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LETTERS TO THE EDITOR

EFFECT OF RADIAL INHOMOGENEITY ON NATURAL FREQUENCIES OF AN ANISOTROPIC HOLLOW SPHERE

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1. INTRODUCTION

One basic problem of elastodynamics is the free vibration of an elastic spherical shell [1]. For the purpose of analyzing the free vibration of a spherically isotropic spherical shell, several separation methods have been proposed. For example, Cohen and Shah [2] used two auxiliary variables to simplify the governing equations; they found that the vibration was eventually divided into two independent classes. Ding and Chen [3], on the other hand, introduced three displacement functions; two independent classes of vibrations were also observed for the coupled vibration of a submerged spherical shell. Chau [4] recently extended Hu's formula [5] to consider the toroidal vibration of a spherically isotropic solid sphere.

Recently, the mechanics of functionally graded materials (FGMs) has been of great interest [6]. In FGM, the material properties always vary along one deep direction continuously. In fact, FGM is a subset of inhomogeneous materials, the study of which has been extensive. As regards the vibration of an inhomogeneous (hollow) sphere, Huston [7] and Sur [8] have studied the radial vibration of isotropic and anisotropic (hollow) spheres respectively. Shulga *et al.* [9] investigated the non-axisymmetric vibrations of a non-homogeneous transversely isotropic hollow sphere and presented a state equation by employing the separation technique as well as the function expansion method. Numerical analysis has also been used for obtaining the vibration frequency of non-homogeneous spheres [10,11]. It is also noted here that Puro [12,13] initiated the separation method for non-homogeneous isotropic and spherically isotropic elasticity to consider some static problems. Earlier, Papport [14] used the separation method to construct general solutions of transversely isotropic inhomogeneous elasticity theory equations.

In this letter, we follow the method proposed in reference [3], using three displacement functions to rewrite the components of displacement. The governing

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equations of motion of a spherically isotropic elastic body with radial inhomogeneity are then turned into an uncoupled partial differential equation and two coupled partial differential equations. For the general non-axisymmetric free vibration problem, the resulting equations are further simplified to the corresponding ordinary differential ones. For the case in which all material constants including the elastic constants and the mass density obey an identical power law in the radial direction, solutions to these equations are given in the paper. In particular, the matrix Frobenius power series method [15] is employed to solve the coupled set. Exact frequency equations are then derived. Finally, numerical results are given to show the effect of the radial inhomogeneity on the natural frequencies.

2. THE SEPARATION TECHNIQUE

In analogy to the homogeneous spherically isotropic elasticity [3], by introducing three displacement functions w, G and ψ to rewrite the displacements as follows,

$$u_\theta = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} - \frac{\partial G}{\partial\theta}, \qquad u_\phi = \frac{\partial\psi}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial G}{\partial\phi}, \qquad u_r = w, \tag{1}$$

the governing equations can be finally transformed into

$$B + (\nabla_2 A_{44})(\nabla_2\psi - \psi) - r^2\rho\frac{\partial^2\psi}{\partial t^2} = 0, \tag{2}$$

$$A + (\nabla_2 A_{44})w - (\nabla_2 A_{44})(\nabla_2 G - G) + r^2\rho\frac{\partial^2 G}{\partial t^2} = 0, \tag{3}$$

$$[L_3 + 2(\nabla_2 A_{13}) + (\nabla_2 A_{33})\nabla_2]w - [L_4 + (\nabla_2 A_{13})]\nabla_1^2 G - r^2\rho\frac{\partial^2 w}{\partial t^2} = 0, \tag{4}$$

where

$$A = L_1 w - L_2 G, \qquad B = [A_{44}\nabla_3^2 - 2A_{44} + A_{11} - A_{12} + \tfrac{1}{2}(A_{11} - A_{12})\nabla_1^2]\psi,$$

