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OPTIMIZATION OF PARAMETRICALLY EXCITED MECHANICAL SYSTEMS AGAINST LOSS OF DYNAMIC STABILITY

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In this paper a variational formulation of optimization problems for mechanical elements like bars or plates, subjected to a parametric excitation force, periodic in time is given. Objective functions characterizing the parametric resonance are introduced. The paper deals with the problem of finding the control function—function of the shape (the area of cross-section of the beam or the thickness of the plate) which maximizes or minimizes one of the objective functions under the constraint of constant volume. In some cases the optimization problems under conditions of parametric resonance resolve into optimization problems with respect to natural frequency. The examples of variational optimization against loss of stability are solved and analyzed in the state of parametric periodic resonance.

1. INTRODUCTION

Usually, the resonances in mechanical systems are undesirable phenomena–especially parametric resonance is very dangerous. There are many items in the literature (references, monographs, books) dealing with the topics mentioned above [1–6]. The Mathieu–Hill differential equation describing parametric resonances is often encountered in engineering and physics problems: see e.g. references [1, 7–9].

The aim is to avoid the resonance states or to minimize their disadvantageous effects. One of the methods leading to this is optimal structural design, in which a range of frequency without resonances is maximized. However, if such a procedure does not lead to avoiding the resonance phenomena, then the resonance effect should be minimized by optimization of some measures of the phenomenon—some objective functions.

In references [10, 11] the parametrical and variational optimization problems for a simply supported beam subjected to a longitudinal force periodic in time were formulated. The author looked for approximate solutions in the steady state of parametric resonance of non-prismatic rods or bars in the form of a series of unknown eigenfunctions of a non-prismatic rod. Galerkin's method gives a set of ordinary differential equations with coefficients which are periodic in time. Some coefficients also depend on a function of the shape.

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The results presented in reference [11] were generalized for a beam with other boundary conditions and for high-dimensional elements, e.g. the optimization of parametrically excited plates problems in a monograph [12].

In the present paper, the problems discussed in the monograph, reference [12], are continued. The mechanical elements under consideration are made of Kelvin–Voigt viscoelastic material. The variational and parametric optimization of parametrically excited systems against a loss of stability are explained and discussed. Problems of systems optimization in periodic parametric resonance are reduced to static considerations. Examples of variational optimization of parametrically loaded beams with respect to the loss of stability also with geometrical constraints are solved and presented.

2. PARAMETRICALLY EXCITED SYSTEMS—EQUATIONS OF MOTION

The equation of motion of "non-prismatic" parametrically excited elastic elements has the form cf. [1]

$$\hat{M}(\mathbf{h})\left[\frac{\partial^2 w}{\partial t^2}\right] + \hat{S}(\mathbf{h})[w] + \beta(t)\hat{P}_\beta[w] = 0, \tag{1}$$

where $\mathbf{h}$ is the vector of control functions or the vector of parameters of shape (e.g. area of the cross-section or the thickness of the plate); $\hat{M}$, $\hat{S}$, $\hat{P}_\beta$ are the inertia, elasticity and stability linear operators. The form of these operators depends on the kinds of mechanical elements to be considered, $w(x, t)$ is a transverse displacement of the vibrating system, $\beta(t)$ generally denotes the parametric load of vibrating elements (e.g. linear force density q (N/m) for continuous for continuous parametric load on the edge of the plate, or longitudinal force $P(t)$ (N) acting on the beam) which is a periodic function of t.

An approximate solution of equation (1) is sought in the form of a series

$$w = \sum_{k=1}^{N} f_k(t)\phi_k(x, y, z), \tag{2}$$

where ϕ_k are the eigen functions of $[\hat{S}(\mathbf{h}) - \omega^2\hat{M}(\mathbf{h})]\phi = 0, f_k(t)$ are the unknown functions of time. The functions ϕ_k will not be found before the optimization procedure. Applying Galerkin's method one obtains a system of ordinary differential equations of the second order in the matrix form, cf. references [1, 12]

$$\mathbf{M}(\mathbf{h})\frac{\mathrm{d}^2\mathbf{f}}{\mathrm{d}t^2} + [\mathbf{S}(\mathbf{h}) + \beta(t)\mathbf{P}_\beta]\mathbf{f} = \mathbf{0}, \tag{3}$$

