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VIBRATION OF THICK CYLINDRICAL SHELLS ON THE BASIS OF THREE-DIMENSIONAL THEORY OF ELASTICITY

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In this paper, an approximate analysis using a layerwise approach is presented to study the vibration of thick circular cylindrical shells on the basis of three-dimensional theory of elasticity. The boundary conditions studied are simply supported–simply supported and clamped–clamped boundary conditions. In this approach, the thick cylindrical shell is discretized into an arbitrary number of thin cylindrical layers in the thickness direction and each layer assumes a three-dimensional stress state. The approach is similar in concept to the finite-strip method. The displacements for each layer are approximated by trigonometric functions in the axial and circumferential directions and by some linear-shape functions in the thickness direction. The governing equation is obtained using an energy minimization principle. Extensive frequency parameters, never seen in the literature before, have been presented for a wide range of thickness-to-radius ratios and thickness-to-length ratios. In addition, the frequency characteristics of thin and thick cylindrical shells are also studied and the displacement fields in the thickness direction for various thickness-to-radius ratios are also presented. The analysis has been verified by comparing results with those in the literature and excellent agreement is obtained.

1. INTRODUCTION

In the literature, many of the shells studied are based on classical shell theories, such as Donnell's [1], Flügge's [2], Love's [3] and Sanders' [4] shell theories, which are based on the four simplifying assumptions of Kirchhoff-Love's hypothesis, see reference [3]. In classical or thin-shell theories, the transverse stress and strain components are ignored. This omission makes the thin-shell theories highly inadequate for the analysis of even slightly thick shells. In recent years, the refinement of thin-shell theories has resulted in a number of the so-called higher order shell theories, see for example, Bhimaraddi [5] and Reddy [6]. The higher order shell theories are better than the thin-shell theories for the analysis of slightly thick shells but are still inadequate for the analysis of moderately thick shells. To analyze moderately thick shells, the transverse normal stress and strain components, which are ignored in the higher ordered shell theories, have to be

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accounted for and only an analysis based on the three-dimensional theory of elasticity would account for all the transverse stress and strain components.

In the literature, the study of shells using three-dimensional theory of elasticity is relatively scarce in comparison to the study of shells using other shell theories because of the complexity involved in considering all six stress and strain components. Studies on shells based on three-dimensional theory of elasticity have been presented by Pochhammer [7] and Chree [8], Greenspon [9], Gazis [10], Herrmann and Mirsky [11], Nelson *et al.* [12], and Armenakas *et al.* [13] for infinitely long cylindrical shells. For finite-length cylindrical shells, Gladwell and Vijay [14], Hutchinson and El-Azhari [15], Singal and Williams [16], Cheung and Wu [17], and Soldatos, Hadjigeorgiou [18], Leissa [19], Jiang [20], and Ye and Soldatos [21] have also presented studies using three-dimensional theory of elasticity.

An exact three-dimensional elasticity analysis of a cylindrical shell is a challenging problem. The difficulty lies in finding a set of displacement fields that would satisfy the specified boundary conditions and the stress-free surface conditions. In the above studies, the three-dimensional elasticity of elasticity solutions are based on the Rayleigh–Ritz method, see for example, Gladwell and Vijay [14], Hutchinson and El-Azhari [15], Singal and Williams [16] and So and Leissa [19], the perturbation method by Jiang [20], and a recursive solution method by Ye and Soldatos [21].

In this paper, an approximate analysis similar to the finite-strip method, see Cheung [22] and Cheung and Tham [23], is employed to study the vibration of thick cylindrical shells on the basis of three-dimensional theory of elasticity. In this approach, the cylindrical shell is discretized into an arbitrary number of thin layers in the thickness direction, and the discretization process is similar to the finite-element method. The displacements in the axial and circumferential directions for each layer are approximated by trigonometric functions that satisfy at least the geometric boundary conditions and the periodicity of harmonic motion respectively. In the thickness direction, the displacements are approximated by some linear-shape functions that are expressed in terms of some generalized co-ordinates. The displacements across the thickness are assumed to be continuous. Using an energy minimization principle, the characteristic eigenvalue equation can be obtained and solved for the natural frequencies and eigenvectors, which are the generalized co-ordinates.

In the literature, many of the studies for finite length thick cylindrical shells using the three-dimensional theory of elasticity are for free–free boundary conditions, see for example, Gladwell and Vijay [14], Hutchinson and El-Azhari [15], Singal and Williams [16] and So and Leissa [19]. In the present work, studies are carried out for simply supported and clamped boundary conditions and extensive frequency parameters are presented which are never found in the literature. Cheung and Wu [17], Soldatos and Hadjigeorgiou [18], Jiang [20] and Ye and Soldatos [21] have presented some studies for either simply supported or clamped boundary conditions. However, in these papers, very few results were presented. The extensive frequency parameters presented would be useful for researchers who can use them to validate against their results. The present analysis has been validated against available published results and the agreement is found to be excellent.

2. FORMULATION

Consider a thick cylindrical shell, see Figure 1(a) having a uniform thickness H, length L and a radius R. In the present approach, the cylinder is discretized into an arbitrary number of thin annular cylindrical layers, nl, see Figure 1(b), and each layer has a uniform thickness h. For the jth layer, the displacements in the x, θ and z directions, defined with respect to a middle surface co-ordinate system (x, θ, z), are denoted by u_x^j, u_θ^j and u_z^j, respectively.

For the jth layer, the displacements u_x^j, u_θ^j and u_z^j are written as

$$u_x^j = \chi_x^j(z,t)\phi_x(x)\cos(n\theta),$$
$$u_\theta^j = \chi_\theta^j(z,t)\phi_\theta(x)\sin(n\theta),$$
$$u_z^j = \chi_z^j(z,t)\phi_z(x)\cos(n\theta), \qquad (1)$$

