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COMBINED FINITE ELEMENT-TRANSFER MATRIX METHOD

N. Bhutani and R. G. Loewy

School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332, *U.S.A.*

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1. INTRODUCTION

The advantages of finite element methods and transfer matrix methods have prompted researchers [1–4] to combine the two methods for dynamic analysis of chain-like structures. But in all these studies [1–4], the inertia effects were included in the finite element formulation. This has the disadvantage that the stiffness matrices are recomputed for every trial value of frequency when a trial and error search is used to determine eigenvalues. The approach presented in this paper proposes explicit formulation of elastic and mass matrices that are independent of each other. It follows that the elastic transfer matrix includes only the elastic properties of the substructure. Inertia forces and moments are introduced at the node points between the elastic transfer matrices by approximating distributed mass properties using a mass transfer matrix with concentrated mass, as is usually done in a one-dimensional transfer matrix method. The approach is demonstrated in this paper for the free vibration analysis of a triangular plate. Extension to analyze other complex structures is straightforward.

2. THEORY

Consider a structure which is divided into n substructures (see Figure 1(a)). The present approach assumes these substructures to be massless elastic bodies which are further discretized into finite elements. The mass of a substructure is lumped at its left and right boundaries. The elastic and mass transfer matrices for each substructure are developed independently of each other. This procedure is outlined as follows.

An assembled elastic matrix $[E_i]$ for an ith substructure (shown in Figure 1(a)) can be formulated using the finite element method and can be written as $[E_i]$ $\{\delta\} = \{F\}$. Since $\{\delta\}$ and $\{F\}$ are composed of displacements and forces on the (i)th and (i')th boundaries. the above equation can be rewritten as

$$[E_i]\begin{Bmatrix}\delta_i\\ \delta_{i'}\end{Bmatrix} = \begin{Bmatrix}F_i\\ F_{i'}\end{Bmatrix}. \tag{1}$$

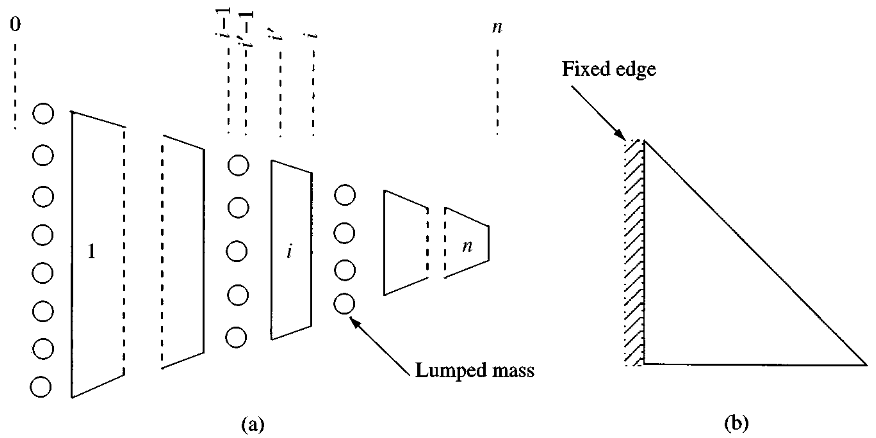

Figure 1. (a) A structure divided into n substructures, (b) a triangular cantilevered plate.

The forces on the right hand side of the above equation can brought to the left hand side and the elastic matrix can be rearranged and partitioned as

$$\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} \begin{Bmatrix} \delta_i \\ F_i \\ \cdots \\ \delta_{i'} \\ F_{i'} \end{Bmatrix} = \{0\}. \tag{2}$$