where $\mathbf{M}$, $\mathbf{S}$, $\mathbf{P}_\beta$ are, respectively, the inertia, elasticity and parametric excitation matrices which depend on the functions or on the parameters of shape, $\mathbf{f}$ is the column matrix of the generalized co-ordinates. The elements of the matrices are

$$M_{ik} = \int_D \phi_i\hat{M}(\mathbf{h})[\phi_k]\mathrm{d}\tau = J_1^{(ik)}, \qquad S_{ik} = \int_D \phi_i\hat{S}(\mathbf{h})[\phi_k]\mathrm{d}\tau = J_2^{(ik)},$$

$$P_{b_{ik}} = \int_D \phi_i\hat{P}_\beta[\phi_k]\mathrm{d}\tau = J_3^{(ik)} \tag{4}$$

Now one can introduce the matrices: $\mathbf{B}(\mathbf{h}) = \mathbf{S}^{-1}\mathbf{P}_\beta$, $\mathbf{C}(\mathbf{h}) = \mathbf{S}^{-1}\mathbf{M}$ (cf. reference [1]).

Continuing, one introduces the damping matrix $\mathscr{E}$. Now problem (1) is described by a system of ordinary equations

$$\frac{\mathrm{d}^2 f_k}{\mathrm{d}t^2} + 2\varepsilon_k(\mathbf{h})\frac{\mathrm{d}f_k}{\mathrm{d}t} + \omega_k^2(\mathbf{h})\left[f_k + \beta(t)\sum_{j=1}^{N} B_{kj} f_j\right] = 0, \quad k = 1, 2, \dots, N, \tag{5}$$

$$[\hat{S}(\mathbf{h}) - \omega^2 \hat{M}(\mathbf{h})]\phi(\mathbf{h}) = 0, \tag{6}$$

where

$$\omega_k^2 = \frac{(\phi_k, \hat{S}(\mathbf{h})[\phi_k])}{(\phi_k, \hat{M}(\mathbf{h})[\phi_k])} = \frac{J_2^{(k)}}{J_1^{(k)}} \tag{7}$$

are the eigenvalues of the problem of natural vibrations (6) (unknown until the optimization procedure), $\mathbf{C}^{-1} = \mathrm{diag}[\omega^2]$, B_{kj} are elements of matrix $\mathbf{B}$, $\varepsilon_{kk} = \varepsilon_k$ are the damping matrix elements. The matrix $\mathscr{E}$ is a function of $\mathbf{C}$. The form of damping matrix must be determined, for the analyzed element, on the basis of theory and experiments. One can prove, e.g., for the Kelvin–Voigt linear viscoelastic material that matrix $\mathscr{E}$ is diagonal and its elements are proportional to $\mathbf{C}^{-1}$ and $\varepsilon_k = \tau\omega_k^2$ (cf. references [1, 12, 13]). The detailed analysis of the influence of damping models on the optimization of parametrically excited system is given in section 4. Equations (5) are a set of coupled linear equations with variable coefficients. One may distinguish two types of instability of the trivial solution of equations(5); the periodic (simple) parametric resonance which occurs in the neighbourhood of the frequencies:

$$\theta = 2\omega_{0s}/k, \quad k = 1, 2, 3, \dots,$$

the combination parametric resonance which occurs in the neighbourhood of the frequencies:

$$\theta = (\omega_{0s} \pm \omega_{0p})/k, \quad k = 1, 2, 3, \dots, s \neq p.$$

Consideration will be confined to periodic parametric resonance only. When the matrix $\mathbf{B}(\mathbf{h}) = \mathbf{S}^{-1}\mathbf{P}_b$ is also diagonal, the analyzed systems are described by a non-coupled set of Mathieu–Hill equations

$$\frac{\mathrm{d}^2 f_k}{\mathrm{d}t^2} + 2\varepsilon_k(\mathbf{h})\frac{\mathrm{d}f_k}{\mathrm{d}t} + \omega_k^2(\mathbf{h})[1 + B_k(\mathbf{h})\Phi(t)]f_k = 0, \quad k = 1, 2, \dots, \tag{8}$$