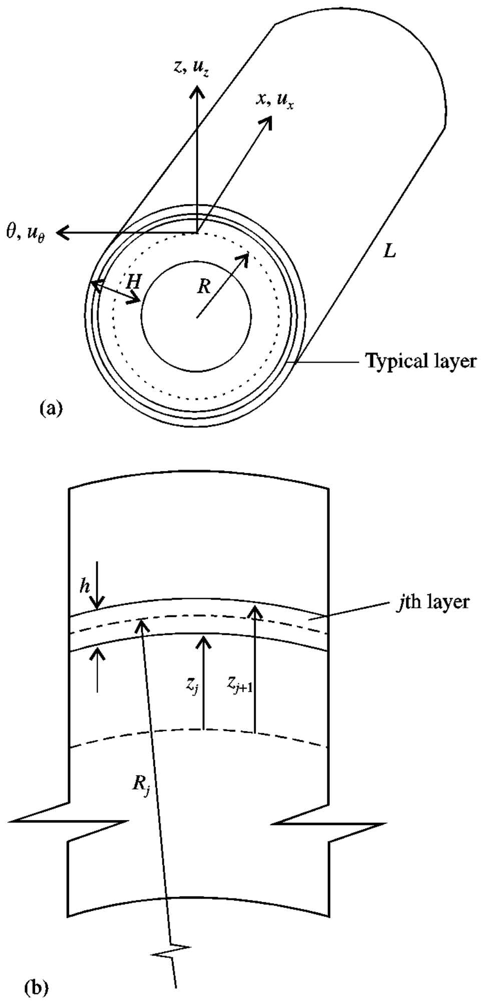

Figure 1. (a) Geometry of a thick cylindrical shell; (b) Geometry of a cylindrical layer.

where

$$\chi_i^j(z,t) = \alpha u_i^j + \beta u_i^{j+1} \tag{2}$$

in which $\beta = (z - z_j)/h$ and $\alpha = 1 - \beta$. z_j is the distance from the middle surface of the cylinder to the lower surface of jth layer and u_i^j are the generalized co-ordinates. $\phi_i(x)$ are some functions which satisfy the geometric boundary conditions, and n is an integer denoting the circumferential wave number.

The displacement fields defined in equation (1) can be written in a matrix form as

$$\mathbf{u} = \mathbf{N} \cdot \mathbf{d}, \tag{3}$$

where

$$\mathbf{u}^{\mathrm{T}} = \{u_x^j \;\; u_\theta^j \;\; u_z^j\}, \tag{4}$$

$$\mathbf{d}^{\mathrm{T}} = \{u_x^j \;\; u_\theta^j \;\; u_z^j \;\; u_x^{j+1} \;\; u_\theta^{j+1} \;\; u_z^{j+1}\}, \tag{5}$$

$$\mathbf{N} = [\mathbf{N}_1 \;\; \mathbf{N}_2] \tag{6}$$

and

$$\mathbf{N}_1 = \begin{bmatrix} \alpha\phi_u \cos(n\theta) & 0 & 0 \\ 0 & \alpha\phi_\theta \sin(n\theta) & 0 \\ 0 & 0 & \alpha\phi_z \cos(n\theta) \end{bmatrix}, \tag{7}$$

$$\mathbf{N}_2 = \begin{bmatrix} \beta\phi_u \cos(n\theta) & 0 & 0 \\ 0 & \beta\phi_\theta \sin(n\theta) & 0 \\ 0 & 0 & \beta\phi_z \cos(n\theta) \end{bmatrix}. \tag{8}$$

For the jth layer, the three-dimensional stress-strain relation is given by Hooke's law as

$$\sigma = [C]\varepsilon, \tag{9}$$

where

$$\sigma^{\mathrm{T}} = \{\sigma_x \;\; \sigma_\theta \;\; \sigma_z \;\; \tau_{\theta z} \;\; \tau_{xz} \;\; \tau_{x\theta}\}, \tag{10}$$

$$\varepsilon^{\mathrm{T}} = \{\varepsilon_x \;\; \varepsilon_\theta \;\; \varepsilon_z \;\; \gamma_{\theta z} \;\; \gamma_{xz} \;\; \gamma_{x\theta}\} \tag{11}$$

and the elastic coefficients matrix $[C]$ for isotropic materials is defined as

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}, \tag{12}$$

where the coefficients C_{ij} are defined as

$$\begin{aligned} C_{11} &= C_{22} = C_{33} = \lambda + 2G, \\ C_{12} &= C_{13} = C_{23} = \lambda, \\ C_{44} &= C_{55} = C_{66} = G, \\ \lambda &= \frac{vE}{(1+v)(1-2v)}, \\ G &= \frac{E}{2(1+v)}. \end{aligned} \tag{13}$$

The three-dimensional elasticity strain–displacement relations can be written in a matrix form as

$$\varepsilon = D \cdot \mathbf{u}, \tag{14}$$

where the differential operator D is defined as

$$D = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \dfrac{1}{R_j}\dfrac{\partial}{\partial \theta} & \dfrac{1}{R_j} \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{\partial}{\partial z} - \dfrac{1}{R_j} & \dfrac{1}{R_j}\dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \\ \dfrac{1}{R_j}\dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial x} & 0 \end{bmatrix} \tag{15}$$

in which R_j is the radius of the jth layer. Equation (14) can be rewritten as

$$\varepsilon = \mathbf{B} \cdot \mathbf{d}, \tag{16}$$

where

$$B = D \cdot N. \tag{17}$$

In the present study, the cylinders are subjected to simply supported and clamped boundary conditions. For simply supported boundary conditions, the conditions at the ends, $x = 0$, L, are defined as

$$u_\theta = u_z = \sigma_x = 0 \tag{18}$$

and for the clamped boundary conditions, the conditions at the ends, $x = 0$, L, are

$$u_x = u_\theta = u_z = \partial u_z / \partial x = 0. \tag{19}$$

In the present analysis, through the use of orthogonal beam functions for ϕ_i, the above boundary conditions are fairly well satisfied.

