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3-D THEORY VERSUS 2-D APPROXIMATE THEORY OF FREE ORTHOTROPIC (ISOTROPIC) PLATE AND SHELL VIBRATIONS, PART 1: DERIVATION OF GOVERNING EQUATIONS

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In Part 1 of the paper the basic assumptions of the shells with added masses considered are introduced. They serve for a derivation of the governing equations, which are analyzed in Part 2 of the paper.

1. INTRODUCTION

A great amount of literature is devoted to the topic of the reduction of three-dimensional (3-D) problems for two-dimensional (2-D) ones. A reason for that is mainly the easier calculations for practically oriented 2-D theories. However, an application range of the approximate theories is defined by a full 3-D theory.

For a general consideration of reduction of 3-D to 2-D theory a reader is referred to reference [1], where a few hundred works have been reviewed.

As has been pointed out in reference [2], especially for thicker plates shear deformation and rotary inertia effects become significant for the lower flexural modes. Form the point of view of 2-D plate theory, a higher order analysis is required to account for these effects. A three-dimensional analysis of the free vibrations of rectangular parallelepipeds was given in reference [3], in which the Ritz method was used.

The twisted parallelepiped was analyzed in reference [4].

During analysis of the attached mass influence on constructions with plates and shells, a few simple and fundamental methods have been used leading to the analysis of "shell-mass" or "plate-mass" models.

In general, it is assumed that masses joined with a plate or a shell are located on small surfaces, which is typical for technical situations. In a majority of the cases considered a contact surface is either a square or a rectangle (a shell's curvature is

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negligible). It has been shown that models with attached masses always have higher corresponding frequencies than purely continuous systems [5].

Both the concentrated mass and the force notions are abstractive. In the theory of plates and shells, the concentrated force notions are interpreted as a sequence of loads acting on the elementary surfaces approaching zero. A similar method is applied while solving the problems with attached masses. For a limited case describing the vibrating system "plate (shell) - attached stiff mass", when joint surfaces approach zero, the differential equations characterizing a point of mass - shell interaction have been derived [6].

One of the most effective methods revealing the dynamical characteristic of plates and shells with added masses is that supported by a generalized function theory. The concentrated masses effects are introduced to the input equations by using δ (Dirac) functions and the mass density is added to the plate density. Therefore, an inertia effect is included [7–9]. For instance, in reference [10] Lagrange's principle and the generalized function theory have been used to produce analyzed the differential equations.

The generalized function method has been also successfully applied to a wide class of different shells: shallow with two curvatures, closed cylindrical, spherical and others [10].

The differential equations obtained have been solved by using different methods. In references [12–16], a method of decomposition for eigenfunctions of the homogeneous problem has been used. In reference [17] the integral Fourier transform has been applied. By using the variational Ritz method many problems connected with the dynamics of shells with discrete masses or orthotropic and the isotropic closed cylindrical shells with attached masses have been solved [18].

In practice, especially at a design stage, simple methods are very profitable for defining the eigenfrequencies. In the literature, it is possible to find a description of approximate method applied to shells with small and large attached masses [19]. In the general case, eigenfrequencies and the corresponding modes of plates and shells with attached masses have been found by using complex algorithms.

Eigenfrequencies estimation of plates and shells on the basis of non-classical theories has led to the conclusion that low construction stiffness has caused an essential difference to the detriment of a classical theory. This indicates the desirability of developing new theories for the kind of problems in real constructions, especially composite constructions.

Many examples of the shell and plate theories from a point of view of asymptotic approaches can also be found in the reference [20].

Because the results presented here are obtained from Kirchhoff and Timoshenko theories, some comments on their nature are included in Appendix A.

2. CURVILINEAR ORTHOGONAL CO-ORDINATES

The following assumptions and hypotheses of the linear theory of an elastic anisotropic body are stated here, for further analysis:

1. A shell is shallow and also becomes shallow after deformations (the conditions of body continuity are not violated). Tension moments are negligible, and the tension tensor is symmetric.

2. Deformations are small; therefore relations between deformation components and their derivatives along co-ordinates are linear.

3. Hooke's principle of stress proportional to strain is valid. This means that relations between tension and deformation components are linear with constant coefficients.

4. The initial (possible) deformations are not taken into account.

Consider now an orthotropic shell with the thickness $2h$, the mean surface of which coincides with curvilinear and orthogonal co-ordinates α, β (α, β cover the main curvature lines of the mean shell surface). γ is the normal to α and β and it describes the distance along the normal from the point (α, β) to the point (α, β, γ).

For this co-ordinate system one obtains the following Lamé coefficients:

$$H_1 = A(1 + K_1\gamma), \qquad H_2 = A(1 + K_2\gamma), \qquad H_3 = 1. \tag{1}$$

Here $A = A(\alpha, \beta)$ and $B = B(\alpha, \beta)$ are the coefficients of the first second power form of the mean surfaces; $K_1 = K_1(\alpha, \beta)$, $K_2 = K_2(\alpha, \beta)$ are the main curvatures of a shell surface along lines $\alpha =$ constant, $\beta =$ constant.

The following formulae [21] define the Lamé coefficients:

$$H_1^2 = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2, \quad (\overleftrightarrow{1, 2, 3}), \quad (\overleftrightarrow{\alpha, \beta, \gamma}). \tag{2}$$

The notation $(\overleftrightarrow{\alpha, \beta, \gamma})$, $(\overleftrightarrow{x, y, z})$ denotes that other formulae are obtained by using a circular shift of the symbols.

The Lamé coefficients are independent and satisfy the following differential equations:

$$\begin{aligned}
&\frac{\partial}{\partial \alpha}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial \beta}\right) + \frac{1}{H_3^2}\frac{\partial H_1}{\partial \gamma}\frac{\partial H_1}{\partial \gamma} = 0,\\
&\frac{\partial}{\partial \beta}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial \beta}\right) + \frac{\partial}{\partial \gamma}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial \gamma}\right) + \frac{1}{H_1^2}\frac{\partial H_2}{\partial \alpha}\frac{\partial H_3}{\partial \alpha} = 0,\\
&\frac{\partial}{\partial \gamma}\left(\frac{1}{H_3}\frac{\partial H_1}{\partial \gamma}\right) + \frac{\partial}{\partial \alpha}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial \alpha}\right) + \frac{1}{H_2^2}\frac{\partial H_3}{\partial \beta}\frac{\partial H_1}{\partial \beta} = 0,\\
&\frac{\partial^2 H_1}{\partial \beta \partial \gamma} - \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\frac{\partial H_1}{\partial \beta} - \frac{1}{H_3}\frac{\partial H_3}{\partial \beta}\frac{\partial H_1}{\partial \gamma} = 0,\\
&\frac{\partial^2 H_2}{\partial \gamma \partial \alpha} - \frac{1}{H_3}\frac{\partial H_3}{\partial \alpha}\frac{\partial H_2}{\partial \gamma} - \frac{1}{H_1}\frac{\partial H_1}{\partial \gamma}\frac{\partial H_2}{\partial \alpha} = 0,\\
&\frac{\partial^2 H_3}{\partial \alpha \partial \beta} - \frac{1}{H_1}\frac{\partial H_1}{\partial \beta}\frac{\partial H_3}{\partial \alpha} - \frac{1}{H_2}\frac{\partial H_2}{\partial \alpha}\frac{\partial H_3}{\partial \beta} = 0.
\end{aligned} \tag{3}$$

Substituting for H_1, H_2, H_3 from equations (1) and (2) yields the Gauss formulae for a shell surface ($\gamma = 0$):

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial B}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial A}{\partial \beta}\right) = -K_1 K_2,$$

$$\frac{\partial}{\partial \beta}(AK_1) = K_2 \frac{\partial A}{\partial \beta}, \quad \frac{\partial}{\partial \alpha}(BK_2) = K_1 \frac{\partial B}{\partial \alpha}. \tag{4}$$

Some formulae necessary for future considerations of the elasticity theory are given below.

The u_1, u_2 and u_3 denote a full shell displacement vector projections for the tangent directions to the co-ordinates α, β, γ.