$$L_1 = (A_{13} + A_{44})\nabla_2 + A_{11} + A_{12} + 2A_{44},$$

$$L_2 = A_{44}\nabla_3^2 - 2A_{44} + A_{11} - A_{12} + A_{11}\nabla_1^2,$$

$$L_3 = A_{33}\nabla_3^2 - 2(A_{11} + A_{12} - A_{13}) + A_{44}\nabla_1^2,$$

$$L_4 = (A_{13} + A_{44})\nabla_2 - A_{44} - A_{11} - A_{12} + A_{13},$$

$$\nabla_2 = r\frac{\partial}{\partial r}, \qquad \nabla_2^2 = r\frac{\partial}{\partial r}r\frac{\partial}{\partial r}, \qquad \nabla_3^2 = \nabla_2^2 + \nabla_2,$$

$$\nabla_1^2 = \frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{5}$$

A_{ij} are the elastic constants and ρ is the mass density. Here we assume that all these material constants are functions of the radial co-ordinate r, i.e. the body is radially inhomogeneous. It is seen that equation (2) is a second order, uncoupled partial differential equation in ψ; equations (3) and (4) form a coupled partial differential equation set in w and G.

For the general non-axisymmetric free vibration problem, it is assumed that

$$[\psi, G, w] = R[U_n(\xi), V_n(\xi), W_n(\xi)]S_n^m(\theta,\phi)\exp(\mathrm{i}\omega t), \tag{6}$$

where $S_n^m(\theta,\phi)$ are spherical harmonics, ω is the circular frequency and $\xi = r/R$ is the non-dimensional radial co-ordinate (R will be taken as the mean radius of a spherical shell in the following). Substituting equation (6) into equations (2)–(4) yields

$$\xi^2 U_n'' + (2 + f_5)\xi U_n' + \{\Omega^2\xi^2 - [2 + (n-1)(n+2)(f_1 - f_2)/2 + f_5]\}U_n = 0, \tag{7}$$

$$\xi^2 W_n'' + (2 + f_6)\xi W_n' + (\Omega^2\xi^2/f_4 + p_1 + 2f_7)W_n - p_2\xi V_n' - [p_3 - n(n+1)f_7]V_n = 0, \tag{8}$$

$$\xi^2 V_n'' + (2 + f_5)\xi V_n' + (\Omega^2\xi^2 + p_4 - f_5)V_n - p_5\xi W_n' - (p_6 + f_5)W_n = 0, \tag{9}$$

where a prime denotes differentiation with respect to ξ, and

$$p_1 = [2(f_3 - f_1 - f_2) - n(n+1)]/f_4, \qquad p_2 = -n(n+1)(f_3 + 1)/f_4,$$

$$p_3 = n(n+1)(f_1 + f_2 + 1 - f_3)/f_4, \qquad p_4 = f_1 - f_2 - n(n+1)f_1 - 2,$$

$$p_5 = f_3 + 1, \qquad p_6 = f_1 + f_2 + 2, \qquad \Omega = \omega R/v_2,$$

$$f_1 = A_{11}/A_{44}, \qquad f_2 = A_{12}/A_{44}, f_3 = A_{13}/A_{44}, \qquad f_4 = A_{33}/A_{44},$$

$$f_5 = (\nabla_2 A_{44})/A_{44}, \qquad f_6 = (\nabla_2 A_{33})/A_{33}, \qquad f_7 = (\nabla_2 A_{13})/A_{33}. \tag{10}$$

Here $v_2 = \sqrt{A_{44}/\rho}$ is the elastic wave velocity. So far, for the free vibration problem, the governing equations have been turned into equations (7)–(9) in a non-dimensional form: equation (7) is an independent, second order, ordinary differential equation in unknown U_n, while equations (8) and (9) are coupled by two unknowns V_n and W_n, and each equation involved is a second order ordinary differential one.