The kinetics T_j and potential V_j energies for the jth layer is obtained by integrating the respective energy density over the layer's volume as follows:

$$T_j = \frac{1}{2} \iiint_{vol} \rho_j((\dot{u}_x^j)^2 + (\dot{u}_\theta^j)^2 + (\dot{u}_z^j)^2) R_j \, \mathrm{d}\theta \, \mathrm{d}x \, \mathrm{d}z, \tag{20}$$

$$V_j = \frac{1}{2} \iiint_{vol} \sigma^{\mathrm{T}} \cdot \varepsilon R_j \, \mathrm{d}\theta \, \mathrm{d}x \, \mathrm{d}z. \tag{21}$$

After substituting equation (3) into equation (20) and substituting equations (9) and (16) into equation (21), the kinetic T_j and potential V_j energies for the jth layer can be written in a matrix form as

$$T_j = \tfrac{1}{2} \dot{\mathbf{d}}^{\mathrm{T}} \cdot M_j \cdot \dot{\mathbf{d}}, \tag{22}$$

$$V_j = \tfrac{1}{2} \mathbf{d}^{\mathrm{T}} \cdot K_j \cdot \mathbf{d}, \tag{23}$$

where M_j and K_j are the mass and stiffness matrices of the jth layer defined as

$$M_j = \iiint_{vol} (\rho_j \, N^{\mathrm{T}} \, N) R_j \, \mathrm{d}\theta \, \mathrm{d}x \, \mathrm{d}z, \tag{24}$$

$$K_j = \iiint_{vol} (B^{\mathrm{T}} \cdot C \cdot B) R_j \, \mathrm{d}\theta \, \mathrm{d}x \, \mathrm{d}z. \tag{25}$$

For the cylinder, the kinetic T and potential V energies are obtained by summing the kinetic T_j and potential V_j energies of all layers. The kinetic T and potential V energies of the cylinder can be written as

$$T = \tfrac{1}{2} \dot{\delta}^{\mathrm{T}} \cdot M \cdot \dot{\delta}, \tag{26}$$

$$V = \tfrac{1}{2} \delta^{\mathrm{T}} \cdot K \cdot \delta, \tag{27}$$

where

$$\delta^{\mathrm{T}} = \{ u_x^1 \;\, u_\theta^1 \;\, u_z^1 \;\, u_x^2 \;\, u_\theta^2 \;\, u_z^2 \cdots u_x^{3(nl+1)} \;\, u_\theta^{3(nl+1)} \;\, u_z^{3(nl+1)} \} \tag{28}$$

in which δ denotes the set of generalized co-ordinates for the cylinder and nl denotes number of layers in the cylinders. M and K are, respectively, the mass and stiffness matrices. The size of these matrices is $3(nl + 1) \times (3(nl + 1)$. Applying an energy minimization principle to an energy function defined as $L = T - V$, one gets

$$K\delta + M\ddot{\delta} = 0. \tag{29}$$

Substituting $\delta = \delta_0 e^{i\omega t}$ into equation (29) will result in an eigenvalue equation of the form

$$\{K - \omega^2 M\}\delta_0 = \{0\}. \tag{30}$$

3. NUMERICAL RESULTS AND DISCUSSION

To validate the present approach, the results are compared with available published results in the literature for a wide range of thickness-to-radius ratios.

Table 1 shows the convergence of the frequency parameter $\Omega = (\omega H/\pi)\sqrt{\rho/G}$ for a thin to a thick cylindrical shell. The results for $H/R = 2{\cdot}0$ correspond to a solid cylindrical shell. The convergence study showed that the frequency parameter Ω converged as the number of layers increased. Table 2 shows the comparison of the frequency parameter $\bar{\omega} = \omega L\sqrt{\rho(1+\nu)/E}$ with three-dimensional elasticity results of Soldatos and Hadjigeorgiou [18]; the results are in good agreement. Tables 3–5 show the comparison of the frequency parameter $\Omega = (\omega H/\pi)\sqrt{\rho/G}$ with the results of Armenakas *et al.* [13] for various thick cylindrical shells. The results are also in good agreement. In all the comparisons, the results compared are for $nl = 25$.

In this paper, the frequency characteristics of thick simply supported–simply supported and clamped–clamped cylindrical shells are studied. Frequency parameters $\Psi = \omega R\sqrt{(1-\nu^2)\rho/E}$, of hollow and solid cylindrical shells are presented for a wide range of H/R and H/L ratios. A study to compare the frequencies obtained using classical shell theory and the present analysis is also presented.

TABLE 1

Convergence of frequency parameters $\Omega = (\omega H/\pi)\sqrt{\rho/G}$ for isotropic cylinders ($m = n = 1$, $H/L = 0{\cdot}2$, $\nu = 0{\cdot}3$)

	H/R			
nl	0·01	0·1	0·2	2·0
5	0·05804	0·07642	0·10723	0·08799
10	0·05782	0·07624	0·10709	0·08734
15	0·05777	0·07620	0·10706	0·08718
20	0·05776	0·07619	0·10705	0·08712
25	0·05775	0·07619	0·10705	0·08709
30	0·05775	0·07618	0·10704	0·08707
35	0·05775	0·07618	0·10704	0·08706
40	0·05774	0·07618	0·10704	0·08705
45	0·05774	0·07618	0·10704	0·08705
50	0·05774	0·07618	0·10704	0·08704
Exact†	0·05774	0·07618	0·10704	0·08703

† Armenakas *et al.* [13].

TABLE 2

Comparison of frequency parameters $\bar{\omega} = \omega L\sqrt{\rho(1+v)/E}$ for isotropic cylinders ($m = 1$, $L/R = 1{\cdot}0$, $v = 0{\cdot}3$)

H/R	n	Soldatos *et al.* [18]	†Armenakas *et al.* [13]	Present
0·1	1	1·06238	1·06226	1·06234
	2	0·88260	0·88233	0·88253
	3	0·80963	0·80925	0·80951
	4	0·89905	0·89877	0·89893
	5	1·12216		1·12209
0·2	1	1·18908	1·18889	1·18898
	2	1·10121	1·10092	1·10107
	3	1·19793	1·19755	1·19777
	4	1·48975	1·48933	1·48971
	5	1·91389		1·91406
0·3	1	1·33761	1·33727	1·33748
	2	1·32371	1·32335	1·32359
	3	1·52805	1·52764	1·52804
	4	1·92695	1·92660	1·92722
	5	2·44628		2·44692

†Adapted from Soldatos *et al.* [18].