The deformable state of a three-dimensional shallow shell is characterized by six deformations $e_{11}, \ldots, e_{22}, \ldots, e_{23}$, connected with the displacement vector components due to the equations [19]

$$e_{11} = \frac{1}{H_1}\frac{\partial u_1}{\partial \alpha} + \frac{1}{H_1 H_2}\frac{\partial H_1}{\partial \beta} u_2 + \frac{\partial H_1}{\partial \gamma} u_3,$$

$$e_{22} = \frac{1}{H_2}\frac{\partial u_2}{\partial \beta} + \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma} u_3 + \frac{1}{H_1 H_2}\frac{\partial H_2}{\partial \alpha} u_1,$$

$$e_{33} = \frac{\partial u_3}{\partial \gamma},$$

$$e_{12} = \frac{H_1}{H_2}\frac{\partial}{\partial \beta}\left(\frac{1}{H_1} u_1\right) + \frac{H_2}{H_1}\frac{\partial}{\partial \alpha}\left(\frac{1}{H_2} u_2\right),$$

$$e_{13} = \frac{1}{H_1}\frac{\partial u_3}{\partial \alpha} + H_1 \frac{\partial}{\partial \gamma}\left(\frac{1}{H_1} u_1\right),$$

$$e_{23} = H_2 \frac{\partial}{\partial \gamma}\left(\frac{1}{H_2} u_2\right) + \frac{1}{H_2}\frac{\partial u_3}{\partial \beta}. \tag{5}$$

The rotary vector $\boldsymbol{\omega}$ has the following co-ordinates:

$$\omega_1 = \frac{1}{2H_2}\left(\frac{\partial u_3}{\partial \beta} - \frac{\partial H_2 u_2}{\partial \gamma}\right) = \frac{1}{2}(\operatorname{rot} \bar{u})_\alpha,$$

$$\omega_2 = \frac{1}{2H_1}\left(\frac{\partial H_1 u_1}{\partial \gamma} - \frac{\partial u_3}{\partial \alpha}\right) = \frac{1}{2}(\operatorname{rot} \bar{u})_\beta,$$

$$\omega_3 = \frac{1}{2H_1}\frac{1}{H_2}\left(\frac{\partial H_2 u_2}{\partial \alpha} - \frac{\partial H_1 u_1}{\partial \beta}\right) = \frac{1}{2}(\operatorname{rot} \bar{u})_\gamma. \tag{6}$$

A motion of the shell element $d\alpha d\beta d\gamma$ is governed by the equations [19]

$$\frac{\partial}{\partial\alpha}(H_2\sigma_{11}) - \sigma_{22}\frac{\partial H_2}{\partial\alpha} + \frac{1}{H_1}\frac{\partial}{\partial\beta}(H_1^2\sigma_{12}) + \frac{1}{H_1}\frac{\partial}{\partial\gamma}(H_1^2 H_2\sigma_{13}) = \varrho H_1 H_2\frac{\partial^2 u_1}{\partial t^2},$$

$$\frac{\partial}{\partial\beta}(H_1\sigma_{22}) - \sigma_{11}\frac{\partial H_1}{\partial\beta} + \frac{1}{H_2}\frac{\partial}{\partial\alpha}(H_2^2\sigma_{21}) + \frac{1}{H_2}\frac{\partial}{\partial\gamma}(H_1 H_2^2\sigma_{23}) = \varrho H_1 H_2\frac{\partial^2 u_2}{\partial t^2},$$

$$\frac{\partial}{\partial\gamma}(H_1H_2\sigma_{33}) - \sigma_{11}H_2\frac{\partial H_1}{\partial\gamma} - \sigma_{22}H_1\frac{\partial H_2}{\partial\gamma} + \frac{\partial}{\partial\alpha}(H_2\sigma_{13}) + \frac{\partial}{\partial\beta}(H_1\sigma_{23}) = \varrho H_1 H_2\frac{\partial^2 u_3}{\partial t^2}, \tag{7}$$

where σ_{11}, $\sigma_{22}, \ldots, \sigma_{23}$ are the tension co-ordinates related to the deformation co-ordinates e_{11}, $e_{22}, \ldots, e_{23}$ by using the general Hooke's principle.

Assume that at every point of a body they meet three perpendicular planes of an elastic symmetry, upon assuming that at every point of an anisotropic body the planes are perpendicular to the corresponding co-ordinates α, β, γ the general Hooke's principle equations have the form

$$\begin{aligned} &e_{11} = a_{1111}\sigma_{11} + a_{1122}\sigma_{22} + a_{1133}\sigma_{33}, \quad (\overleftrightarrow{1, 2, 3}),\\ &e_{12} = a_{1212}\sigma_{12}, \quad e_{13} = a_{1313}\sigma_{13}, \quad e_{23} = a_{2323}\sigma_{23}. \end{aligned} \tag{8}$$

In that case the number of independent elasticity constant coefficients a_{ijkl} is equal to nine and they are obtained from the relations

$$\begin{aligned} &a_{1111} = \frac{1}{E_1}, \quad a_{1122} = -\frac{\nu_{12}}{E_2}, \quad a_{1133} = -\frac{\nu_{13}}{E_3}, \quad a_{2211} = -\frac{\nu_{21}}{E_1},\\ &a_{2222} = \frac{1}{E_2}, \quad a_{2233} = -\frac{\nu_{23}}{E_3}, \quad a_{3311} = -\frac{\nu_{31}}{E_1}, \quad a_{3322} = -\frac{\nu_{32}}{E_2},\\ &a_{3333} = \frac{1}{E_3}, \quad a_{1313} = -\frac{1}{G_{13}}, \quad a_{2323} = \frac{1}{G_{23}}, \quad a_{1212} = \frac{1}{G_{12}}. \end{aligned} \tag{9}$$

Because the above equations are symmetric one has

$$E_2\nu_{21} = E_1\nu_{12}, \quad E_3\nu_{32} = E_2\nu_{23}, \quad E_1\nu_{13} = E_3\nu_{31}. \tag{10}$$

One defines a body as an orthotropic one if at each point three mutually perpendicular planes of an elastic symmetry meet.

One defines a plane as an isotropic one when all directions of its points are equivalent because of the elasticity properties. If the isotropic plane is attached to each body's point, then the material is defined as a transversely isotropic one, and the number of independent constant elasticity coefficients a_{ijkl} is reduced to the following ones:

$$a_{1111} = \frac{1}{E}, \quad a_{1122} = -\frac{v}{E}, \quad a_{1133} = -\frac{v'}{E'}, \quad a_{2222} = a_{1111},$$
$$a_{2233} = a_{1133}, \quad a_{3333} = \frac{1}{E'}, \quad a_{3322} = a_{2233},$$
$$a_{3311} = a_{1133}, \quad a_{1313} = \frac{1}{G'}, \quad a_{2323} = \frac{1}{G'}, \quad a_{1212} = \frac{2(1+v)}{E}. \tag{11}$$

Here E denotes Young's modulus for the isotropic plane directions and E' is the Young's modulus for the perpendicular to the isotropic plane directions; v is Poisson's coefficient characterizing the shortening in the isotropic plane due to the extension in the normal direction of the same plane; v' is an analogical Poisson coefficient in the direction normal to that plane, G' is the shear modulus for planes normal to the isotropic plane; $G = E/2(1+v)$ is the shear modulus for the planes parallel to the isotropic plane.

