\gamma)$.

Since the line $1{\cdot}02(2\delta + 1{\cdot}08) = 1 + \gamma$ does not cross the region, determined by $d_3 - 0$, in the (γ, δ) plane, the delay-independent stable region is governed by $d_3 < 0$ only as shown in Figure 4.

Figure 5 shows the delay-independent stable region of the active tendon structure in the (γ, δ) plane for $\alpha = 2$, and $\xi = 0{\cdot}5$.

4.3. STABILITY OF AN ACTIVE SUSPENSION OF QUARTER CAR MODEL

Finally, consider the delay-independent stability of a four-dimensional system described by

$$\begin{aligned} m_b x'' + \bar{c}_s(x' - y') + \bar{k}_s(x - y) + u &= 0 \\ m_t y'' - \bar{c}_s(x' - y') - \bar{k}_s(x - y) - u + \bar{k}_t(y - f) &= 0 \end{aligned} \tag{39}$$

This is the linearized model of an active suspension for a quarter car model. Here, $(\ )' \equiv \mathrm{d}/\mathrm{d}\bar{t}(\ )$, x is the vertical displacement of the vehicle body m_b, y is the vertical

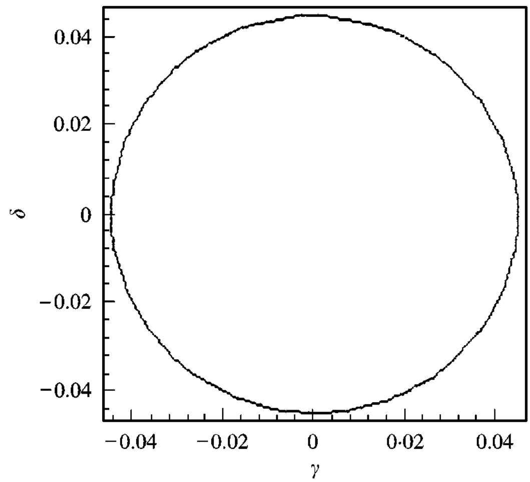

Figure 4. The delay-independent stable region of active-tendon structure (γ, δ) plane when $\alpha = 2$, $\xi = 0{\cdot}02$.

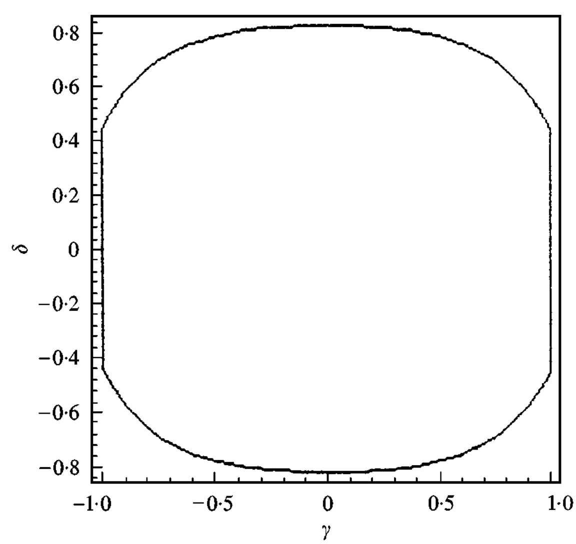

Figure 5. The delay-independent stable region of active tendon structure in (γ, δ) plane when $\alpha = 2$, $\xi = 0{\cdot}5$.

displacement of the unsprung mass m_t, f is the road disturbance, $\bar{c}_s \geqslant 0$, $\bar{k}_s \geqslant 0$ and $\bar{k}_t \geqslant 0$ are the damping coefficient, the stiffness of spring, and the stiffness of tire, respectively. To reduce the vibration of the vehicle body, an active control force u is introduced in the form of linear state feedback of the vehicle body with a time delay caused mainly by the hydraulic actuator

$$u = \bar{g}_1 x(\bar{t} - \bar{\tau}) + \bar{g}_2 x'(\bar{t} - \bar{\tau}). \tag{40}$$