Now we assume that all material constants have the same power function distribution along the radial direction, i.e. $A_{ij} = A_{ij}^0\xi^\alpha$ and $\rho = \rho_0\xi^\alpha$. Such a power law distribution is very common in usual FGMs [6]. Then we have the non-dimensional elastic constants as $f_5 = f_6 = \alpha, f_7 = \alpha f_3/f_4$ and $f_1 = A_{11}^0/A_{44}^0$, etc. The solution to equation (7) can thus be easily obtained as

$$U_n(\xi) = \xi^{-(1+\alpha)/2}[B_{n1} J_\eta(\Omega\xi) + B_{n2} Y_\eta(\Omega\xi)] \qquad (n \geqslant 1), \tag{11}$$

where J_η and Y_η are the first and second kinds of Bessel functions respectively, B_{n1} and B_{n2} are arbitrary constants, and $\eta^2 = \frac{1}{4}[9 + 2(n^2 + n - 2)(f_1 - f_2) + \alpha(6 + \alpha)] > 0$.

It should be pointed out that $n = 0$ is a special case for which the function $V_n(\xi)$ contributes nothing to the elastic field. In fact, equations (8) and (9) will degenerate into the following single equation for $n = 0$:

$$\xi^2 W_0'' + (2 + \alpha)\xi W_0' + (1/f_4)(\Omega^2\xi^2 + 2f_3 - 2f_1 - 2f_2 + 2\alpha f_3)W_0 = 0, \tag{12}$$

with the following solution:

$$W_0(\xi) = \xi^{-(1+\alpha)/2}[C_{01} J_\zeta(\gamma\xi) + C_{02} Y_\zeta(\gamma\xi)] \quad (n = 0), \tag{13}$$

where $\zeta^2 = (1 + \alpha)^2/4 + 2(f_1 + f_2 - f_3 - \alpha f_3)/f_4 > 0$, C_{0i} are two arbitrary constants, and $\gamma = \Omega/\sqrt{f_4}$.

When $n \geqslant 1$, it is seen that $\xi = 0$ is a regular singular point for the coupled system. To obtain the solution to this ordinary differential equation system, the matrix Frobenius power series method developed in Ding *et al.* [15] is employed. Details are, however, omitted here for the sake of simplicity. The general solution can finally be expressed as the linear combination of four independent solutions as follows:

$$W_n(\xi) = \sum_{j=1}^{4} C_{nj} W_{nj}(\xi), \qquad V_n(\xi) = \sum_{j=1}^{4} C_{nj} V_{nj}(\xi) \quad (n \geqslant 1), \tag{14}$$

where C_{nj} are arbitrary constants, W_{nj} and V_{nj} are convergent, infinite series in variable ξ.

3. EXACT FREQUENCY EQUATIONS

Suppose that the inner and outer radii of the spherical shell are a and b, respectively, and both surfaces are free from tractions, i.e. $\sigma_r = \tau_{r\theta} = \tau_{r\phi} = 0$, $(r = a, b)$. Using the results obtained above, one can find that the vibration of an inhomogeneous spherically isotropic elastic spherical shell is also separated into two independent classes. The first class, which corresponds to an equivoluminal motion of the shell, is characterized by the absence of radial component of displacement while for the second class, the displacement has, in general, both transverse and radial components, but the rotation has no radial component. One can then derive two sets of linear homogeneous algebriac equations of undetermined constants B_{ni} and C_{ni} respectively. For non-trivial solutions to exist, the coefficient determinants of the two systems should vanish so that the corresponding frequency equations are obtained.

3.1. FREQUENCY EQUATION OF THE FIRST CLASS ($n \geqslant 1$)

$$|E_{ij}^1| = 0 \quad (i, j = 1, 2), \tag{15}$$

where

$$E_{11}^1 = [\eta - (3 + \alpha)/2] J_\eta(\Omega\xi_1) - \Omega\xi_1 J_{\eta+1}(\Omega\xi_1),$$

$$E_{12}^1 = [\eta - (3 + \alpha)/2] Y_\eta(\Omega\xi_1) - \Omega\xi_1 Y_{\eta+1}(\Omega\xi_1),$$