where

$$\beta_{\mathrm{cr}} \cong -\frac{1}{B_k} = -\frac{(\phi_k, \hat{S}(\mathbf{h})[\phi_k])}{(\phi_k, \hat{P}_\beta[\phi_k])} = -\frac{J_2^{(k)}}{J_3^{(k)}} \tag{9}$$

are the eigenvalues of the eigenvalue problem of static stability and the quantities ε_k, ω_k, B_k depend on the function of shape. To simplify further considerations, only the first term of series (2) is taken into account (one Mathieu–Hill equation) and the discussion confined to periodic parametric resonance (the first type of instability) only. Taking $\beta(t) = \beta_0 + \beta_t \cos\theta t$, after some transformations one has

$$\frac{d^2 f_k}{dt^2} + 2\varepsilon_k(\mathbf{h})\frac{df_k}{dt} + \omega_k^2(\mathbf{h})[1 + B_k(\mathbf{h})\,\beta_0]\left(1 + \frac{B_k(\mathbf{h})\beta_t}{(1 + B_k(\mathbf{h})\beta_0}\cos\theta t\right)f_k = 0. \tag{10}$$

Next, one can transform equation (10) to the form (cf. reference [12])

$$\ddot{f}_k + 2\varepsilon_k(\mathbf{h})\dot{f}_k + \Omega_k^2(\mathbf{h})(1 - 2\mu_k(\mathbf{h})\cos\theta t)f_k = 0, \quad k = 1, 2, \ldots, N, \tag{11}$$

where for the kth mode (kth form of vibration)

$$\Omega_k(\mathbf{h}) = \omega_k(\mathbf{h})\sqrt{1 + \beta_0 B_k(\mathbf{h})}, \qquad \mu_k(\mathbf{h}) = -\frac{\beta_t B_{kk}(\mathbf{h})}{2(1 + \beta_0 B_k(\mathbf{h}))} = \frac{\beta_t}{2(\beta_{cr}(\mathbf{h}) - \beta_0)} \tag{12}$$

for non-prismatic elements by analogy to that for a prismatic one. The quantity μ_k—the parameter of excitation–was introduced by Bolotin in reference [1] for parametric excitation: $P(t) = P_0 + P\cos\theta t$, acting on prismatic beam—i.e. with constant area of cross-section h. In formula (12) $\mathbf{h}$ may depend on x, y, z.

3. SOLUTIONS OF EQUATIONS OF MOTION AND STABILITY

The most popular and very effective method of receiving the instability region and amplitudes is Bolotin's method. In applying it one first assumes the solution at the stability limits to be of the form of a truncated Fourier series, and next the harmonic balance method is applied. So the solution with the period $2T(T = 2\pi/\theta)$ is assumed in the form

$$f(t) = \sum_{k=1,3,5,\ldots}^{\infty}\left(a_k \sin\frac{k\theta t}{2} + b_k\cos\frac{k\theta t}{2}\right), \tag{13}$$

and the solution with the period T in the form

$$f(t) = b_0 + \sum_{k=2,4,6,\ldots}^{\infty}\left(a_k \sin\frac{k\theta t}{2} + b_k\cos\frac{k\theta t}{2}\right). \tag{14}$$

The non-zero solution of the linear equation (5) exists if the proper determinants are equal to zero (cf. reference [1]):

$$W_\infty^{(2T)} = F[\theta, \mathscr{E}, \beta_t B_{ij}(\mathbf{h})) = 0, \qquad W_\infty^{(T)} = F'[\theta, \mathscr{E}, \beta_t B_{ij}(\mathbf{h})) = 0. \tag{15,16}$$

On the basis of the general theory of differential equations with variable coefficients one concludes that on the boundary of the first instability region the periodic solution with the period $2T$ exists. Now assume the solution of equation

(11) to be of the form (cf. reference (13))

$$f_k(t) = a_k \sin\frac{\theta t}{2} + b_k \cos\frac{\theta t}{2}. \qquad (17)$$

