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AN EXPLICIT FORM GENERAL SOLUTION FOR OSCILLATORS WITH A NON-SMOOTH RESTORING FORCE, $\ddot{x} + sign(x) f(x) = 0$

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Let us consider a non-linear conservative free oscillator

$$\ddot{x} + sign(x) f(x) = 0, \quad x \in R, \tag{1}$$

where $f(x)$ is even and a sufficiently smooth function. When $f(0) = 0$, the characteristic of oscillator (1) is at least a continuous odd function, and hence, solutions of the differential equation must possess at least two continuous derivatives. If $f(0) \neq 0$, the characteristic has a step-wise discontinuity at zero $x = 0$. In this case, the function $x(t)$ will have the only first continuous derivative at those instants of time t for which $x(t) = 0$, and the equality in equation (1) has to be understood correctly in terms of distributions. In this letter, a one-parametric (except time translation) family of periodic solutions will be constructed. Then, taking into account that the system admits the group of time translations, $t \rightarrow t + t_0$, the second arbitrary parameter will be introduced, and the general periodic solution obtained.

First, let consider the special case of the piece-wise constant characteristic, when $f(x) \equiv 1$ and the differential equation (1) takes the form [1]

$$\ddot{x} + sign(x) = 0. \tag{2}$$

Since either $\ddot{x} = -1$ or $\ddot{x} = 1$ for the positive and negative co-ordinate x, respectively, an exact analytical piecewise continuous solution can be obtained by matching the two parabolas over the one period of vibration with a future periodic extension of the solution on all infinite-time regions [2]. In order to get a "single function" solution, basically the Fourier series of the periodic solution [2] was proposed in recent work [3]‡. It should be noted that the Fourier series form gives, in principal, an approximate solution since it is impossible to account for the infinite number of terms. As long as one can keep "any number of terms", the above remark is not so important for the smooth time histories. However, it becomes very important when dealing with either a discontinuous function $x(t)$ or its

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‡It has not been explained how to expand function $sign(x)$ into the Fourier series with respect to time t, before solution $x(t)$ is known.

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discontinuous derivatives. It is known that the trigonometric series appear to be "bad working" around the discontinuities due to the Gibbs phenomenon (see for instance reference [4, p. 602]). In terms of acceleration, the series performs an oscillating error near those points of time t at which the acceleration $\ddot{x}(t)$ has the step-wise discontinuities switching its value from -1 to 1 or back as it is dictated by equation (2).

A closed-form analytical solution will be obtained below by making use the simplest version of the saw-tooth transformation of time (STTT) [5]. A special feature of the solution is that it does not involve any trigonometric series at all, and at the same time it is expressed by a "single function".

The basic step of the STTT consists of introduction of a new time parameter τ as

$$x(t) = X(\tau), \qquad \tau = \tau(\omega t), \tag{3}$$

where ω is an arbitrary parameter τ is a periodic saw-tooth function which is given by one of the two equivalent expressions,

$$\tau(\xi) = \begin{cases} \xi, & -1 \leqslant \xi \leqslant 1, \\ -\xi + 2, & 1 \leqslant \xi \leqslant 3, \end{cases} \quad \tau(\xi) \overset{\forall \xi}{=} \tau(\xi + 4), \tag{4}$$

$$\tau(\xi) = \frac{2}{\pi} \arcsin \sin \frac{\pi \xi}{2}. \tag{5}$$

The amplitude and the period of the function $\tau(\xi)$ ($|\tau(\xi)| \leqslant 1$) are normalized in such a manner that the expression

$$|\tau'(\xi)]^2 = 1 \tag{6}$$

holds at least for almost all $\xi \in (-\infty, \infty)$.

Function $\tau(\xi)$ can be called a *saw-tooth sine* and viewed as a standard elementary periodic function by both physical and mathematical treatments [5].

On the next step, substituting representation (3) into the differential equation (2), one obtains

$$\omega^2 \frac{\mathrm{d}^2 X}{\mathrm{d}\tau^2} + sign(X) = 0, \qquad \left.\frac{\mathrm{d}X}{\mathrm{d}\tau}\right|_{\tau = \pm 1} = 0, \tag{7}$$

where expression (6) has been taken into account, the boundary conditions are used to eliminate the singular term $\omega^2 (\mathrm{d}X/\mathrm{d}\tau)\tau''(\omega t)$ from the expression for second derivative, $\ddot{x}(t)$. This term includes the Dirac functions and has to be eliminated since $\ddot{x}(t)$ can only have step-wise discontinuities.