This control is an extension of the so-called "sky-hook" damper. In the stability analysis of the steady state motion of a linear system it is not necessary to take the external excitation, i.e., the road disturbance here, into account. Let $w_s = \sqrt{\bar{k}_s/m_b}$, $\beta = m_b/m_t$, $\tau = \bar{\tau}w_s$, $k_t = \bar{k}_t/\bar{k}_s$, $c_s = \bar{c}_s/\sqrt{m_b\bar{k}_s}$, $g_1 = \bar{g}_1//\bar{k}_s$, $g_2 = \bar{g}_2/\sqrt{m_b\bar{k}_s}$, and $t = w_s\bar{t}$, one can cast equation (39) without f into

$$\ddot{x} + c_s(\dot{x} - \dot{y}) + (x - y) + g_1 x(t - \tau) + g_2 \dot{x}(t - \tau) = 0$$

$$\ddot{y} - c_s\beta(\dot{x} - \dot{y}) - \beta(x - y) + k_t\beta y - g_1\beta x(t - \tau) - g_2\beta\dot{x}(t - \tau) = 0 \quad (41)$$

The characteristic equation of equation (41) reads

$$\lambda^4 + c_s(1 + \beta)\lambda^3 + (1 + \beta + k_t\beta)\lambda^2 + c_s k_t\beta\lambda + k_t\beta + (g_2\lambda + g_1) \times (\lambda^2 + k_t\beta)\exp(-\lambda\tau) = 0. \quad (42)$$

4.3.1. *General results*

If $\tau = 0$, the characteristic equation is

$$\lambda^4 + (c_s(1 + \beta) + g_2)\lambda^3 + (1 + \beta + k_t\beta + g_1)\lambda^2 + k_t\beta(c_s + g_2)\lambda + k_t\beta(1 + g_1) = 0. \quad (43)$$

By using the Routh–Hurwitz criterion, one can readily know that the system with $\tau = 0$ is stable if and only if

$$g_1 + 1 > 0, \qquad g_2 + c_s > 0,$$

$$g_1 g_2 + (1 + \beta)g_2 + c_s(1 + \beta)g_1 + \beta^2 c_s(1 + k_t) + 2\beta c_s + c_s > 0,$$

$$g_2^2 + c_s(\beta + \beta k_t)g_2 - c_s g_1 g_2 - c_s^2(1 + \beta)g_1 + \beta c_s^2 k_t + g_2 c_s > 0. \quad (44)$$

When $\tau > 0$, the function $F(w)$ in equation (18) is in the form

$$F(w) = w^8 + b_1 w^6 + b_2 w^4 + b_3 w^2 + b_4, \quad (45)$$

where

$$b_1 = -2 - 2\beta - 2k_t\beta + c_s^2 + 2c_s^2\beta + c_s^2\beta^2 - g_2^2,$$

$$b_2 = 1 + 4k_t\beta + 2k_t\beta^2 + k_t^2\beta^2 + 2g_2^2 k_t\beta - 2c_s^2 k_t\beta^2 - g_1^2 - 2c_s^2 k_t\beta + 2\beta + \beta^2,$$

$$b_3 = -k_t\beta(-c_s^2 k_t\beta + 2k_t\beta + 2\beta + g_2^2 k_t\beta - 2g_1^2 + 2),$$

$$b_4 = (1 - g_1^2)k_t^2\beta^2. \quad (46)$$

To guarantee the delay-independent stability of the system, b_4 must be positive, namely, $|g_1| < 1$. By using *discr*, one obtains the discriminant sequence

$$1, \quad d_0, \quad d_0 d_1, \quad d_1 d_2, \quad d_2 d_3, \quad d_3 d_4, \quad d_4 d_5, \quad d_5^2 d_6, \tag{47}$$