TABLE 3

Comparison of frequency parameters $\Omega = (\omega H/\pi)\sqrt{\rho/G}$ for isotropic cylinders ($m = 1$, $H/R = 0{\cdot}4$, $v = 0{\cdot}3$)

n	H/L	Armenakas *et al.* [13]	Present
0	0·01	0·01612	0·01002
	0·1	0·15289	0·10000
	0·2	0·20495	0·20000
	0·4	0·27540	0·27544
	0·6	0·42022	0·42035
	0·8	0·60009	0·60033
	1·0	0·79274	0·79314
2	0·01	0·06217	0·06219
	0·1	0·07456	0·07458
	0·2	0·12494	0·12496
	0·4	0·27193	0·27199
	0·6	0·44144	0·44158
	0·8	0·62520	0·62546
	1·0	0·81660	0·81701

Figure 2 shows the variation of the frequency parameters ψ against H/R ratios for frequencies computed using classical Love's shell theory, see Lam and Loy [24] and Loy *et al.* [25], and the present three-dimensional analysis. For $H/R \leqslant 0{\cdot}05$,

TABLE 4

Comparison of frequency parameters $\Omega = (\omega H/\pi)\sqrt{\rho/G}$ for isotropic cylinders ($m = 1$, $H/R = 1{\cdot}0$, $\nu = 0{\cdot}3$)

n	H/L	Armenakas *et al.* [13]	Present
1	0·01	0·00040	0·00041
	0·1	0·03563	0·03563
	0·2	0·11368	0·11368
	0·3	0·20451	0·20451
	0·4	0·29836	0·29836
	0·6	0·48693	0·48698
	0·8	0·67583	0·67597
	1·0	0·86589	0·86618
3	0·01	0·65850	0·65890
	0·1	0·65957	0·65997
	0·2	0·66496	0·66535
	0·3	0·67876	0·67914
	0·4	0·70383	0·70421
	0·6	0·78912	0·78952
	0·8	0·91240	0·91285
	1·0	1·06030	1·06084

TABLE 5

Comparison of frequency parameters $\Omega = (\omega H/\pi)\sqrt{\rho/G}$ for isotropic solid cylinders ($m = n = 1$, $H/R = 2{\cdot}0$, $\nu = 0{\cdot}3$)

H/L	Armenakas *et al.* [13]	Present
0·01	0·000253	0·000258
0·1	0·02423	0·02434
0·2	0·08703	0·08709
0·4	0·26658	0·26659
0·6	0·46827	0·468256
0·8	0·67316	0·67316
1·0	0·87660	0·87665

there is no difference between classical shell theory and the present three-dimensional approach. Incidentally, $H/R = 0{\cdot}05$ represents the limit for classical shell theories. However, for $H/R > 0{\cdot}05$, the two sets of results begin to diverge, with the present three-dimensional approach giving lower frequencies than the classical shell theory.

Figures 3 and 4 show the variation of the frequency parameters ψ against the circumferential wave number n for a thin simply supported cylindrical shell, $H/R = 0{\cdot}05$, and a thick simply supported cylindrical shell, $H/R = 0{\cdot}3$ and $0{\cdot}5$, for

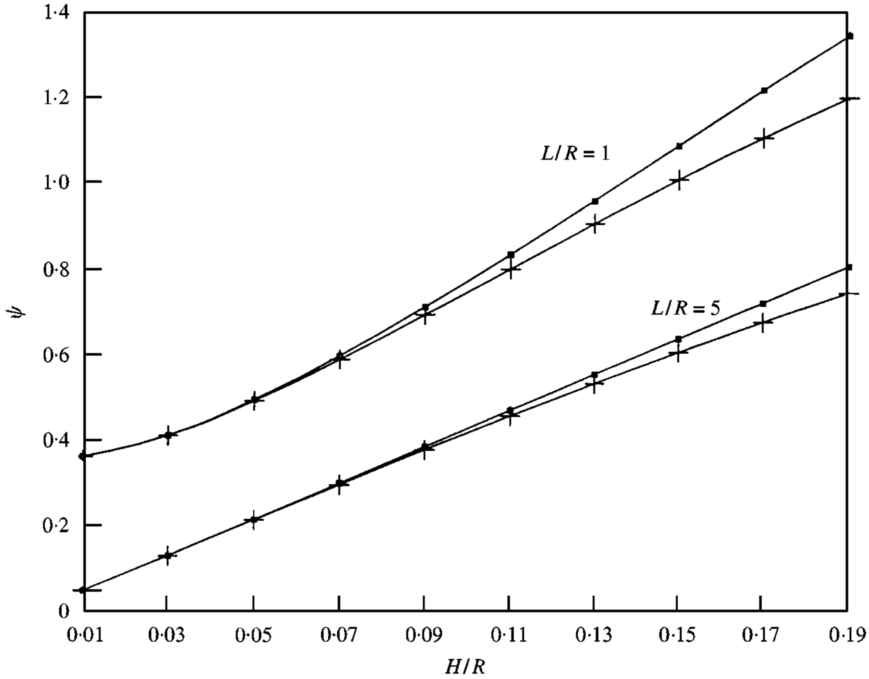

Figure 2. Variation of frequency parameters $\Psi = \omega R\sqrt{(1 - \nu^2)\rho/E}$ against H/R ratios for simply supported cylinders. ($m = 1$, $n = 4$). -■- 2D; + 3D.

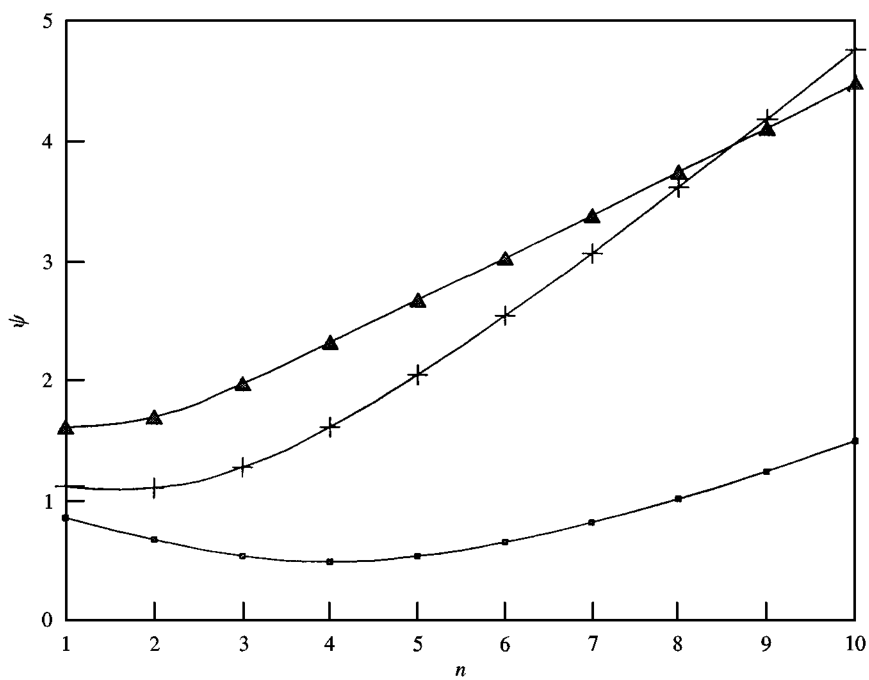

Figure 3. Variation of frequency parameters $\Psi = \omega R\sqrt{(1 - \nu^2)\rho/E}$ against circumferential wave number n for simply supported cylinders. ($m = 1$, $L/R = 1$). -■- $H/R = 0{\cdot}05$; + $H/R = 0{\cdot}3$; -▲- H/R 1·0.