Solving the equations of the generalized Hooke's law according to the tension components $\sigma_{11}, \sigma_{22}, \ldots, \sigma_{23}$ yields the inverse formulas

$$\begin{aligned} &\sigma_{11} = A_{1111}e_{11} + A_{1122}e_{22} + A_{1133}e_{33}, \quad (\overleftrightarrow{1,2,3}),\\ &\sigma_{12} = A_{1212}e_{12}, \quad \sigma_{13} = A_{1313}e_{13},\\ &\sigma_{23} = A_{2323}e_{23}, \end{aligned} \tag{12}$$

where the stiffness coefficients have the forms

$$\begin{aligned} &A_{1111} = (a_{2222}a_{3333} - a_{2233}a_{3322})/\Delta, \quad (\overleftrightarrow{1,2,3}),\\ &A_{1122} = (a_{1133}a_{3322} - a_{1122}a_{3333})/\Delta, \quad (\overleftrightarrow{1,2,3}),\\ &A_{1212} = \frac{1}{a_{1212}}, \quad A_{1313} = \frac{1}{a_{1313}}, \quad A_{2323} = \frac{1}{a_{2323}},\\ &A_{2211} = A_{1122}, \quad (\overleftrightarrow{1,2,3}), \quad \Delta = \det[a_{iijj}]_{i,j=\overline{1,3}}. \end{aligned} \tag{13}$$

For transversely isotropic materials the above coefficients are

$$\begin{aligned} &A_{1111} = E(1 - v'v'')/\Omega, \quad A_{1122} = E(v'v'' + v)/\Omega,\\ &A_{1133} = Ev'(1+v)/\Omega, \quad A_{3333} = E'(1 - v^2)/\Omega,\\ &A_{2222} = A_{1111}, \quad A_{2233} = A_{1133}, \quad A_{1313} = G'\\ &A_{2323} = G', \quad A_{1212} = E/2(1+v), \quad v''E' = v'E,\\ &\quad \Omega = (1+v)(1 + 2v'v'' - v). \end{aligned} \tag{14}$$

Following a typical approach leading to the governing equations formations [19–21], one needs to apply Hamilton's principle expressed by volume integrals.

3. FUNDAMENTAL RELATIONS AND HYPOTHESES

Here the fundamental relationships further utilized during the dynamic model "shallow shell - attached mass" creation will be given. The necessary relations are obtained from the general equations of elasticity theory by taking into account many additional conditions.

Consider a shallow orthotropic shell which has a projected rectangular shape with sides a and b. Suppose that x and y are the Cartesian co-ordinates of the shell surface; then the second power of the linear element in the plane $x0y$ given by the relation

$$ds^2 = dx^2 + dy^2, \tag{15}$$

which defines the coefficients of the first second power form

$$A = B = 1. \tag{16}$$

For the shallow shell considered with the curvilinear orthogonal co-ordinate system α, β, γ one has

$$ds^2 = d\alpha^2 + d\beta^2, \tag{17}$$

which means that $A \approx 1$, $B \approx 1$.

Suppose that a mean surface is defined by $z = f(x, y)$. Thus, taking a mesh defined by $x =$ constant, $y =$ constant, from equations (16) one obtains

$$A = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2}, \quad B = \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2}. \tag{18}$$

A shell will be shallow enough [23] if at each point of the mean surface one has

$$(\partial f/\partial x)^2 \ll 1, \qquad (\partial f/\partial y)^2 \ll 1. \tag{19}$$

Therefore, in all relations of the previous section one can take $\alpha = x$, $\beta = y$, and the first second power form coefficients are equal to one. From differential geometry it is known that for the shallow shells one then has

$$k_1 = -\partial^2 f/\partial x^2, \quad k_2 = -\partial^2 f/\partial y^2, \quad k_{12} = \partial^2 f/\partial x \partial y, \tag{20}$$

where $k_{12} = 0$ if the co-ordinate axes coincide with the main curvature lines. In the latter case one has

$$k_1 = -\partial^2 f/\partial x^2, \quad k_2 = -\partial^2 f/\partial y^2, \quad k_{12} = 0. \tag{21}$$

The following additional assumptions are also made.

(a) In the first two equations the terms $k_1\sigma_{11}$ and $k_2\sigma_{12}$ are negligible.
(b) In relations between shear deformations and displacements the terms with curvature coefficients are negligible.

Finally, one obtains the following relationships and equations for a shallow shell

Motion equations:

$$\frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\sigma_{12}}{\partial y}+\frac{\partial\sigma_{13}}{\partial z}=\varrho\frac{\partial^2 u}{\partial t^2},\quad \overleftrightarrow{(1,2)},\overleftrightarrow{(u,v)},$$

$$\frac{\partial\sigma_{13}}{\partial x}+\frac{\partial\sigma_{23}}{\partial y}+\frac{\partial\sigma_{33}}{\partial z}-k_1\sigma_{11}-k_2\sigma_{22}=\varrho\frac{\partial^2 w}{\partial t^2}. \tag{22}$$

Geometrical relationships:

$$e_{11}=\frac{\partial u}{\partial x}+k_1 w,\quad \overleftrightarrow{(1,2)},\overleftrightarrow{(x,y)},$$

$$e_{12}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \overleftrightarrow{(1,2,3)},\overleftrightarrow{(x,y,z)},\quad e_{33}=\frac{\partial w}{\partial z}. \tag{23}$$

Rotary vector components:

$$\omega_1=\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right),\quad (1,2,3),(x,y,z). \tag{24}$$

The shell tensions are defined by relations (12). For $k_1=k_2=0$, equations (22)–(24) govern the plates' behaviour.

Suppose that a shallow moderately thick shell is loaded by an arbitrary number of the attached masses, situated on the rectangular elements ΔS $(i=1,\dots,N)$ on the shell's top surface, which are bounded by two line segment pairs $x=x_i-\tilde{c}_1^i$, $x=x_1-\tilde{\tilde{c}}_1^i$ and $y=y_i-\tilde{c}_2^i$, $y=y_i-\tilde{\tilde{c}}_2^i$, where x_i, y_i are the first two co-ordinates of the attached mass centre $O^i(x_i,y_i,z_i)$ (see Figure 1). Denoting $\tau_1^i=\tilde{\tilde{c}}_1^i/\tilde{c}_1^i$, $\tau_2^i=\tilde{\tilde{c}}_2^i/\tilde{c}_2^i$ (where τ_1^i, τ_2^i are characterized by a degree of deviation from its geometrical centre), for a homogeneous material one has $\tilde{\tilde{c}}_1^i=\tilde{c}_1^i$, $\tilde{\tilde{c}}_2^i=\tilde{c}_2^i$, because of the symmetry, which means that $\tau_1^i=\tau_2^i=1$. Denoting the ith added mass height by h_i does not introduce any additional constraints. A contact surface dimension $\tilde{c}_1^i(1+\tau_1^i)\tilde{c}_2^i(1+\tau_2^i)$ is small in comparison with the shell surface.

4. VARIATIONAL EQUATIONS

The variational Hamilton's principle will be used in a derivation of the differential equations governing a shell's dynamics as well as the boundary and initial conditions.

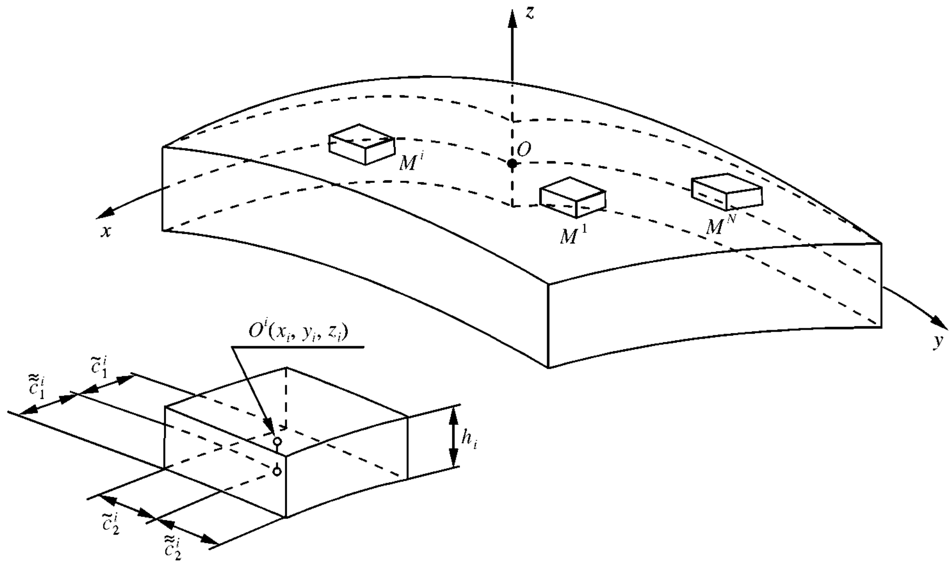

Figure 1. A moderately thick shell loaded by an arbitrary number of the added masses

Consider a motion process between two time points t_0 and t. One compares the different trajectories between these points. “Real” trajectories are defined by the condition

$$\int_{t_0}^{t} (\delta K + \delta A - \delta \Pi)\,\mathrm{d}t = 0. \tag{25}$$

Here δK denotes the kinetic energy variation, δA denotes the external forces work variation and $\delta \Pi$ is the potential energy deformation variation.