where

$$\begin{aligned} d_0 &= -b_1, \qquad d_1 = -8b_2 + 3b_1^2, \qquad d_2 = b_1^2 b_2 + 3b_1 b_3 - 4b_2^2, \\ d_3 &= -3b_1^3 b_3 + b_1^2 b_2^2 - 6b_1^2 b_4 + 14 b_1 b_2 b_3 - 4b_2^3 + 16 b_2 b_4 - 18 b_3^2, \\ d_4 &= -b_1^2 b_2^2 b_3 - 18 b_1 b_2 b_3^2 + 7b_1^2 b_3 b_4 + 12 b_1 b_2^2 b_4 - 48 b_2 b_3 b_4 + 4b_2^3 b_3 \\ &\quad + 16 b_1 b_4^2 + 27 b_3^3 + 4b_1^3 b_3^2 - 3b_1^3 b_2 b_4, \\ d_5 &= -27 b_1^4 b_4^2 + 18 b_1^3 b_2 b_3 b_4 - 4b_1^3 b_3^3 - 4b_1^2 b_2^3 b_4 + b_1^2 b_2^2 b_3^2 + 144 b_1^2 b_2 b_4^2 \\ &\quad - 6b_1^2 b_3^2 b_4 - 80 b_1 b_2^2 b_3 b_4 + 18 b_1 b_2 b_3^3 - 192 b_1 b_3 b_4^2 + 16 b_2^4 b_4 \\ &\quad - 4b_2^3 b_3^2 - 128 b_2^2 b_4^2 + 144 b_2 b_3^2 b_4 + 256 b_4^3 - 27 b_3^4, \\ d_6 &= b_4 > 0. \end{aligned} \tag{48}$$

One can easily list the possible cases, subject to $l = 2v$ for $v = 1, 2, 3, 4$, of the modified sign tables. The system has totally 41 possible cases (which is omitted for saving space of sign tables of the discriminant sequence that make the system delay-independent stable!

Equation (39) or (41) is delay-independent stable if and only if (i) the conditions in equation (44) hold and, (ii) one of the 41 cases of the *modified* sign tables to be hold true.

For $m_b = 290$, $m_t = 59$, $\bar{k}_s = 16812$, $\bar{k}_t = 19000$, and $\bar{c}_s = 100$, one has $\beta = 4{\cdot}9153$, $k_t = 1{\cdot}1301$, $c_s = 0{\cdot}0453$. It is easy to find that $d_0 > 0$, and $d_1 > 0$ always hold true. Thus, in order that the system is delay-independent stable, at least one of the values of d_2, and d_3 should be negative. Otherwise, the sign table of the discriminant sequence cannot change its signs twice. It is also easy to get $|g_2| < 0{\cdot}4943$. If this is the case one has $d_2 > 0$. Therefore, the system is delay-independent stable only for $d_4 < 0$ and $d_5 > 0$. Drawing the graphs of $d_3 = 0$, $d_4 = 0$, and $d_5 = 0$, one can easily get the delay-independent stable region shown in Figure 6.

4.3.2. *Example*: *a vehicle equipped with sky-hook damper*

Now, consider the delay-independent stability of the vehicle model equipped with the so-called sky-hook damper. When the sky-hook damper is introduced, the delayed state feedback control force is in the form $u = g_2 \dot{x}(t - \tau)$. The system parameters are given in Table 4.

From Table 4, one obtains the dimensionless parameters of the vehicle

$$\beta = 4{\cdot}9153, \qquad k_t = 1{\cdot}1301, \qquad c_s = 0 \sim 0{\cdot}4438. \tag{49}$$

The corresponding expressions, in terms of g_2 and c_s, of d_i, $(i = 0, 1, 2, \ldots, 6)$ are listed in Appendix B. It is easy to see that if $c_s = 0$, then $d_i > 0$, $(i = 0, 1, 2, 3, 5, 6)$ for all given g_2. Hence, the system cannot be delay-independent stable. This