Equation (2) is partitioned so that $[E_1]$ is a square matrix and the top partition consists of as many equations as there are variables on the (i)th boundary. The upper partition of equation (2) can be solved for the state vector on the (i)th boundary to give

$$\begin{Bmatrix} \delta_i \\ F_i \end{Bmatrix} = -[E_1]^{-1}[E_2] \begin{Bmatrix} \delta_{i'} \\ F_{i'} \end{Bmatrix} \tag{3}$$

and can be substituted into remaining equations in the lower partition of equation (2) to yield

$$\begin{Bmatrix} \delta \\ F \\ 0 \end{Bmatrix} = [K_i] \begin{Bmatrix} \delta \\ F \end{Bmatrix}_{i'}, \quad \text{where} \quad [K_i] = \begin{bmatrix} -E_1^{-1}E_2 \\ E_4 - E_3E_1^{-1}E_2 \end{bmatrix}. \tag{4}$$

In the above equation, $[K_i]$ is the elastic transfer matrix for the ith substructure. The zero elements in the state vector on the left-hand side of equation (4) result when there are an unequal number of nodes on the left and the right boundaries of the substructure. The subscript of the parentheses, i.e., (i) or (i'), identifies the station or the boundary. It may be noted here that equation (3) requires inversion of a matrix $[E_1]$, which, in some cases may be singular. This problem can usually be overcome by interchanging rows, as their arrangement is arbitrary. Most important, the elastic transfer matrix $[K_i]$ developed above is independent of the trial frequency, hence avoiding the unnecessary calculation of elastic matrices for every trial value of the natural frequency.

The mass of the substructure is assumed to be lumped at the substructure interface (see Figure 1(a)) and the state vectors of the two adjacent substructures are related by the influence of a mass transfer matrix, $[M_i]$ [5]. For example, the mass matrix between the $(i'-1)$th and the $(i-1)$th stations can be written as

$$\begin{Bmatrix} \delta \\ F \end{Bmatrix}_{i'-1} = [M_i] \begin{Bmatrix} \delta \\ F \end{Bmatrix}_{i-1}, \quad \text{where } [M_i] = \begin{bmatrix} [I] & \vdots & [0] \\ \cdots & \cdots & \cdots \\ [\nwarrow m\omega^2 \searrow] & \vdots & [I] \end{bmatrix} \tag{5}$$

Here, ω is the vibratory frequency and m are the lumped masses which account for the mass of the surrounding elements of the adjacent substructures. For bending cases, $[M_i]$ could have additional terms representing the rotational mass moment of inertia.

Equations (4) and (5) formulated the elastic and mass transfer matrices, respectively, for any ith substructure. These equations can be combined to satisfy equilibrium and continuity conditions at (i')th and $(i'-1)$th stations to give the following relationship between (i)th and $(i-1)$th stations:

$$\begin{Bmatrix} \delta \\ F \\ 0 \end{Bmatrix}_{i} = [K_i][M_i] \begin{Bmatrix} \delta \\ F \end{Bmatrix}_{i-1}. \tag{6}$$

Similarly, we can formulate the elastic and mass transfer matrices of all the n substructures and successively substitute for intermediate state vectors to obtain a matrix which relates the state vector at the (n)th boundary to that at the (0)th boundary of the entire structure. This will appear as

$$\begin{Bmatrix} \delta \\ F \\ 0 \end{Bmatrix}_{n} = [D\} \begin{Bmatrix} \delta \\ F \end{Bmatrix}_{0}, \tag{7}$$

where $[D]$ is the matrix product $[K_n][M_n] \ldots [K_i][M_i] \ldots [K_1][M_1]$. Applying boundary conditions at the (0)th and (n)th boundaries yields the characteristic equation

$$[\bar{D}]\{U\}_0 = \{0\}, \tag{8}$$

where vector $\{U\}_0$ is a vector of the unknown displacements and forces, and $[\bar{D}]$, which is a submatrix of $[D]$, is a function of the natural frequencies of the entire structure. The correct natural frequencies are those at which the determinant of matrix $[\bar{D}]$ vanishes. A numerical example which demonstrates this approach is considered next.