$$E_{21}^1 = [\eta - (3 + \alpha)/2] J_\eta(\Omega\xi_2) - \Omega\xi_2 J_{\eta+1}(\Omega\xi_2),$$

$$E_{22}^1 = [\eta - (3 + \alpha)/2] Y_\eta(\Omega\xi_2) - \Omega\xi_2 Y_{\eta+1}(\Omega\xi_2), \tag{16}$$

and $\xi_1 = a/R = 1 - t^*/2$, $\xi_2 = b/R = 1 + t^*/2$, $R = (a + b)/2$. Here $t^* = (b - a)/R$ is the thickness-to-mean radius ratio of the shell. Note that when $n = 1$, frequency equation (15) corresponds to a torsional or rotary mode of the shell. It is interesting to consider the case when $\Omega\xi_i (i = 1, 2)$ are large (high frequency) and the spherical shell is thin, for which the asymptotic expansions of Bessel functions can be used. We can thus derive the following frequency equation:

$$\frac{\tan(\Omega t^*)}{\Omega t^*} = \frac{4\eta^2 + 15 + 4\alpha}{8\Omega^2\xi_1\xi_2 - 4\eta^2 + 33 + 16\alpha + 2\alpha^2}. \tag{17}$$

Equation (17) is identical to that obtained by Cohen and Shah [2] if the inhomogeneity is not considered, i.e., when $\alpha = 0$.

3.2. FREQUENCY EQUATION OF THE SECOND CLASS ($n \geqslant 0$)

$$|E_{ij}^2| = 0 \quad (i, j = 1, 2) \tag{18}$$

for $n = 0$, where

$$E_{11}^2 = \{2 + (f_4/f_3)[\zeta - (1 + \alpha)/2]\} J_\zeta(\gamma\xi_1) - (f_4/f_3)\gamma\xi_1 J_{\zeta+1}(\gamma\xi_1),$$

$$E_{12}^2 = \{2 + (f_4/f_3)[\zeta - (1 + \alpha)/2]\} Y_\zeta(\gamma\xi_1) - (f_4/f_3)\gamma\xi_1 Y_{\zeta+1}(\gamma\xi_1),$$

$$E_{21}^2 = \{2 + (f_4/f_3)[\zeta - (1 + \alpha)/2]\} J_\zeta(\gamma\xi_2) - (f_4/f_3)\gamma\xi_2 J_{\zeta+1}(\gamma\xi_2),$$

$$E_{22}^2 = \{2 + (f_4/f_3)[\zeta - (1 + \alpha)/2]\} Y_\zeta(\gamma\xi_2) - (f_4/f_3)\gamma\xi_2 Y_{\zeta+1}(\gamma\xi_2). \tag{19}$$

Obviously, frequency equation (18) corresponds to the purely radial vibration,

$$|E^3_{ij}| = 0 \quad (i, j = 1, 2, \ldots, 4) \tag{20}$$

for $n \geqslant 1$, where

$$\begin{aligned}
E^3_{1i} &= n(n+1)V_{ni}(\xi_1)/\xi_1 + 2W_{ni}(\xi_1)/\xi_1 + (f_4/f_3)W'_{ni}(\xi_1) \\
E^3_{2i} &= W_{ni}(\xi_1)/\xi_1 + V_{ni}(\xi_1)/\xi_1 - V'_{ni}(\xi_1) \qquad\qquad (i = 1, 2, \ldots, 4). \\
E^3_{3i} &= W_{ni}(\xi_2)/\xi_2 + V_{ni}(\xi_2)/\xi_2 - V'_{ni}(\xi_2) \\
E^3_{4i} &= n(n+1)V_{ni}(\xi_2)/\xi_2 + 2W_{ni}(\xi_2)/\xi_2 + (f_4/f_3)W'_{ni}(\xi_2)
\end{aligned} \tag{21}$$

It is seen that the integer m, included in the spherical harmonics and representing the non-axisymmetric motion ($m \neq 0$) of the sphere, does not appear in the frequency equations. The reason, explained by Silbiger [16] for the same phenomenon for an empty, thin isotropic spherical shell, is also valid here. In fact, since both the spherical isotropy and the radial inhomogeneity do not violate the spherical symmetry of the shell, the non-axisymmetric modes of vibrations can be obtained by the superposition of axisymmetric ones of identical natural frequency.