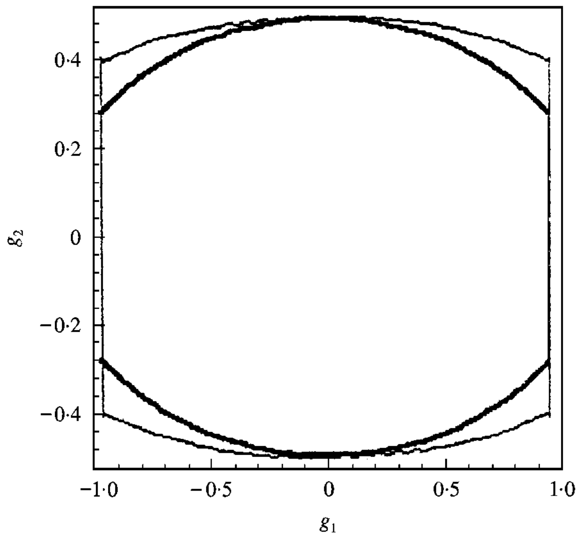

Figure 6. The delay-independent stable region is the region bounded by the two thick curves and $|g_1| = 1$, when $m_b = 290$, $m_t = 59$, $\bar{k}_s = 16\,812$, $k_t = 19\,000$, and $\bar{c}_s = 100$.

TABLE 4
The system parameters of a vehicle

m_b	m_t	$\bar{c}_s$	$\bar{k}_s$	$\bar{k}_t$
290 (kg)	59 (kg)	0 ~ 980 (N s/m)	16 812 (N/m)	19 000 (N/m)

demonstrates that it is, here again, the damping that makes delay-independent stability possible when the sky-hook damper is introduced.

The conditions in equation (44) can be simplified to

$$6{\cdot}2294 \times 10 c_s + 5{\cdot}9153 g_2 > 0, \tag{50a}$$

$$g_2^2 + 1{\cdot}1470 \times 10 g_2 c_s + 5{\cdot}5548 c_s^2 > 0. \tag{50b}$$

By solving the above inequalities, one gets

$$g_2 > -\,1{\cdot}0531 \times 10 c_s \tag{50c}$$

and

$$g_2 < -\,1{\cdot}0963 \times 10 c_s \quad \text{or} \quad g_2 > -\,0{\cdot}5067 c_s. \tag{50d}$$

Equation (50c) and the first inequality in equation (50d) cannot hold true simultaneously, thus, to guarantee the delay-independent stability, one must have

$$g_2 > -\,0{\cdot}5067 c_s. \tag{51}$$

As c_s varies from 0 to 0·4438, one always has $d_o > 0$ and $d_\varepsilon > 0$. Thus, in order that the system is delay-independent stable, at least one of the values of d_1, d_2, and d_3 should be negative. By drawing the graphs of d_i's ($i = 1, 2, 3, 4, 5$) and checking the

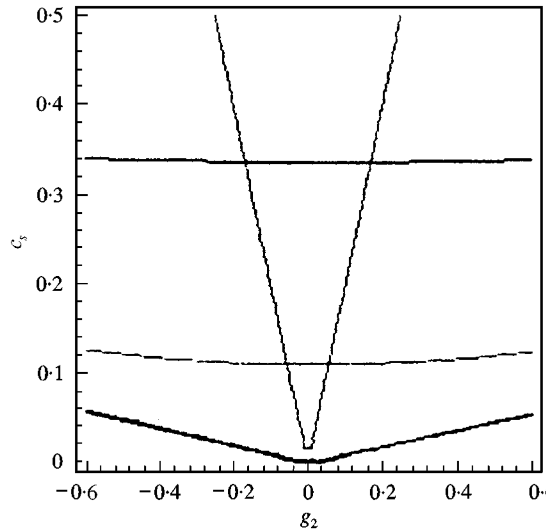

Figure 7. The delay-independent stable region in $\{(g_2, c_s):|g_2| < 0{\cdot}6,\ 0 < c_s < 0{\cdot}5\}$ of vehicle with "sky-hook" damper. Only the points in the narrow V-shape region make the system delay-independent stable.

sign tables of the discrimination sequence, we see that only the points in the narrow V-shape region, shown in Figure 7, make the system delay-independent stable. The boundary of the region is given by $d_5 = 0$.