In the next considerations the focus is on the eigenfrequencies of either “shell - mass” or “plate - mass” system. Free vibration type of thin-walled structures characterizes its internal properties occurring as a result of external load action. Because of that observation and assuming a lack of external forces, one can take $\delta A = 0$. Taking into account the above assumptions one obtains the following Hamilton’s principle:

$$\int_{t_0}^{t} \delta(K - \Pi)\,\mathrm{d}t = 0. \tag{26}$$

Here $L = K - \Pi$ is th Lagrange function. This function in the case of “shell - mass” vibrating system, possesses the form

$$L = \frac{1}{2}\iiint_V \Big\}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$
-\frac{1}{2}\iiint_V (\sigma_{11}e_{11} + \sigma_{22}e_{22} + \sigma_{33}e_{33} + \sigma_{12}e_{12} + \sigma_{13}e_{13} + \sigma_{23}e_{23})\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{27}
$$

V denotes the total volume occupied by the shell and the attached masses. An original discrete-continuous construction, consisting of a moderately thick shell with constant thickness and the attached elements, is "changed" to one continuous model: a certain shell with changeable thickness. The L function is given by

$$
L = \frac{1}{2}\int_{-h}^{z^*}\int_0^a\int_0^b \varrho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
- \frac{1}{2}\int_{-h}^{z^*}\int_0^a\int_0^b (\sigma_{11}e_{11} + \sigma_{22}e_{22} + \sigma_{33}e_{33} + \sigma_{12}e_{12} + \sigma_{13}e_{13} + \sigma_{23}e_{23})\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{28}
$$

where

$$
z^* = h + \sum_{i=1}^{N} h_i \Theta^*(x, x_i, \tilde{c}_1^i, \tilde{\tilde{c}}_1^i)\cdot\Theta^{**}(y, y_i, \tilde{c}_2^i, \tilde{\tilde{c}}_2^i), \tag{29}
$$

$$
\begin{aligned}
\Theta^*(x, x_i, \tilde{c}_1^i, \tilde{\tilde{c}}_1^i) &= \Theta_1[x - (x_i - \tilde{c}_1^i)] - \Theta_2[x - (x_i + \tilde{\tilde{c}}_1^i)],\\
\Theta^{**}(y, y_i, \tilde{c}_2^i, \tilde{\tilde{c}}_2^i) &= \Theta_1[y - (y_i - \tilde{c}_2^i)] - \Theta_2[y - (y_i + \tilde{\tilde{c}}_2^i)],
\end{aligned} \tag{30}
$$

with Θ_1, Θ_2 being the characteristic functions with the properties

$$
\Theta_1 = \begin{Bmatrix} 0, & x < x_i - \tilde{c}_1^i \\ 1, & x \geqslant x_i - \tilde{c}_1^i \end{Bmatrix}, \qquad \Theta_2 = \begin{Bmatrix} 0, & x < x_i - \tilde{\tilde{c}}_1^i \\ 1, & x \geqslant x_i - \tilde{\tilde{c}}_1^i \end{Bmatrix}. \tag{31}
$$

Integrals occurring in equation (28) can be transformed to the forms

$$
L_1 = \frac{1}{2}\int_{-h}^{h}\int_0^a\int_0^b \varrho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
+ \frac{1}{2}\sum_{i=1}^{N}\int_h^{h+h_i}\int_0^a\int_0^b \varrho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\Theta^*\Theta^{**}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{32}
$$

$$L_2 = \frac{1}{2}\int_{-h}^{h}\int_0^a\int_0^b(\sigma_{11}e_{11} + \sigma_{22}e_{22} + \sigma_{33}e_{33} + \sigma_{12}e_{12} + \sigma_{13}e_{13}$$

$$+ \sigma_{23}e_{23})\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \frac{1}{2}\sum_{i=1}^{N}\int_{h}^{h+h_i}\int_0^a\int_0^b(\sigma_{11}e_{11}$$

$$+ \sigma_{22}e_{22} + \sigma_{33}e_{33} + \sigma_{12}e_{12} + \sigma_{13}e_{13}$$

$$+ \sigma_{23}e_{23})\Theta^*\Theta^{**}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{33}$$

As an example consider the second integral of equation (28) in the form

$$\frac{1}{2}\int_{-h}^{z^*}\int_0^a\int_0^b(\ldots)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{2}\int_{-h}^{h}\int_0^a\int_0^b(\ldots)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \frac{1}{2}\int_{-h}^{z^*}\int_0^a\int_0^b(\ldots)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{34}$$

where $(\ldots) = (\sigma_{11}e_{11} + \sigma_{22}e_{22} + \sigma_{33}e_{33} + \sigma_{12}e_{12} + \sigma_{13}e_{13} + \sigma_{23}e_{23})$.

The second term of equation (34) on the basis of equation (29) and upon using characteristic function properties, is expressed by

$$\frac{1}{2}\int_{h}^{z^*}\int_0^a\int_0^b(\ldots)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{2}\sum_{i=1}^{N}\int_{-h}^{h+h_i}\int_0^a\int_0^b(\ldots)\Theta^*\Theta^{**}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{35}$$

From equations (35) and (34) one obtains equations (32) and (33).

It must be emphasized that during the derivation of relationships (32) and (33) additional conditions are introduced because of the system “shallow shell - mass”, which proves that the considerations here are general. In addition, different shells and plates with added masses classified on the basis of different deformations could be obtained in the frame of the theory introduced here. And finally, this method can serve as a tool for the investigation of newly developed models and for accuracy investigations of existing ones.

To continue the investigations further and to build a “shell -mass” model, it is necessary to introduce certain physical and geometrical simplications. As the attached element one may take an absolutely stiff mass concentrated on a small surface. One is going to get high-accuracy results by taking additional forced terms caused by attached masses interaction. Up to now, this has been regarded as rather negligible by other researchers. Of course, this direction of investigations is not the only possibility since other proposed models are also available.

The assumptions about the added mass joints, which lead to the omitting of the masses internal deformation, allow for cancellation of the second integral in equation (33). It characterizes deformation energy of the attached masses. We are also going to transform the second term of equation (32), which characterizes the kinetic energy of the attached masses. First, we investigate a problem dealing with velocity distributions of the points inside a small volume covered by the attached mass in relation to a certain point $O^i(x_i, y_i, z_i)$ being the mass centre. It is assumed that the velocity field is continuous and has first order derivatives.

Suppose that the point $O^i(x_i, y_i, z_i)$ velocity is equal to $\bar{\xi}_0^i$, and the velocity of the point of the added mass is equal to $\bar{\xi}_{1i}^i(x, y, z)$. One can develop $\bar{\xi}_1^i$ in a neighbourhood of O^i taking account only of linear terms of ϱ_i, where $\varrho_i = |\bar{u} - \bar{u}_i|$, $\bar{u}_i = u(x_i, y_i, z_i)$ (curvature of the joint surface is negligible):

$$\xi_{1x}^i = \xi_{0x}^i + \frac{\partial \xi_{1x}^i}{\partial x}(x - x_i) + \frac{\partial \xi_{1x}^i}{\partial y}(y - y_i) + \frac{\partial \xi_{1x}^i}{\partial z}(z - z_i), \overleftrightarrow{(1, 2, 3)},$$

$$i = 1, \ldots, N \tag{36}$$

Thus, the following relationships are obtained from equation (36) for a displacement of an arbitrarily taken added mass point:

$$u_1^i = u_0^i + \frac{\partial \xi_{1x}^i}{\partial x}(x - x_i)\Delta t + \frac{\partial \xi_{1x}^i}{\partial y}(y - y_i)\Delta t + \frac{\partial \xi_{1x}^i}{\partial z}(z - z_i)\Delta t,$$

$$(\overleftrightarrow{1, 2, 3}), \quad i = 1, \ldots, N \tag{37}$$

Defining an arbitrarily taken point velocity by the mass centre velocity O^i from equation (36) one obtains

$$\xi_{1x}^i = \xi_{0x}^i + \frac{\partial \xi_{1x}^i}{\partial x}(x - x_i) + \frac{1}{2}\left(\frac{\partial \xi_{1x}^i}{\partial y} + \frac{\partial \xi_{1y}^i}{\partial x}\right)(y - y_i) + \frac{1}{2}\left(\frac{\partial \xi_{1x}^i}{\partial y} - \frac{\partial \xi_{1y}^i}{\partial x}\right)(y - y_i)$$

$$+ \frac{1}{2}\left(\frac{\partial \xi_{1x}^i}{\partial z} - \frac{\partial \xi_{1z}^i}{\partial x}\right)(z - z_i) + \frac{1}{2}\left(\frac{\partial \xi_{1x}^i}{\partial z} - \frac{\partial \xi_{1z}^i}{\partial x}\right)(z - z_i), (\overleftrightarrow{1, 2, 3}),$$

$$i = 1, \ldots, N. \tag{38}$$

In the above relationships both symmetric and antisymmetric tensors are used, which are defined in the three-dimensional vector by

$$\dot{\omega}_1^i = \frac{1}{2}\left(\frac{\partial \xi_{1z}^i}{\partial y} - \frac{\partial \xi_{1y}^i}{\partial z}\right), \quad (\overleftrightarrow{1, 2, 3}). \tag{39}$$