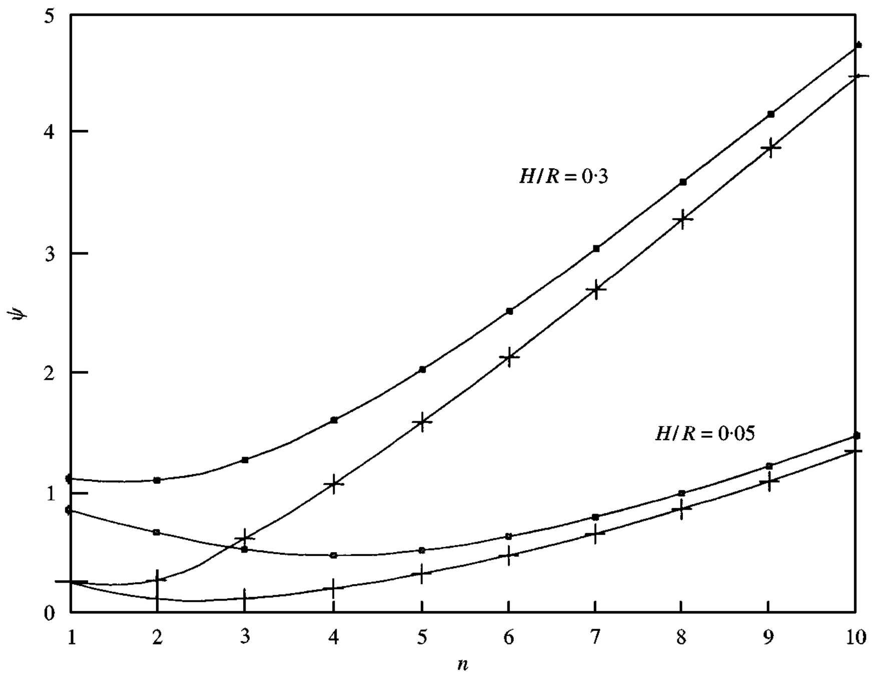

Figure 4. Variation of frequency parameters $\Psi = \omega R\sqrt{(1-\nu^2)\rho/E}$ against circumferential wave number n for simply supported cylinders. ($m = 1$). -■- $L/R = 1$; + $L/R = 4$.

Table 6

Frequency parameters $\Psi = \omega R\sqrt{(1-\nu^2)\rho/E}$ *for simply supported–simply supported isotropic cylinders* ($m = 1$, $n = 0$, $\nu = 0{\cdot}3$)

	H/L					
H/R	0·1	0·2	0·4	0·6	0·8	1·0
0·2	0·92930	1·06948	2·04718	3·62123	5·39832	7·24317
0·4	0·46466	0·92930	1·27983	1·95313	2·78942	3·68531
0·6	0·30979	0·61954	1·08352	1·45184	1·95976	2·52904
0·8	0·23238	0·46468	0·92931	1·23445	1·57390	1·97392
1·0	0·18597	0·37177	0·74346	1·11517	1·36267	1·65852
1·2	0·15507	0·30986	0·61958	0·92933	1·23274	1·45956
1·4	0·13304	0·26566	0·53110	0·79659	1·06209	1·32058
1·6	0·11659	0·23254	0·46476	0·69704	0·92935	1·16166
2·0	0·09416	0·18648	0·37203	0·55778	0·74359	0·92942

the axial wave number $m = 1$. From Figure 3, when the H/R ratio is increased, the circumferential wave number n at which the fundamental frequency occurred is observed to decrease. For example, the fundamental frequency for $H/R = 0{\cdot}05$ occurred at $n = 4$, for $H/R = 0{\cdot}3$ at $n = 2$ and for $H/R = 0{\cdot}5$ at $n = 1$. This means that when H/R is beyond a certain value, the fundamental frequencies would occur at $n = 1$. In Figure 4, the difference in frequencies for a long cylindrical shell, $L/R = 4$, and a short cylindrical shell, $L/R = 1$, is comparatively significant for thick cylindrical shells than for thin cylindrical shells. Both figures highlighted the interesting characteristics of thick shells and also the difference in the frequency characteristics between a thick shell and a thin shell.

Tables 6–8 present the frequency parameters Ψ for simply supported isotropic cylindrical shells. Table 6 gives the frequency parameters for axisymmetric vibrations while Tables 7 and 8 give the frequency parameters for asymmetric

TABLE 7

Frequency parameters $\Psi = \omega R\sqrt{(1 - v^2)\rho/E}$ *for simply supported–simply supported isotropic cylinders* ($m = 1$, $n = 1$, $v = 0{\cdot}3$)

	H/L					
H/R	0·1	0·2	0·4	0·6	0·8	1·0
0·2	0·58597	0·99477	2·06648	3·64526	5·41921	7·26078
0·4	0·26463	0·61887	1·23515	1·96994	2·81661	3·71242
0·6	0·14793	0·40327	0·90637	1·41530	1·96735	2·55004
0·8	0·09462	0·28341	0·69696	1·11052	1·53354	1·96882
1·0	0·06622	0·21128	0·55453	0·90510	1·25635	1·60987
1·2	0·04945	0·16469	0·45411	0·75670	1·05878	1·35951
1·4	0·03874	0·13291	0·38063	0·64536	0·91020	1·17269
1·6	0·03150	0·11027	0·32509	0·55920	0·79446	1·02738
2·0	0·02262	0·08093	0·24774	0·43515	0·62557	0·81467

TABLE 8

Frequency parameters $\Psi = \omega R\sqrt{(1 - v^2)\rho/E}$ *for simply supported–simply supported isotropic cylinders* ($m = 1$, $n = 2$, $v = 0{\cdot}3$)