To see this, three cases are checked as follows:

Case 1: $\bar{c}_s = 50$, namely $c_s = 0{\cdot}0226$.
The corresponding expressions, in terms of g_2, of d_i, $(i = 0, 1, 2, \ldots, 6)$, read

$$
\begin{aligned}
d_0 &= 2{\cdot}2923 \times 10 + g_2^2,\\
d_1 &= 4{\cdot}3518 \times 10^2 + 4{\cdot}8656 \times 10 g_2^2 + 3 g_2^4,\\
d_2 &= 2{\cdot}3250 \times 10^3 + 2{\cdot}2033 \times 10^3 g_2^2 + 2{\cdot}5083 \times 10^2 g_2^4 + 1{\cdot}1110 \times 10 g_2^6,\\
d_3 &= -\,8{\cdot}6414 \times 10^3 + 1{\cdot}4475 \times 10^5 g_2^2 + 2{\cdot}1153 \times 10^4 g_2^4 + 1{\cdot}3942 \times 10^3 g_2^6,\\
&\quad + 3{\cdot}0858 \times 10 g_2^8,\\
d_4 &= -\,1{\cdot}0524 \times 10^6 + 1{\cdot}6809 \times 10^7 g_2^2 + 5{\cdot}1766 \times 10^6 g_2^4\\
&\quad + 4{\cdot}0799 \times 10^5 g_2^6 + 1{\cdot}6955 \times 10^4 g_2^8 + 2 \times 10^{-6} g_2^{10},\\
d_5 &= 6{\cdot}431 \times 10^3 - 4{\cdot}8902 \times 10^7 g_2^2 + 7{\cdot}8158 \times 10^8 g_2^4\\
&\quad + 1{\cdot}8088 \times 10^6 g_2^6 + 1{\cdot}3609 \times 10^7 g_2^8 + 5{\cdot}1261 \times 10^5 g_2^{10},\\
d_6 &= 3{\cdot}0858 \times 10.
\end{aligned}
\tag{52}
$$

Since the first four signs of the discriminant sequence are 1, 1, 1, and 1, the last four signs must be -1, 1, -1, and 1 to ensure the delay-independent stability. Hence, in order that the system is delay-independent stable, one must have $d_3 < 0$, $d_4 < 0$, $d_5 > 0$. They are certainly true for small $|g_2|$. If $|g_2| > \sqrt{8{\cdot}6414/1{\cdot}4475 \times 10^2} = 0{\cdot}5970$, then $d_3 > 0$. It follows that the system is not delay-independent stable.

Case 2: $\bar{c}_s = 500$, namely $c_s = 0{\cdot}2264$.

The corresponding expressions, in terms of g_2, of d_i, $(i = 0, 1, 2, \ldots, 6)$ are in the forms

$$\begin{aligned} d_0 &= 2{\cdot}1146 \times 10 + g_2^2 > 0, \quad d_1 = 2{\cdot}2704 \times 10^2 + p_1(g_2^2) > 0, \\ d_2 &= -\,7{\cdot}3482 \times 10^3 + p_2(g_2^2), \\ d_5 &= -\,8{\cdot}1478 \times 10^5 + 2{\cdot}5649 \times 10^4 g_2^2 + 1{\cdot}4962 \times 10^4 g_2^4 \\ &\quad + 1{\cdot}3796 \times 10^3 g_2^6 + 3{\cdot}0857 \times 10 g_2^8, \\ d_4 &= -\,9{\cdot}8379 \times 10^7 + p_4(g_2^2), \qquad d_5 = 6{\cdot}0000 \times 10^7 + p_5(g_2^2), \\ d_6 &= 3{\cdot}0858 \times 10, \end{aligned} \tag{53}$$

where $p_i(x)$ are polynomials with $p_i(0) = 0$. If the control gain $|g_2|$ is small then the system is delay-independent stable since the number of variation of signs of the discriminant sequence $[1, 1, 1, -1, 1, -1, 1]$ is 4.