Introducing the following notation

$$\hat{\xi}_x^i = \frac{\partial \xi_{1x}^i}{\partial x}(x - x_i) + \frac{1}{2}\left(\frac{\partial \xi_{1x}^i}{\partial y} + \frac{\partial \xi_{1y}^i}{\partial x}\right)(y - y_i)$$

$$+ \frac{1}{2}\left(\frac{\partial \xi_{1x}^i}{\partial z} + \frac{\partial \xi_{1z}^i}{\partial x}\right)(z - z_i), \quad (\overleftrightarrow{1, 2, 3}), \tag{40}$$

and taking into account equations (39) and (40) in equation (38) one obtains

$$\xi^i_{1x} = \xi^i_{0x} + \hat{\xi}^i_x + \dot{\omega}^i_2(z - z_i) - \dot{\omega}^i_3(y - y_i), \quad (\overleftrightarrow{1, 2, 3}), \tag{41}$$

By using the symbolic notation

$$\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k} = \frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \xi^i_{1x} & \xi^i_{1y} & \xi^i_{1z} \end{vmatrix}. \tag{42}$$

equation (41) can be expressed in the form

$$\xi^i_{1x} = \xi^i_{0x} + \xi^i_x + [\boldsymbol{\omega} \times (\mathbf{u} - \mathbf{u}_i)]_x, \quad (\overleftrightarrow{1, 2, 3}). \tag{43}$$

To conclude, velocities of the attached mass points can be defined in the form of a three-component sum. The first one $\boldsymbol{\xi}^i_0$ does not depend on the co-ordinates x, y, z and the translatory motion velocity of a whole body is equal to the mass centre velocity. The second component is related to the relative extension and shear deformation velocities between the element and the added mass. If there are no deformations inside the added mass one has $\hat{\xi}^i_x = \hat{\xi}^i_y = \hat{\xi}^i_z = 0$ (velocity deformation tensor components are equal to zero).

The third component in the relationship (43) defines the components of the vector $\boldsymbol{\omega}^i$ describing an instantaneous angular velocity vector of the body treated as absolutely stiff. Taking into account the latter observation relationships (43) yields the form

$$\xi^i_{1x} = \xi^i_{0x} + [\boldsymbol{\omega}^i \times (\mathbf{u} - \mathbf{u}_i)]_x, \quad (\overleftrightarrow{1, 2, 3}), \tag{44}$$

Now, taking into account (44), one obtains the kinetic energy

$$\begin{aligned} L'' &= \frac{1}{2} \sum_{i=1}^{N} \int_{-h}^{h+h_i} \int_0^a \int_0^b \varrho \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right] \Theta^* \Theta^{**} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= \frac{1}{2} \sum_{i=1}^{N} \int_{-h}^{h+h_i} \int_0^a \int_0^b \varrho \{ (\xi^i_{0x})^2 + (\xi^i_{0y})^2 + (\xi^i_{0z})^2 + (\dot{\omega}^i_2)^2 [(z - z_i)^2 \\ &\quad + (x - x_i)^2] + (\dot{\omega}^i_1)^2 [(z - z_i)^2 + (y - y_i)^2] + (\dot{\omega}^i_3)^2 [(y - y_i)^2 \\ &\quad + (x - x_i)^2] - 2\dot{\omega}^i_2 \dot{\omega}^i_3 (z - z_i)(y - y_i) - 2\dot{\omega}^i_3 \dot{\omega}^i_1 (x - x_i)(y - y_i) \\ &\quad - 2\dot{\omega}^i_1 \dot{\omega}^i_2 (y - y_i)(x - x_i) \} \Theta^* \Theta^{**} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \end{aligned} \tag{45}$$

The ith added mass dimension with the co-ordinates x_i, y_i, z_i is given by $\tilde{c}^i_1(1 + \tau^i_1)\tilde{c}^i_2(1 + \tau^i_2)$. Multiplying and dividing equation (45) by that quantity and remembering that $\mathrm{d}V = \tilde{c}_1^i(1+\tau_1^i)\tilde{c}_2^i(1+\tau_2^i)\,\mathrm{d}z$, after integration of equation (45) one gets

$$L_1'' = \frac{1}{2}\sum_{i=1}^{N}\int_0^a\int_0^b \{\tilde{M}^i[(\xi_{0x}^i)^2 + (\xi_{0y}^i)^2 + (\xi_{0z}^i)^2] + \tilde{J}_{xx}^i(\dot{\omega}_2^i)^2 + \tilde{J}_{yy}^i(\dot{\omega}_2^i)^2$$

$$+ \tilde{J}_{zz}^i(\dot{\omega}_3^i)^2 - 2\tilde{J}_{xy}^i\dot{\omega}_1^i\dot{\omega}_2^i - 2\tilde{J}_{xz}^i\dot{\omega}_1^i\dot{\omega}_3^i - 2\tilde{J}_{yx}^1\dot{\omega}_2^i\dot{\omega}_3^i\}\,\Theta^*\Theta^{**}\,\mathrm{d}x\,\mathrm{d}y. \quad (46)$$

Here $\tilde{M}^i$, $\tilde{J}_{xx}^i$, $\tilde{J}_{yy}^i$, $\tilde{J}_{zz}^i$, $\tilde{J}_{xy}^i$, $\tilde{J}_{yz}^i$, $\tilde{J}_{xz}^i$, denote masses and mass inertia moments of the attached mass related to a unit joint contact surface uniquely distributed on it.

In a limited case, when the concentrated masses are located on a shell one can use δ functions. This means that in equation (46) one can take $\tilde{c}_1^i \to 0$, $\tilde{c}_2^i \to 0$, and in addition obtain

$$\lim_{\tilde{c}_1^i\to 0}\frac{\Theta^*(x, x_i, \tilde{c}_1^i, \tilde{\tilde{c}}_1^i)}{\tilde{c}_1^i(1+\tau_1^i)} = \delta(x - x_i), \quad \lim_{\tilde{c}_2^i\to 0}\frac{\Theta^{**}(y, y_i, \tilde{c}_2^i, \tilde{\tilde{c}}_2^i)}{\tilde{c}_2^i(1+\tau_2^i)} = \delta(y - y_i). \quad (47)$$

Then, L_1'' is transformed to the form

$$L_1'' = \frac{1}{2}\sum_{i=1}^{N}\int_0^a\int_0^b \{M^i[(\xi_{0x}^i)^2 + (\xi_{0y}^i)^2 + (\xi_{0z}^i)^2] + J_{xx}^i(\dot{\omega}_1^i)^2$$

$$+ J_{yy}^i(\dot{\omega}_2^i)^2 + J_{zz}^i(\dot{\omega}_3^i)^2 - 2J_{xy}^i\dot{\omega}_1^i\dot{\omega}_2^i - 2J_{xz}^i\dot{\omega}_1^i\dot{\omega}_3^i$$

$$- 2J_{yx}^1\dot{\omega}_2^i\dot{\omega}_3^i\}\,\delta(x - x_i)\delta(y - y_i)\mathrm{d}x\,\mathrm{d}y, \quad (48)$$

where ω_J^i are defined by equation (39).