Conserving a generality (see the result below), we seek a solution such that the co-ordinate $X(\tau)$ is negative when τ is negative, and $X(\tau)$ is positive when τ is positive, and else $X(-\tau) = -X(\tau)$. In this case due to the continuity one has $X(0) = 0$. Under these conditions the boundary value problem (7) is split into the two subproblems, respectively,

$$\frac{\mathrm{d}^2 X}{\mathrm{d}\tau^2} = \omega^{-2}, \qquad X|_{\tau=0} = 0, \qquad \left.\frac{\mathrm{d}X}{\mathrm{d}\tau}\right|_{\tau=-1} = 0 \tag{8}$$

and

$$\frac{d^2X}{d\tau^2} = -\omega^{-2}, \qquad X|_{\tau=0} = 0, \qquad \left.\frac{dX}{d\tau}\right|_{\tau=+1} = 0. \tag{9}$$

The solutions of problems (8) and (9) are found, respectively, as

$$X = \omega^{-2}(\tau + \tau^2/2) \quad \text{and} \quad X = \omega^{-2}(\tau - \tau^2/2).$$

These can be represented in a single expression form,

$$X = \omega^{-2}\left[\tau - sign(\tau)\frac{\tau^2}{2}\right], \qquad \tau = \tau(\omega t). \tag{10}$$

The period is $T = 4/\omega$, and the amplitude value is given by $x_{\max} = X(1) = \omega^{-2}/2$. Instead of the arbitrary parameter ω the parameter of amplitude could be used.

So expression (10) gives a one-parameter family of solutions. As was meantioned above, the second arbitrary parameter, say t_0, can be introduced into the solution by replacing $\tau(\omega t) \to \tau(\omega(t + t_0))$. This means the general solution of equation (2) has been obtained.

Now consider the oscillator of a more general form (1). In this case, equations (8) and (9) should be modified as

$$\frac{d^2X}{d\tau^2} = \omega^{-2}f(X) \quad \text{and} \quad \frac{d^2X}{d\tau^2} = -\omega^{-2}f(X) \tag{11}$$

under the same boundary conditions as the ones in equations (8) and (9) respectively.

Equations (11) cannot be solved so easily as equations (8) and (9). However taking into account that the new temporal variable τ is bounded as $|\tau| \leqslant 1$, one can obtain the solutions iteratively in a power series from with respect to τ. First, one should replace the function $f(X)$ by its truncated Maclaurin's series with respect to X. Then one can seek solution in the power series form

$$X = A\tau + c_2\tau^2 + c_3\tau^3 + c_4\tau^4 + c_5\tau^5 + \cdots .$$

A constant term of the series has been omitted due the boundary condition $X(0) = 0$. In the iterative process, the coefficient A will remain unknown and play the role of an arbitrary constant. All other coefficients, $c_2, c_3, \ldots$, are determined by the standard way after substitution the series into the differential equations and collecting the terms of the same power of τ. Note that due to the symmetry, one can obtain a solution for the second of the two equations (11) only. To make it valid for both equations, one should multiply all terms involving even degrees of the series by the function $sign(\tau)$ as it has been done in the particular case (10). A reason of this step is to provide the symmetry, $X(-\tau) = -X(\tau)$. The final result is

$$\begin{aligned} X = {} & A\tau + \frac{Af(0)f''(0)\tau^5}{40\omega^4} \\ & - sign(\tau)\left[\frac{f(0)\tau^2}{2\omega^2} + \frac{A^2f''(0)\tau^4}{24\omega^4}\right] + O(\tau^6), \\ \tau = {} & \tau(\omega(t + t_0)), \end{aligned} \tag{12}$$

where the parameters A and ω are coupled by the boundary condition at $\tau = 1$ (or $\tau = -1$) as

$$A - \frac{f(0)}{\omega^2} - \frac{A^2 f''(0)}{6\omega^4} + \frac{Af(0)f''(0)}{8\omega^4} = 0. \tag{13}$$

In equation (13), all terms generated by the terms $O(\tau^6)$ in equation (12) have been neglected.

It is seen that solution (12) includes two independent arbitrary constants, t_0 and either A or ω. Note that the parameter A gives a certain estimation for the amplitude of vibration, but it is not the exact amplitude (a correct value of the amplitude is given by $x_{\max} = X(1)$). In addition, the parameter ω is not a regular (trigonometric) frequency of the the oscillator because of the special normalization of the period or the saw-tooth sine, $\tau(\xi)$. The trigonometric frequency, ω_{trig}, is calculated by equating the periods, $T = 4/\omega = 2\pi/\omega_{trig}$. However, algebraic relation (13) can play a certain role in the non-linear frequency response.

Let us give the final remarks. Representation (3) in applicable in many other cases of oscillating systems as well. However, it is important to note that *the above transformation of time*, $t \rightarrow \tau$, *is not invertible over all periods of motion and hence representation* (3) *is not always correct*. For example, let us assume that the differential equation $\ddot{x} + f(x, \dot{x}) = 0$ possesses a periodic solution (the limit circle). In this case, representation (3) is applicable if function $f(x, \dot{x})$ includes only even powers of $\dot{x}$. In fact, due to equation (6), the even powers are functions of τ, whereas odd powers are not: $\dot{x}^{2n} = \omega^{2n}(\mathrm{d}X/\mathrm{d}\tau)^{2n}$, $\dot{x}^{2n-1} = \omega^{2n-1}(\mathrm{d}X/\mathrm{d}\tau)^{2n-1}\tau'(\omega t)$ ($n = 1, 2, \ldots$). For *any periodic motion*, a more general, two-components form of the representation [5] should be used.

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