Case 3: $\bar{c}_s = 980$, namely $c_s = 0{\cdot}4438$.

In this case, we have the following expressions:

$$\begin{aligned} d_0 &= 1{\cdot}6048 \times 10 + g_2^2 > 0, \qquad d_1 = -\,2{\cdot}6524 \times 10^2 + q_1(g_2^2), \\ d_2 &= -\,2{\cdot}8067 \times 10^4 - 2{\cdot}6557 \times 10^3 g_2^2 + q_2(g_2^2), \\ d_3 &= -\,2{\cdot}6154 \times 10^6 - 3{\cdot}3765 \times 10^5 g_2^2 - 5{\cdot}4472 \times 10^3 g_2^4 \\ &\quad + 1{\cdot}3377 \times 10^3 g_2^6 + 3{\cdot}0858 \times 10 g_2^8, \\ d_4 &= -\,3{\cdot}0774 \times 10^8 - 1.0070 \times 10^8 g_2^2 \\ &\quad - 6{\cdot}9429 \times 10^6 g_2^4 + q_4(g_2^2), \\ d_5 &= 7{\cdot}2028 \times 10^8 + q_5(g_2^2), \\ d_6 &= 3.0858 \times 10, \end{aligned} \tag{54}$$

where $q_i(x)$ are polynomials with $q_i(0) = 0$. If $|g_2|$ is small, then the system is also delay-independent stable since the number of variation of sign table of the discriminant sequence $[1, 1, -1, 1, 1, 1, -1, 1]$ is 4.

4.4. COMMENTS ON RETARDED STATE FEEDBACK CONTROL

The above analysis shows that it is the damping that makes the delay-independent stability of these systems possible. When a system is damped, and retarded feedback control is performed, the delay-independent stable regions in parameter space of concern are bounded.

5. CONCLUDING REMARKS

The paper presents a systematic approach to the delay-independent stability analysis of linear Md.o.f. dynamic systems with two time delays. The approach makes the stability analysis much simpler than before. To complete the stability test, one needs to complete some algebraic computations only.

The approach is applicable to the theoretical analysis for the delay-independent stability of Md.o.f. dynamic systems with two time delays. It gives sufficient and necessary conditions for delay-independent stability. Once the discriminant sequence of a system is determined through the use of MAPLE routine *discr* given in section 2.2, the sufficient and necessary conditions for delay-independent stability can be written by hand. It is only required to find out all the possible modified sign tables. In general, the number of modified sign tables may become very large when the system dimension increases, but every sign table can be constructed by following a very simple rule, and all the cases can be treated in a unified way.

In addition, the number of modified sign tables may be very small if only a few uncertain parameters are involved in the system, which enables the sufficient and necessary conditions be in terms of a few inequalities. By drawing the graphs of d_i's, one can easily obtain the delay-independent stable region in the parameter space of concern.

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REFERENCES

1. G. Stepan 1989 *Retarded Dynamical Systems: Stability and Characteristic Functions*, Essex: Longman Scientific and Technical.
2. Y. X. Qin, Y. Q. Liu, L. Wang and Z. X. Zhen 1989 *Stability of Dynamic Systems with Time Delays*, Beijing: The Science Press.
3. S. Arusawatwung 1996 *International Journal of Control* **65**, 347–367. Stability of retarded delay differential systems.
4. H. Y. Hu and Z. H. Wang 1998 *Journal of Sound and Vibration* **214**, 213–225. Stability analysis of damped Sd.o.f. systems with two time delays in state feedback.
5. L. Yang, J. Z. Zhang and X. R. Hou 1996 *Nonlinear Algebraic Equation System and Automated Theorem Proving*. Shanghai: Shanghai Scientific and Technological Education Publishing House.
6. L. Zhang, C. Y. Yang, M. J. Chajes and A. H.- D. Cheng 1993 *Journal of Engineering Mechanics* **119**, 1017–1024. Stability of active-tendon structural control with time delay.
7. L. Palkovics and P. J. Th. Venhovens 1992 *Vehicle System Dynamics* **21**, 269–296. Investigation on stability and possible chaotic motions in the controlled wheel suspension system.
8. A. Heck 1996 *Introduction to Maple*, New York: Springer.
9. L. Yang, X. R. Hou and Z. B. Zeng 1996, Science in China (Series E) **26**(10), 424–441. A complete discrimination system for polynomial.