	H/L					
H/R	0·1	0·2	0·4	0·6	0·8	1·0
0·2	0·42083	0·92122	2·13264	3·71801	5·48184	7·31352
0·4	0·34653	0·58064	1·26379	2·05181	2·90617	3·79623
0·6	0·42526	0·52734	0·95010	1·48277	2·05183	2·63821
0·8	0·50848	0·55701	0·81368	1·19274	1·61427	2·05130
1·0	0·58193	0·60585	0·76098	1·03069	1·35246	1·69482
1·2	0·64421	0·65489	0·74815	0·93894	1·18618	1·45957
1·4	0·69206	0·69461	0·74953	0·88478	1·07552	1·29592
1·6	0·71819	0·71560	0·74697	0·84477	0·99411	1·17411
2·0	0·692070	0·68648	0·69772	0·75645	0·85715	0·98627

TABLE 9

Frequency parameters $\Psi = \omega R\sqrt{(1-\nu^2)\rho/E}$ for clamped–clamped isotropic cylinders ($m = 1$, $n = 0$, $\nu = 0{\cdot}3$)

	H/L					
H/R	0·1	0·2	0·4	0·6	0·8	1·0
0·2	1·03567	1·47766	3·27761	5·32309	7·39131	9·46118
0·4	0·51877	1·03754	1·84188	2·78976	3·78864	4·80347
0·6	0·34587	0·69170	1·38339	2·00293	2·63252	3·28708
0·8	0·25944	0·51879	1·03755	1·55631	2·07508	2·55996
1·0	0·20761	0·41506	0·83005	1·24506	1·66007	2·07508
1·2	0·17309	0·34593	0·69173	1·03756	1·38340	1·72924
1·4	0·14848	0·29657	0·59294	0·88936	1·18579	1·48222
1·6	0·13008	0·25958	0·51886	0·77822	1·03758	1·29696
2·0	0·10486	0·20806	0·41529	0·62271	0·83017	1·03765

TABLE 10

Frequency parameters $\Psi = \omega R\sqrt{(1-\nu^2)\rho/E}$ for clamped–clamped isotropic cylinders ($m = 1$, $n = 1$, $\nu = 0{\cdot}3$)

	H/L					
H/R	0·1	0·2	0·4	0·6	0·8	1·0
0·2	0·69319	1·39567	3·27162	5·32557	7·39564	9·46597
0·4	0·36463	0·79219	1·74212	2·74975	3·77012	4·79481
0·6	0·23543	0·54159	1·19726	1·87447	2·55528	3·23727
0·8	0·16803	0·40698	0·90887	1·42129	1·93471	2·44804
1·0	0·12725	0·32348	0·73081	1·14309	1·55548	1·96754
1·2	0·10040	0·26659	0·61014	0·95512	1·29969	1·64384
1·4	0·08171	0·22525	0·52289	0·81965	1·11556	1·41099
1·6	0·06818	0·19379	0·45659	0·71722	0·97661	1·23542
2·0	0·05030	0·14813	0·36120	0·57102	0·77929	0·98685

vibrations. The result show that the frequencies would decrease when H/R is increased and increase when H/L is increased.

Tables 9–11 present the frequency parameters Ψ for clamped isotropic cylindrical shells. Table 9 gives the frequency parameters for axisymmetric vibrations, and Tables 10 and 11 give the frequency parameters for asymmetric vibrations. The results also show that the frequencies would decrease when H/R is increased and increase when H/L is increased. When the results of simply supported and clamped boundary conditions are compared it is found that the frequencies of clamped boundary conditions are higher than those for simply supported boundary conditions.

TABLE 11

Frequency parameters $\Psi = \omega R\sqrt{(1-\nu^2)\rho/E}$ *for clamped–clamped isotropic cylinders* $(m = 1,\ n = 2,\ \nu = 0{\cdot}3)$

	H/L					
H/R	0·1	0·2	0·4	0·6	0·8	1·0
0·2	0·56971	1·33769	3·27748	5·34072	7·41226	9·48240
0·4	0·41326	0·78170	1·73559	2·75292	3·77897	4·80702
0·6	0·45211	0·64728	1·23845	1·89793	2·57231	3·25193
0·8	0·52113	0·62996	1·02049	1·48953	1·98156	2·48296
1·0	0·58878	0·65257	0·91839	1·26565	1·64284	2·03355
1·2	0·64836	0·68706	0·87208	1·13459	1·43149	1·74544
1·4	0·69484	0·71876	0·85065	1·05236	1·29016	1·54752
1·6	0·72023	0·73528	0·83271	0·99138	1·18547	1·40027
2·0	0·69329	0·70046	0·76283	0·87287	1·01364	1·17400

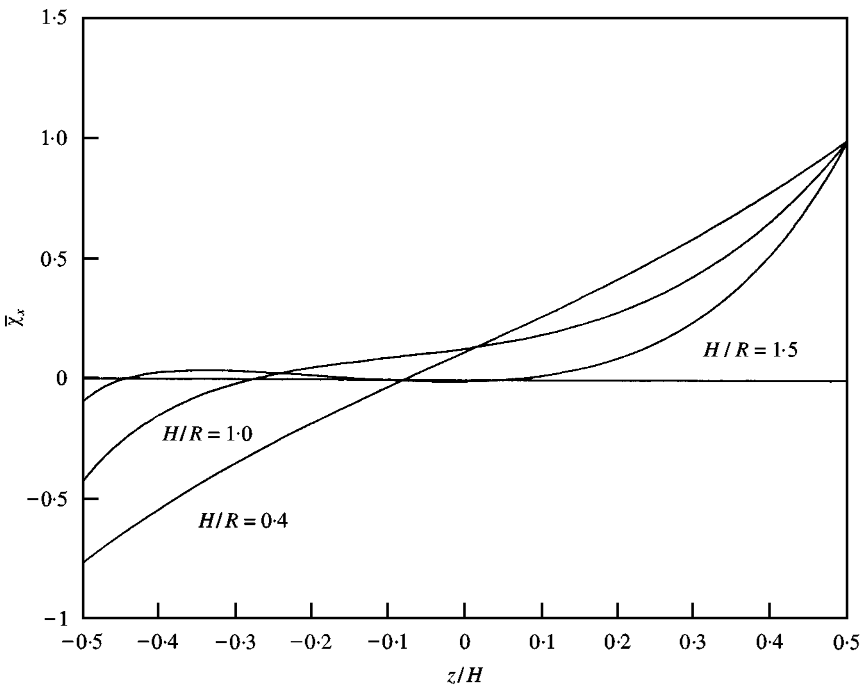

Figure 5. Distribution of normalized χ_x against z/H.