Substituting equation (46) to equation (32) and equation (33) into equation (28) yields the Lagrange function of the "shallow shell - concentrated mass" system:

$$L = \frac{1}{2}\int_{-h}^{h}\int_0^a\int_0^b \varrho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$+ \frac{1}{2}\sum_{i=1}^{N}\int_0^a\int_0^b \{M^i[(\xi_{0x}^i)^2 + (\xi_{0y}^i)^2 + (\xi_{0z}^i)^2] + J_{xx}^i(\dot{\omega}_1^i)^2$$

$$+ J_{yy}^i(\dot{\omega}_2^i)^2 + J_{zz}^i(\dot{\omega}_3^i)^2 - 2J_{xy}^i\dot{\omega}_1^i\dot{\omega}_2^i - 2J_{xz}^i\dot{\omega}_1^i\dot{\omega}_3^i$$

$$- 2J_{yx}^1\dot{\omega}_2^i\dot{\omega}_3^i\}\,\delta(x - x_i)\delta(y - y_i)\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_{-h}^{h}\int_0^a\int_0^b(\sigma_{11}e_{11}$$

$$+ \sigma_{22}e_{22} + \sigma_{33}e_{33} + \sigma_{12}e_{12} + \sigma_{13}e_{13} + \sigma_{23}c_{23})\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \quad (49)$$

A variational principle (25) related to equation (49) should be formulated by using the following additional conditions: relationships of the generalized Hooke's law (12) and the geometrical relationships (33). In addition, one needs expressions for the mass centre velocity vector components ξ_0^i and the angular velocity vector components $\boldsymbol{\omega}^i$.

To conclude, one has obtained an important relationship (49) fully described by energetic characteristics of the investigated system in the three-dimensional space of displacements.

Next, one can derive motion equations of a shallow shell with attached concentrated masses.

First one defines potential energy variations $\delta \Pi$ due to the shell's deformation. Taking into account the relationships

$$\delta\left(\frac{\partial u}{\partial x}\right) = \frac{\partial(\delta u)}{\partial x}$$

and the integration by parts rule one gets

$$\iiint_V \left[\frac{\partial(\delta u)}{\partial x}\sigma_{11} + \left(\frac{\partial(\delta u)}{\partial y} + \frac{\partial(\delta v)}{\partial x}\right)\sigma_{12} + \left(\frac{\partial(\delta u)}{\partial z} + \frac{\partial(\delta w)}{\partial x}\right)\sigma_{13} + \cdots\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \iint_s [(\sigma_{11}l + \sigma_{12}m + \sigma_{13}n)\delta u + \cdots]\mathrm{d}x\,\mathrm{d}y$$
$$- \iiint_V \left[\left(\frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\sigma_{22}}{\partial y} + \frac{\partial\sigma_{13}}{\partial z}\right)\delta u + \cdots\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Grouping the terms in δu, δv and δw one obtains

$$\delta\Pi = -\int_{-h}^{h}\int_0^a\int_0^b \left[\left(\frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\sigma_{12}}{\partial y} + \frac{\partial\sigma_{13}}{\partial z}\right)\delta u + \left(\frac{\partial\sigma_{12}}{\partial x} + \frac{\partial\sigma_{22}}{\partial y} + \frac{\partial\sigma_{23}}{\partial z}\right)\delta v\right.$$
$$\left. + \left(\frac{\partial\sigma_{13}}{\partial x} + \frac{\partial\sigma_{23}}{\partial y} + \frac{\partial\sigma_{33}}{\partial z} - k_1\sigma_{11} - k_2\sigma_{22}\right)\delta w\right] + \int_{-h}^{h}\int_0^b [\sigma_{11}\delta u$$
$$+ \sigma_{12}\delta v + \sigma_{13}\delta w]|_0^a\,\mathrm{d}y\,\mathrm{d}z + \int_{-h}^{h}\int_0^a [\sigma_{22}\delta v + \sigma_{12}\delta u + \sigma_{23}\delta w]|_0^b\,\mathrm{d}x\,dz$$
$$+ \int_0^a\int_0^b [\sigma_{33}\delta w + \sigma_{13}\delta u + \sigma_{23}\delta v]|_{-h}^{h}\,\mathrm{d}x\,\mathrm{d}y. \qquad (50)$$

Consider a variation of the kinetic energy of the system "shallow shell - mass". From equations (45) and (32) one obtains the following value of the kinetic energy:

$$K = \frac{1}{2}\int_{-h}^{h}\int_0^a\int_0^b \varrho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$+\sum_{i=1}^{N}\int_0^a\int_0^b\{M^i[(\xi_{0x}^i)^2+(\xi_{0y}^i)^2+(\xi_{0z}^i)^2]+J_{xx}^i(\dot\omega_1^i)^2$$

$$+J_{yy}^i(\dot\omega_2^i)^2+J_{zz}^i(\dot\omega_3^i)^2-2J_{xy}^i\dot\omega_1^i\dot\omega_2^i-2J_{xz}^i\dot\omega_1^i\dot\omega_3^i$$

$$-2J_{yx}^1\dot\omega_2^i\dot\omega_3^i\}\,\delta(x-x_i)\delta(y-y_i)\mathrm{d}x\,\mathrm{d}y. \tag{51}$$

One can now characterize the values ξ_{0x}^i, ξ_{0y}^i, ξ_{0z}^i, $\dot\omega_1^i$, $\dot\omega_2^i$, $\dot\omega_3^i$. The following law of the displacements change because the ith mass thickness is assumed:

$$u_i=u^h-(z_i-h)\frac{\partial w^h}{\partial x},\quad v_i=v^h-(z_i-h)\frac{\partial w^h}{\partial y},\quad w_i=w^h, \tag{52}$$

where u^h, v^h, w^h are the displacements of a top shell surface in contact with the ith attached mass. Taking into account equation (39) one obtains the relationships

$$\xi_{0x}^i=\frac{\partial u^h}{\partial t}-(z_i-h)\frac{\partial^2 w^h}{\partial x\partial t},$$

$$\xi_{0y}^i=\frac{\partial y^h}{\partial t}-(z_i-h)\frac{\partial^2 w^h}{\partial y\partial t},$$

$$\xi_{0z}^i=\frac{\partial w^h}{\partial t},\quad \dot\omega_1^i=\frac{\partial^2 w^h}{\partial y\partial t},$$

$$\dot\omega_2^i=\frac{\partial^2 w^h}{\partial x\partial t},\quad \dot\omega_3^i=\frac{1}{2}\left[\frac{\partial^2 v^h}{\partial x\partial t}-\frac{\partial^2 u^h}{\partial y\partial t}\right]. \tag{53}$$

Coming back to integral of the kinetic energy variation of a shell with attached masses described by equations (51) and taking into account equations (53) one gets

$$\int_{t_0}^{t}\delta K\,\mathrm{d}t=\int_{t_0}^{t}\Bigg(\sum_{i=1}^{N}\int_0^a\int_0^b\Bigg\{M^i\Bigg[\frac{\partial u}{\partial t}\frac{\partial(\delta u)}{\partial t}+(z_i-h)^2\frac{\partial^2 w}{\partial x\partial t}\frac{\partial^2(\delta w)}{\partial x\partial t}$$

$$-\frac{\partial(\delta u)}{\partial t}(z_i-h)\frac{\partial^2 w}{\partial x\partial t}-\frac{\partial u}{\partial t}(z_i-h)\frac{\partial^2(\delta w)}{\partial x\partial t}+\frac{\partial v}{\partial t}\frac{\partial(\delta v)}{\partial t}+(z_i-h)^2\frac{\partial^2 w}{\partial y\partial t}$$

$$\times\frac{\partial^2(\delta w)}{\partial y\partial t}-\frac{\partial(\delta w)}{\partial t}(z_i-h)\frac{\partial^2 w}{\partial y\partial t}-\frac{\partial^2 v}{\partial t^2}(z_i-h)\frac{\partial^2(\delta w)}{\partial y\partial t}+\frac{\partial w}{\partial t}\frac{\partial(\delta w)}{\partial t}\Bigg]$$

$$+J_{xx}^i\frac{\partial^2 w}{\partial y\partial t}\frac{\partial^2(\delta w)}{\partial y\partial t}+J_{yy}^i\frac{\partial^2 w}{\partial x\partial t}\frac{\partial^2(\delta w)}{\partial x\partial t}+\frac{1}{4}J_{zz}^i\left[\frac{\partial^2 v}{\partial x\partial t}-\frac{\partial^2 u}{\partial y\partial t}\right]\Bigg(\frac{\partial^2(\delta v)}{\partial x\partial t}$$