APPENDIX A

The possible sign tables of the discriminant sequence of real polynomial of orders 6 and 8 and positive leading coefficient (sections 2.3 and 4.2) are shown in Tables 2 and 3.

TABLE 2

Sign tables of the discriminant sequence of real polynomial of order 6 and with positive leading coefficient (section 2.3 and 4.2)

$D_1(f)$	$D_2(f)$	$D_3(f)$	$D_4(f)$	$D_5(f)$	$D_6(f)$	$l = 2v$
1	−1	0	0	0	0	$v = 1$
1	−1	1	1	0	0	
1	−1	−1	1	0	0	$v = 2$
1	1	−1	1	0	0	
1	−1	1	1	1	−1	
1	−1	−1	−1	1	−1	
1	1	1	−1	1	−1	
1	−1	1	1	−1	−1	
1	−1	−1	1	1	−1	$v = 3$
1	1	−1	1	1	−1	
1	1	−1	−1	1	−1	
1	−1	1	1	−1	−1	
1	−1	1	−1	−1	−1	
1	1	−1	1	−1	−1	

TABLE 3

Sign tables of the discriminant sequence of a real polynomial of order 8 (section 4.1)

$D_1(f)$	$D_2(f)$	$D_3(f)$	$D_4(f)$	$D_5(f)$	$D_6(f)$	$D_7(f)$	$D_8(f)$	$l = 2v$
1	−1	−1	1	0	0	0	0	$v = 2$
1	−1	−1	1	−1	−1	0	0	$v = 3$
1	−1	−1	1	1	−1	0	0	
1	−1	−1	1	−1	−1	−1	1	
1	−1	−1	−1	1	−1	−1	1	
1	−1	−1	−1	−1	1	−1	1	
1	−1	−1	1	1	−1	−1	1	
1	−1	−1	−1	1	1	−1	1	
1	−1	−1	1	−1	1	1	1	$v = 4$
1	−1	−1	1	1	1	−1	1	
1	−1	−1	1	−1	−1	1	1	
1	−1	−1	−1	1	−1	1	1	
1	−1	−1	1	1	−1	1	1	

APPENDIX B

The expressions of d_i ($i = 0, 1, 2, \ldots, 6$) of the discriminant sequence of vehicle with sky-hook damper (Section 4.3):

$$d_0 = g_2^2 + 2{\cdot}2940 \times 10 - 3{\cdot}4990 \times 10 c_s^2,$$

$$d_1 = 3g_2^4 + (4{\cdot}8763 \times 10 - 2{\cdot}0994 \times 10^2 c_s^2) g_2^2 + 4{\cdot}3738 \times 10^2 - 4{\cdot}2904 \times 10^3 c_s^2 + 3{\cdot}6729 \times 10^3 c_s^4,$$

$$d_2 = 1{\cdot}1110 \times 10 g_2^6 + (2{\cdot}5126 \times 10^2 - 8{\cdot}4319 \times 10^2 c_s^2) g_2^4 + (2{\cdot}2179 \times 10^3 - 2{\cdot}8326 \times 10^4 c_s^2 + 1{\cdot}8201 \times 10^4 c_s^4) g_2^2 + 2{\cdot}4296 \times 10^3 - 2{\cdot}0412 \times 10^5 c_s^2 + 2{\cdot}6615 \times 10^5 c_s^4 - 8{\cdot}0459 \times 10^4 c_s^6,$$