Figures 5–7 show the distributions of $\bar{\chi}_x$, $\bar{\chi}_\theta$ and $\bar{\chi}_z$, the normalized values of χ_x, χ_θ and $\bar{\chi}_z$ with respect to their respective maximum values, with z/H. The distributions for χ_x, χ_θ and χ_z are different from one another and are found to vary for different H/R ratios. It appeared that when H/R ratio is large, higher order polynomials are needed to describe the distributions of χ_x, χ_θ and χ_z.

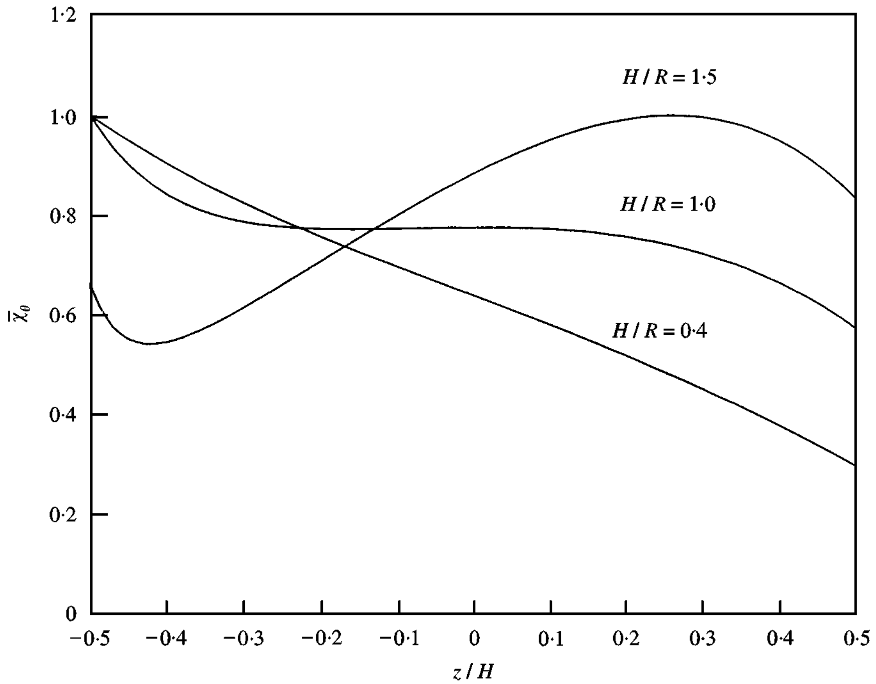

Figure 6. Distribution of normalized χ_θ against z/H.

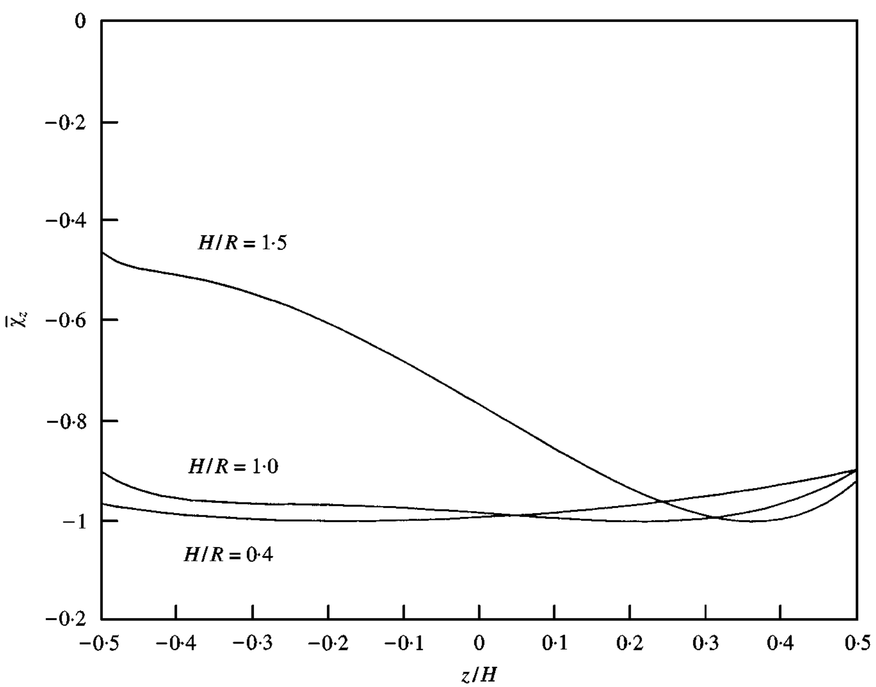

Figure 7. Distribution of normalized χ_z against z/H.

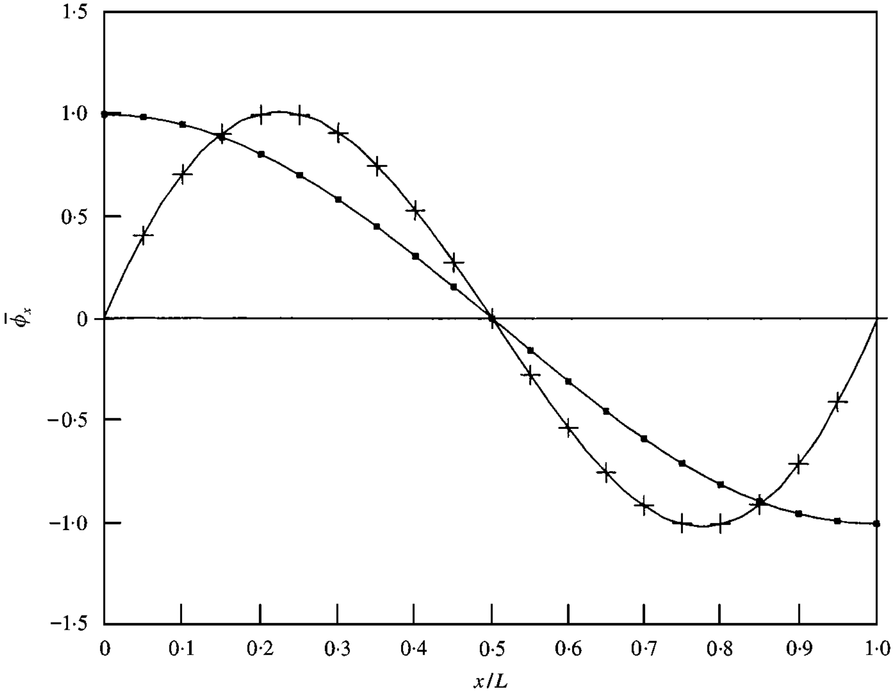

Figure 8. Distribution of normalized ϕ_x against x/L. -■- SS-SS; + C-C.