$$-\frac{\partial^2(\delta u)}{\partial y\partial t}\Bigg)+J_{xy}^i\frac{\partial^2(\delta w)}{\partial y\partial t}\frac{\partial^2 w}{\partial x\partial t}+J_{xy}^i\frac{\partial^2 w}{\partial y\partial t}\frac{\partial^2(\delta w)}{\partial x\partial t}-J_{xz}^i\frac{\partial^2(\delta w)}{\partial y\partial t}\Bigg[\frac{\partial^2 v}{\partial x\partial t}$$

$$-\frac{\partial^2 u)}{\partial y \partial t}\bigg] - J^i_{xz}\frac{\partial^2 w}{\partial y \partial t}\bigg[\frac{\partial^2(\delta v)}{\partial x \partial t} - \frac{\partial^2(\delta u)}{\partial y \partial t}\bigg] + J^i_{yz}\frac{\partial^2(\delta w)}{\partial x \partial t}\bigg[\frac{\partial^2 v}{\partial x \partial t} - \frac{\partial^2 u}{\partial y \partial t}\bigg]$$

$$+ J^i_{yz}\frac{\partial^2 w}{\partial x \partial t}\bigg[\frac{\partial^2(\delta v)}{\partial x \partial t} - \frac{\partial^2(\delta u)}{\partial y \partial t}\bigg]\bigg\}\bigg|_{z=h}\delta(x - x_i)\delta(y - y_i)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$+ \int_{t_0}^{t}\int_{-h}^{h}\int_0^a\int_0^b \varrho\bigg[\frac{\partial u}{\partial t}\frac{\partial(\delta u)}{\partial t} + \frac{\partial v}{\partial t}\frac{\partial(\delta v)}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial(\delta w)}{\partial t}\bigg]\bigg|_{z=h}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t. \tag{54}$$

After equation (54) components transformations using an integral by parts rule and substituting the obtained expression together with equation (50) into equation (26) one has

$$\int_{t_0}^{t}\int_0^a\int_0^b\int_{-h}^{h}\bigg[\bigg(\frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\sigma_{12}}{\partial y} + \frac{\partial\sigma_{13}}{\partial z} - \varrho\frac{\partial^2 u}{\partial t^2}\bigg)\delta u + \bigg(\frac{\partial\sigma_{12}}{\partial x} + \frac{\partial\sigma_{22}}{\partial y} + \frac{\partial\sigma_{23}}{\partial z} - \varrho\frac{\partial^2 v}{\partial t^2}\bigg)\delta v$$

$$+ \bigg(\frac{\partial\sigma_{13}}{\partial x} + \frac{\partial\sigma_{23}}{\partial y} + \frac{\partial\sigma_{33}}{\partial z} - k_1\sigma_{11} - k_2\sigma_{22} - \varrho\frac{\partial^2 w}{\partial t^2}\bigg)\delta w\bigg] - \int_{t_0}^{t}\int_0^b\int_{-h}^{h}(\sigma_{11}\delta u$$

$$+ \sigma_{12}\delta v + \sigma_{13}\delta w)|_0^a\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t - \int_{t_0}^{t}\int_0^a\int_{-h}^{h}(\sigma_{22}\delta v + \sigma_{12}\delta u + \sigma_{23}\delta w)|_a^b\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t$$

$$- \int_{t_0}^{t}\int_0^a\int_0^b\bigg\{\sigma_{13} - \sum_{i=1}^{N}\bigg\{M^i\bigg[-\frac{\partial^2 u}{\partial t^2} + (z_i - h)\frac{\partial^3 w}{\partial x \partial t^2}\bigg]\delta(x - x_i)\delta(y - y_i)$$

$$-\frac{1}{4}J^i_{zz}\frac{\partial}{\partial y}\bigg[\delta(y - y_i)\bigg(\frac{\partial^3 v}{\partial x \partial t^2} - \frac{\partial^3 u}{\partial y \partial t^2}\bigg)\bigg] + \frac{1}{2}J^i_{xz}\frac{\partial}{\partial y}\bigg[\delta(y - y_i)\frac{\partial^3 w}{\partial y \partial t}\bigg]\delta(x - x_i)$$

$$-\frac{1}{2}J^i_{yz}\frac{\partial}{\partial y}\bigg[\delta(y - y_i)\frac{\partial^3 w}{\partial x \partial t^2}\bigg]\delta(x - x_i)\bigg\}\delta u\bigg|_{z=h} - \sigma_{13}\delta u|_{z=-h} + \bigg\{\sigma_{23} - \sum_{i=1}^{N} M^i$$

$$\times\bigg[-\frac{\partial^2 v}{\partial t^2} + (z_i - h)\frac{\partial^3 w}{\partial y \partial t^2}\bigg]\delta(x - x_i)\delta(y - y_i) + \frac{1}{4}J^i_{zz}\frac{\partial}{\partial x}\bigg[\delta(x - x_i)\bigg(\frac{\partial^3 v}{\partial x \partial t^2}$$

$$-\frac{\partial^3 u}{\partial y \partial t^2}\bigg)\bigg]\delta(y - y_i) - \frac{1}{2}J^i_{xz}\frac{\partial}{\partial x}\bigg[\delta(x - x_i)\frac{\partial^3 w}{\partial y \partial t^2}\bigg]\delta(y - y_i) + \frac{1}{2}J^i_{yz}\frac{\partial}{\partial x}\bigg[\delta(x - x_i)$$

$$\times\frac{\partial^3 w}{\partial x \partial t^2}\bigg]\delta(y - y_i)\bigg\}\delta v\bigg|_{z=h} - \sigma_{23}\delta v|_{z=-h} + \bigg\{\sigma_{33} - \sum_{i=1}^{N} M^i\bigg[-\frac{\partial^2 w}{\partial t^2} + (z_i - h)^2$$

$$\times\frac{\partial}{\partial x}\bigg[\delta(x - x_i)\frac{\partial^3 w}{\partial x \partial t^2}\bigg]\delta(y - y_i) - (z_i - h)\frac{\partial}{\partial x}\bigg[\delta(x - x_i)\frac{\partial^2 u}{\partial t^2}\bigg]\delta(y - y_i)$$

$$\times (z_i - h)^2 \frac{\partial}{\partial y}\left[\delta(y - y_i)\frac{\partial^3 w}{\partial y \partial r^2}\right]\delta(x - x_i) - (z_i - h)\frac{\partial}{\partial y}\left[\delta(y - y_i)\frac{\partial^2 v}{\partial t^2}\right]\delta(x - x_i)$$

$$+ J^i_{xx}\frac{\partial}{\partial y}\left[\delta(y - y_i)\frac{\partial^3 w}{\partial y \partial t^2}\right]\delta(x - x_i) + J^i_{yy}\frac{\partial}{\partial x}\left[\delta(x - x_i)\frac{\partial^3 w}{\partial x \partial t^2}\right]\delta(y - y_i)$$

$$+ J^i_{xy}\frac{\partial}{\partial y}\left[\delta(y - y_i)\frac{\partial^3 w}{\partial x \partial t^2}\right]\delta(x - x_i) + J^i_{xy}\frac{\partial}{\partial x}\left[\delta(x - x_i)\frac{\partial^3 w}{\partial y \partial t^2}\right]\delta(y - y_i)$$

$$-\frac{1}{2}J^i_{xz}\frac{\partial}{\partial y}\left[\delta(y - y_i)\left(\frac{\partial^3 v}{\partial x \partial t^2} - \frac{\partial^3 u}{\partial y \partial t}\right)\right]\delta(x - x_i)$$

$$+\frac{1}{2}J^i_{yz}\frac{\partial}{\partial x}\left[\delta(x - x_i)\left(\frac{\partial^3 v}{\partial x \partial t^2} - \frac{\partial^3 u}{\partial y \partial t}\right)\right]\delta(y - y_i)\Bigg\}\delta w\Bigg|_{z=h} - \sigma_{33}\delta w|_{z=-h}\Bigg\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$+ \int_{-h}^{h}\int_0^a\int_0^b \left[\varrho\left(\frac{\partial u}{\partial t}\delta u + \frac{\partial v}{\partial t}\delta v + \frac{\partial w}{\partial t}\delta w\right)\right]_{t=0}^{t} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (55)$$

5. CONCLUSIONS

The assumptions and hypotheses of the three-dimensional theory of orthotropic shallow shell with attached masses have been formulated. Then the variational Hamilton's principle has been used for a derivation of the differential equations governing a shell's dynamics. Special attention has been used for a derivation of the differential equations governing a shell's dynamics. Special attention has been paid to providing a proper model of the concentrated stiff mass additives (the masses and mass inertial moments related to a unit joint contact surface have been taken in account). Finally, the variational equations are derived from which the motion equations, can be derived. These will be obtained and analyzed further in Part 2 of the paper.