4. NUMERICAL EXAMPLES

Since the frequency equations are three dimensional, there is an infinite number of frequencies. In what follows, only the smallest natural frequency that is of practical significance is given in the calculation. The material is taken to be a hypothetical one exhibiting substantial anisotropy [2,3], whose non-dimensional elastic constants are $f_1 = 20$, $f_2 = 12$ and $f_3 = f_4 = 2$. *For the convenience of future comparison*, all results are given in tabular forms.

Table 1 gives the lowest natural frequencies of the first class for two values of the thickness-to-mean radius ratio t^*. From Table 1, we can find that the lowest non-dimensional frequency $\Omega = \omega R\sqrt{\rho_0/A^0_{44}}$ for the torsional mode ($n = 1$) increases with the increase of the inhomogeneity parameter α, while it decreases for other higher modes ($n > 1$). It is also noted here that for the torsional mode, there is no difference for the non-dimensional frequency between different materials if the homogeneity parameter α is the same. It is obvious since the parameter η is only related to α for this case.

Table 2 gives the lowest natural frequencies of the second class also for two values of t^*. Note that there is a zero natural frequency for mode $n = 1$ of the second class that corresponds to the rigid-body translation and hence results are not given for $n = 1$. From the results, it is seen that the lowest natural frequency decreases with the increase of the inhomogeneity parameter α for all modes except for $n = 1$.

For all cases, it is shown that the inhomogeneity has a greater effect for the thick shell than for the thin shell.

TABLE 1

The lowest natural frequencies $\Omega = \omega R\sqrt{\rho_0/A_{44}^0}$ of the first class

n	t^*	$\alpha = -2.0$	$\alpha = -1.5$	$\alpha = -1.0$	$\alpha = -0.5$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 1.5$	$\alpha = 2.0$
1	0·2	15·772	15·798	15·827	15·861	15·899	15·940	15·982	16·034	16·087
	1·2	3·0239	3·1688	3·3249	3·4897	3·6611	3·8372	4·0166	4·1982	4·3812
2	0·2	3·9917	3·9851	3·9785	3·9720	3·9655	3·9592	3·9528	3·9466	3·9404
	1·2	3·1991	3·1453	3·0967	3·0526	3·0129	2·9771	2·9448	2·9157	2·8895
3	0·2	6·3074	6·2970	6·2867	6·2764	6·2663	6·2563	6·2463	6·2365	6·2269
	1·2	4·8009	4·7528	4·7080	4·6662	4·6273	4·5911	4·5575	4·5263	4·4974

TABLE 2

The lowest natural frequencies $\Omega = \omega R\sqrt{\rho_0/A_{44}^0}$ of the second class

n	t^*	$\alpha = -2.0$	$\alpha = -1.5$	$\alpha = -1.0$	$\alpha = -0.5$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 1.5$	$\alpha = 2.0$
0	0·2	7·6159	7·6035	7·5910	7·5784	7·5658	7·5532	7·5406	7·5279	7·5153
	1·2	7·1241	6·9792	6·8401	6·7070	6·5799	6·4592	6·3448	6·2369	6·1353
2	0·2	2·5394	2·5353	2·5312	2·5270	2·5227	2·5184	2·5141	2·5098	2·5054
	1·2	3·0274	2·8803	2·7440	2·6192	2·5065	2·4057	2·3165	2·2383	2·1701
3	0·2	3·3657	3·3609	3·3559	3·3506	3·3451	3·3394	3·3334	3·3272	3·3207
	1·2	3·9377	3·7880	3·6467	3·5145	3·3918	3·2788	3·1754	3·0815	2·9966

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