3. NUMERICAL EXAMPLE

The example considered here is a flat, cantilevered, triangular plate shown in Figure 1 (b). The material and geometric properties are: Young's modulus $= 2{\cdot}07 \times 10^{11}$ Pa mass density $= 7850$ kg/m^3. Possion's ratio $= 0{\cdot}3$, isosceles side length $= 0{\cdot}254$ m, and plate thickness $= 1{\cdot}55 \times 10^{-3}$ m. The experimental value of the fundamental frequency given by Gustafson [6] is 34·5 Hz. Bathe and his colleagues [7] applied the standard finite element method based on the discrete

TABLE 1

A comparison of the present FE-TM method results with previous studies and ABAQUS for the case of a flat triangular cantilevered plate (Figure 1(b))

No. of segments/side	No. of elements	Mass d.o.f.	Fundamental frequency (Hz)		
				Present study	
			Previous studies	ABAQUS	FE-TMM
1	1	1	-	17·77	17·76
2	4	3	-	28·98	29·03
3	9	6	-	32·91	32·94
4	16	10	34·5 [7]	34·49	34·49
5	25	15	35·1 [4]	35·25	35·46
Experimental	—	—	34·5 [6]	—	—

Kirchhoff theory using a mesh with four segments per side. The value of the first natural frequency obtained was also 34·5 Hz. Chang [4] obtained a value of 35·1 Hz for this case, with five equal segments per side, using a combined Finite Element-Transfer Matrix (FE-TM) method. He used a distributed mass model with inertial matrices embedded in the finite element elastic formulation.

For the present study, five cases with increasing number of substructures were studied. For the first case, the plate structure was modeled as a single substructure; for the second case the structure is divided into two substructures and so on. Each substructure was further discretized into triangular elements such that the plate has an equal number of segments per side. A three-node plate element [8] was used to formulate the element elastic matrices for each substructure. A lumped mass representation as shown in Figure 1(a) was used. The mass of each substructure was assumed to be concentrated at the substructure interfaces only. The nodal mass m in the mass transfer matrix was calculated as $m = \frac{1}{3}m_t$, where m_t is the total mass of all the triangular elements connected to that node. Table 1 shows agreement between the frequencies calculated using the present FE-TM method with those from previous studies [4, 6, 7] and ABAQUS results. The frequency initially increases when the number of substructures is increased and then converges to 34·5 Hz. This effect is attributed to an increase in stiffness due to an increase in number of boundary nodes at the fixed edge.

4. CONCLUDING REMARKS

The present approach, a variation of the combined FE-TM method has shown satisfactory results for the fundamental natural frequency of a plate structure. The method can be used to extract higher frequencies and mode shapes [9]. This approach differs from the earlier versions of the FE-TM method in which it allows the formulation of mass and stiffness properties of a substructure as separate transfer matrices which can be stored as separate databases. The stiffness matrix for each substructure need only be calculated and stored once. During the root finding process, only the mass matrices need be recalculated, as only they are functions of the natural frequencies of the structure or of the trial frequencies used in the search procedure. Hence, using an explicit form for the mass matrices avoids repetitive computations. Storing data in separate databases can also facilitate the evaluation of specific design changes in a design optimization procedure. For example, partial transfers with assumed trial frequencies can be stored separately for the different substructures or components of complex structures, thereby making for very efficient calculation of dynamic characteristics associated with design changes (namely, geometric or material properties). Besides its potential for application in design optimization procedures, this approach can also be viewed as a reduction technique. Since the size of the characteristic matrix depends only on the number of the boundary nodes of the structure, an automatic reduction in the size of the final characteristic matrix is achieved. For example, for the fifth case of the cantilevered plate problem, the size of the characteristic matrix $[\bar{D}]$ is 18×18 as opposed to 45×45, if a standard finite element method is used. The computational efficiency and numerical accuracy aspects of the present approach are discussed in reference [9] and will be presented in subsequent publications.

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