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APPENDIX A

Because some kinematic models of continuous systems will be used further, their fundamental assumptions are briefly discussed.

A.1. BEAMS (BERNOULLI–EULER HYPOTHESIS)

The x axis has the same direction as the beam axis, whereas the y and x-axis correspond to the main cross-sectional axes (a right-hand rectangular co-ordinate system is applied). Because, according to the beam definition, a longitudinal beam diameter is considerably larger than its two other diameters, the following assumptions are applied.

First, the following stress components σ_y, σ_z and τ_{yz} are negligible with

$$\sigma_y = \sigma_z = \tau_{yz} = 0. \tag{A1}$$

Second, the cross-sections perpendicular to the beam axis before bending become flat and perpendicular to a new bended axes and they are not deformable in their plane:

$$\mathbf{U} = \mathbf{U}_0 + z(\mathbf{n} - \mathbf{i}_3), \tag{A2}$$

where $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are the unit vectors in the x, y, z directions, respectively; $\mathbf{n}$ is the unit normal to the deformable beam axis, $\mathbf{U}_0 = \mathbf{U}\mathbf{i}_1 + \mathbf{W}\mathbf{i}_3$, where $U(x)$, $W(x)$.

If only the linear theory is used, then equation (A2) has the form

$$v = U - zW', \quad V = 0, \quad w = W.$$

Next, one can formulate a theory taking into account the deformation associated with transverse shear.

The displacement vector $\mathbf{U}$ can be developed into a series in the neighbourhood of $z = 0$:

$$\mathbf{U}(x, y, z) = \mathbf{U}(x, y, 0) + z\left(\frac{\partial \mathbf{U}}{\partial z}\right)_{z=0} + \frac{1}{2!}z^2\left(\frac{\partial^2 \mathbf{U}}{\partial z^2}\right)_{z=0} + \cdots. \tag{A3}$$

This means that a simple expression including transverse shear deformation has the form

$$\mathbf{U} = \mathbf{U}_0 + z\mathbf{U}_1, \tag{A4}$$

where $\mathbf{U}_1$ is defined as

$$\mathbf{U}_1 = U_1\mathbf{i}_1 + W_1\mathbf{i}_3 \tag{A5}$$

and U_1, W_1 depend on x. These are four degrees of freedom: U, W, U_1 and W_2. If again equation (A1) and the known stress and deformation are used, then the equation

$$2e_{zz} = U_1^2 + (1 + W_1)^2 - 1 = 0 \tag{A6}$$

can be used as an additional geometrical constraint, decreasing the number of degrees of freedom to three. Both equations (A4) and (A6) imply that the transversal cross-sections perpendicular to the non-deformable axis are flat and they are not deformable in their plane, although they are perpendicualr to the deformable axis.

For small displacements (linearization defined by equation (A6)) one has

$$W_1 = 0, \tag{A7}$$

$$v = U + zU_1, \quad V = 0, \quad w = W, \tag{A8}$$

which leads to the following non-zero deformation components

$$\varepsilon_x = U' + zU_1, \quad \gamma_{xz} = W' + U_1. \tag{A9}$$

The Bernoulli–Euler hypothesis leads to the following constraint: $U_1 = W'$.
The hypotheses described can be generalized for models of plates and shells.

A.2. PLATES

The co-ordinates of the z- and y-axis lie in one plane, the z-axis overlaps the normal to this plane, and x, y, z create a right-hand co-ordinate system (in our case the left-hand co-ordinates are used).

The following Kirchhoff hypotheses are used.

1. A transversal normal stress is negligible in comparison with other stress components:

$$\sigma_z = 0. \tag{A10}$$

2. A linear plate element, initially perpendicular to the average surface, becomes perpendicular to the deformable average surface and does not undergo any stretching. Relation (A2) is still valid, where $U(x, y)$, $v(x, y)$, $W(x, y)$. There are three-degrees-of-freedom (U, v, W). For small displacements (see equations (A2)) one has

$$v = U - zW'_x, \quad V = v - zW'_y, \quad w = W, \tag{A11}$$

whereas the deformations are

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0. \tag{A12}$$

Thus, the theory taking into account shear transverse deformation is simply a generalization of the shear transversal deformation beam theory. Equations (A3) and (A4) are valid, but $\mathbf{U}_1$ is defined as

$$\mathbf{U}_1 = U_1\mathbf{i}_1 + v_1\mathbf{i}_2 + W_1\mathbf{i}_3. \tag{A13}$$

where $U_1(x, y)$, $v_1(x, y)$, $W_1(x, y)$. Now, one has six-degrees of freedom (U, v, W, U_1, v_1, W_1). Equation (A8) is transformed into the form

$$2e_{zz} = U_1^2 + v_1^2 + (1 + W_1)^2 - 1 = 0, \tag{A14}$$

which reduces the number of degrees of freedom to five. This means that linear elements perpendicular to the non-deformable average surface become straight and non-deformable, but they are not perpendicualr to the deformable average surface.

For small displacements, equation (A14) is linearized, $W_1 = 0$, and

$$v = U + zU_1, \quad V = v + zV_1, \quad w = W_1. \tag{A15}$$

A.3. SHELLS

Now, a system of the curvilinear co-ordinates α and β located on the average surface S_m overlapping the main curvature lines is applied.

One has the relation

$$\mathbf{i}^{(0)} = \mathbf{i}_0^{(0)}(\alpha, \beta) + \zeta_1 \mathbf{n}^{(0)}(\alpha, \beta), \tag{A16}$$

where ζ_1 is a distance between a point on the shell and the average surface, $\mathbf{i}^{(0)}$ is the radius vector of this point, and $\mathbf{n}^{(0)}$ is the unit vector perpendicular to the average surface S_m. The hypothesis takes into account the effect of shear transversal deformation for shells:

$$\sigma_\zeta = 0, \quad \mathbf{U} = \mathbf{U}_0 + \zeta \mathbf{U}_1. \tag{A17, A18}$$

The following geometric constraint is applied:

$$e_{\zeta\zeta} = U_1^2 + v_1^2 + (1 + W_1)^2 - 1 = 0. \tag{A19}$$

In a case of small deformation (linearization defined by equation (A19)), one obtains

$$W_1 = 0, \tag{A20}$$

$$v = U + \zeta U_1, \quad V = v + \zeta v_1, \quad w = W, \tag{A21}$$

which generalizes the results of equation (A15).

In the frame of the Kirchhoff–Love hypothesis an arbitrary point of the average surface with the co-ordinates $(\alpha, \beta, 0)$ before $\mathbf{i}_0^{(0)}$ and after $(\mathbf{i}_0)$ deformation is governed by the equations

$$\mathbf{i}_0 = \mathbf{i}_0^{(0)} + \mathbf{U}_0, \quad \mathbf{i} = \mathbf{i}_0 + \zeta \mathbf{n}, \tag{A22, A23}$$

where $\mathbf{n}$ is the unit vector normal to the deformable average surface. For the shells where curvilinear co-ordinate systems overlap the co-ordinates on the plane x, y, all relations discussed for plates are valid. The theories discussed, taking into account transversal shear deformation effects, are related in our paper to the theories of Timoshenko type, since this famous work was published in 1921.

The generalized Timoshenko models are related to the case where the initial linear elements are perpendicular to the non-deformable average surface and become curvilinear after deformation and are not perpendicular to the deformable average surface. In this work the generalized Timoshenko model is defined by equation (A3), where the next series terms are taken.