$$d_3 = 3{\cdot}0858 \times 10 g_2^8 + (1{\cdot}3944 \times 10^3 - 2{\cdot}8796 \times 10^2 c_s^2) g_2^6 + (2{\cdot}1214 \times 10^4 - 1{\cdot}1720 \times 10^5 c_s^2 - 9{\cdot}2107 \times 10^4 c_s^4) g_2^4 + (1{\cdot}4592 \times 10^5 - 2{\cdot}2847 \times 10^6 c_s^2 - 1{\cdot}3000 \times 10^6 c_s^4 + 2{\cdot}2157 \times 10^6 c_s^6) g_2^2 + 2{\cdot}0 \times 10^{-2} - 1{\cdot}6862 \times 10^7 c_s^2 + 1{\cdot}9247 \times 10^7 c_s^4 - 5{\cdot}5733 \times 10^6 c_s^6 + 1{\cdot}3219 \times 10^6 c_s^8,$$

$$d_4 = 2{\cdot}0 \times 10^{-6} g_2^{10} + (1{\cdot}6908 \times 10^4 + 9{\cdot}2018 \times 10^4 c_s^2) g_2^8 + (4{\cdot}0803 \times 10^5 - 7{\cdot}5475 \times 10^4 c_s^2 - 6{\cdot}5315 \times 10^6 c_s^4) g_2^6 + (5{\cdot}1954 \times 10^6 - 3{\cdot}6548 \times 10^7 c_s^2 - 1{\cdot}5074 \times 10^8 c_s^4 + 1{\cdot}1910 \times 10^8 c_s^6) g_2^4 + (1{\cdot}7118 \times 10^7 - 6{\cdot}0289 \times 10^8 c_s^2 - 8{\cdot}8559 \times 10^7 c_s^4 + 5{\cdot}9568 \times 10^8 c_s^6 - 1{\cdot}1266 \times 10^8 c_s^8) g_2^2 - 7{\cdot}0 \times 10^{-1} - 2{\cdot}0537 \times 10^9 c_s^2 + 2{\cdot}6868 \times 10^9 c_s^4 - 1{\cdot}0113 \times 10^9 c_s^6 + 1{\cdot}9267 \times 10^8 c_s^8,$$

$$d_5 = (5{\cdot}1116 \times 10^5 + 2{\cdot}8394 \times 10^6 c_s^2) g_2^{10} + (1{\cdot}3606 \times 10^7 + 5{\cdot}9698 \times 10^6 c_s^2 - 2{\cdot}0438 \times 10^8 c_s^4) g_2^8 + (1{\cdot}8146 \times 10^8 - 1{\cdot}1296 \times 10^9 c_s^2 - 5{\cdot}2176 \times 10^9 c_s^4 + 3{\cdot}8766 \times 10^9 c_s^6) g_2^6 + (7{\cdot}9224 \times 10^8 - 2{\cdot}0776 \times 10^{10} c_s^2 - 1{\cdot}3361 \times 10^{10} c_s^4 + 3{\cdot}2239 \times 10^{10} c_s^6 - 7{\cdot}1515 \times 10^9 c_s^8) g_2^4 + (2{\cdot}0 \times 10^2 - 9{\cdot}5430 \times 10^{10} c_s^2 + 1{\cdot}1869 \times 10^{11} c_s^4 - 3{\cdot}4610 \times 10^{10} c_s^6 - 5{\cdot}4125 \times 10^8 c_s^8 + 3{\cdot}4764 \times 10^9 c_s^{10}) g_2^2 + 13 - 2{\cdot}56 \times 10^2 c_s^2 + 2{\cdot}4447 \times 10^{10} c_s^4 - 3{\cdot}2411 \times 10^{10} c_s^6 + 1{\cdot}3533 \times 10^{10} c_s^8 - 3{\cdot}2646 \times 10^9 c_s^{10},$$

$$d_6 = 3{\cdot}0858 \times 10.$$