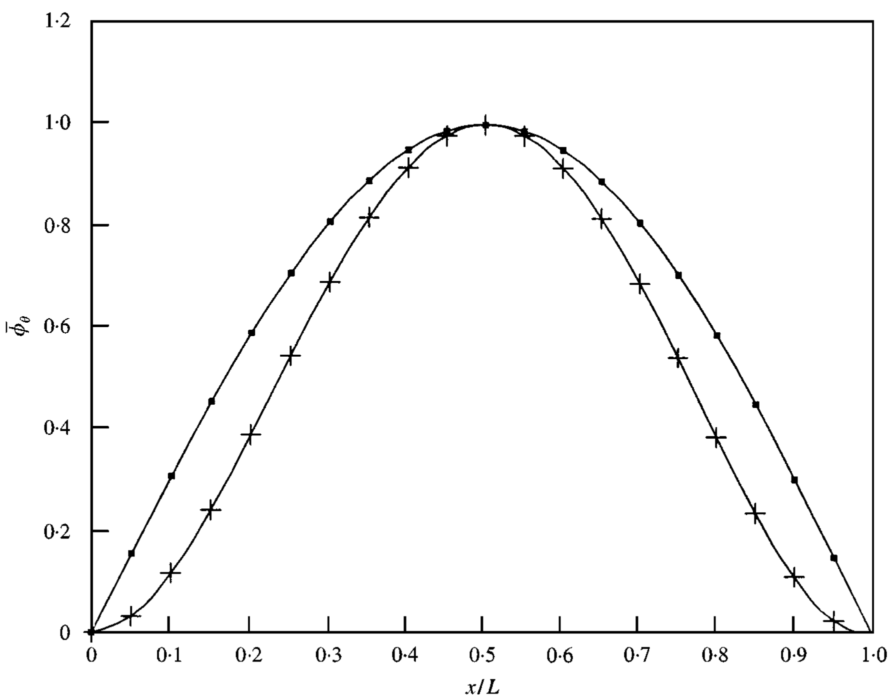

Figure 9. Distribution of normalized ϕ_θ against x/L. -■- SS-SS; + C-C.

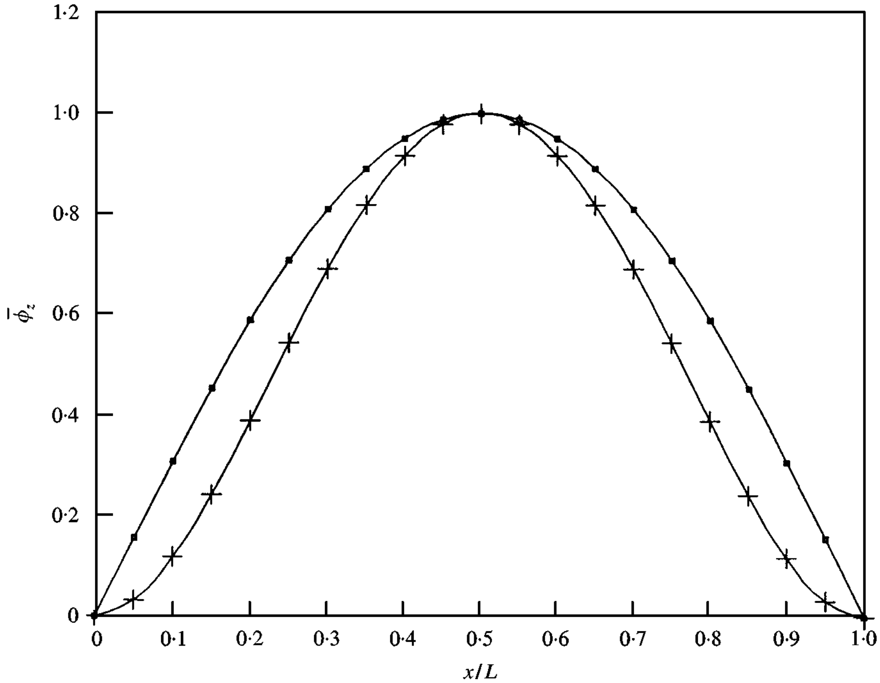

Figure 10. Distribution of normalized ϕ_z against x/L. -■- SS-SS; + C-C.

Figures 8–10 show the distributions of $\bar{\phi}_x$, $\bar{\phi}_\theta$ and $\bar{\phi}_z$, the normalized values of ϕ_x, ϕ_θ and ϕ_z with respect to their respective maximum values, against x/L. From the figures, the geometric boundary conditions for both the simply supported and clamped boundary conditions are satisfied.

The present study highlighted a few things. Firstly, classical shell theories are only accurate for $H/R \leqslant 0{\cdot}05$. Secondly, the distributions for χ_x, χ_θ and χ_z are different from one another and they vary with the H/R ratio. When H/R ratio is large, the distributions of χ_x, χ_θ and χ_z are such that higher order polynomials are needed to describe them.

4. CONCLUSIONS

A study of the vibration of thick isotropic cylindrical shells on the basis of three-dimensional theory of elasticity has been presented for simply supported and clamped boundary conditions. In the study, a thick cylindrical shell is discretized into an arbitrary number of thin layers in the thickness direction. The displacements through the thickness are approximated by some linear-shape functions that are expressed in terms of some generalized coordinates. The displacements in the axial and circumferential directions are approximated by trigonometric functions that are chosen to satisfy the specific geometric boundary conditions and the periodicity of the motion. The characteristic eigenvalue governing equation is obtained using an energy minimization principle and is solved to yield the natural frequencies and the generalized co-ordinates. Extensive frequency parameters, never published in the literature before, for a wide range of thickness-to-radius and thickness-to-length ratios, have been presented. The distribution of the displacement fields in the thickness direction for various thickness-to-radius ratios and a study comparing the frequency characteristics for thin and thick cylindrical shells are also presented. Comparisons of results with published results in the literature have been carried out and good agreement is observed.

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