Inserting this function into the first of equations (11) and comparing the coefficients of $\sin\theta t/2$ and $\cos\theta t/2$ one gets a system of algebraic equations for the coefficients. The non-zero solution of these equations exists if the determinant equals zero. Solving it and neglecting the higher powers of $\Delta(\mathbf{h})/\pi$, one has

$$\theta \cong 2\Omega(\mathbf{h})\sqrt{1 \pm \sqrt{\mu^2(\mathbf{h}) - (\Delta(\mathbf{h})/\pi)^2}}, \qquad (18)$$

where

$$\Delta(\mathbf{h}) = \frac{2\pi\varepsilon(C(\mathbf{h}))}{\omega(\mathbf{h})\sqrt{1-\beta_0/\beta_{cr}(\mathbf{h})}} = \frac{2\pi\varepsilon(C(\mathbf{h}))}{\Omega(\mathbf{h})} \qquad (19)$$

is a decrement rate of the vibration for the non-prismatic element loaded by the constant part of the parametric excitation β_0, $\beta_{cr} \cong -1/B_k$ (cf. reference (9)), the quantity $\Omega = \Omega_1$ is defined in equation (12).

4. OBJECTIVE FUNCTION

The main purpose of the paper is to determine and define the proper measures of periodic parametric resonance. These measures are the objective functions in the variational optimization procedure. Four physically motivated quantities characterizing parametrically excited systems are introduced.

The periodic parametric resonance occurs if in the parametrically exciting system the proper relations between the frequency of external excitation θ and natural frequencies take place. The most dangerous, main parametric periodic resonance occurs in the neighborhood of the doubled value of the first natural frequency $\theta = 2\omega$. The square of the first natural frequency is the proper objective function in the optimization procedure when one maximizes the non-resonance region $0 \leqslant \theta \leqslant 2\omega$. The objective function has the form of the non-additive functional

$$\omega^2 = J_2/J_1 = R(J_1, J_2), \qquad (20)$$

where

$$J_1 = J_1^{(11)} = \int_D \phi_1 \hat{M}(\mathbf{h})[\phi_1]\,d\tau. \qquad J_2 = J_2^{(11)} = \int_D \phi_1 \hat{S}(\mathbf{h})[\phi_1]\,d\tau, \qquad (21)$$

cf. reference (4). If one analyzes the transverse vibrations of non-prismatic beams the operators $\hat{M}(\mathbf{h})$, $\hat{S}(\mathbf{h})$ take the forms (cf. references (1) and [11])

$$\hat{M}(\mathbf{h}) = \rho(x)h(x), \qquad \hat{S}(\mathbf{h}) = \frac{\partial^2}{\partial x^2}\left[K_\alpha h^\alpha \frac{\partial^2}{\partial x^2}\right], \qquad (22)$$

where $h(x)$ is the area of cross-section of the beam, $K_\alpha = EA_\alpha$, E is Young's modulus, A_α is a constant connected with the geometry of cross-section and depending on α ($\alpha = 1, 2, 3$), and ρ is the mass density. The function of state ϕ_1

satisfies the equation of state of natural transverse vibrations of the non-prismatic rod without damping:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[K_\alpha h^\alpha(x)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right]-\rho h(x)\omega^2\phi(x)=0. \tag{23}$$

The present paper is devoted to the optimization problems of mechanical systems described by the equations of motion in modal form with damping proportional to $\mathrm{d}f_k/\mathrm{d}t$; cf. equation (5). For one mode the equation of motion takes form (11). Formula (18) gives the boundary of the instability region in the $(\mu, \theta/2\Omega)$ plane, see Figure 1. If the expression under the inner square root is positive, formula (18) gives two real values of critical frequency. On the basis of formula (18) the critical value of the excitation parameter denoted by μ^* is (cf. references [1, 11, 12])

$$\mu^*(\mathbf{h})=\frac{\Delta(\mathbf{h})}{\pi}=\frac{2\varepsilon(C(\mathbf{h}))}{\omega(\mathbf{h})\sqrt{1-\beta_0/\beta_{cr}(\mathbf{h})}}=F_1(J_1,J_2,J_3), \tag{24}$$

where $\Delta(\mathbf{h})$ is defined in equation (19). So μ^* is the special value of μ which characterizes the energy dissipation in parametric systems (cf. reference [1]) and depends also on other material properties like E and ρ. In this paper $2\varepsilon(C(\mathbf{h}))$ is generally some function of the matrix $\mathbf{C}=\mathbf{S}^{-1}\mathbf{M}$, and its form must be determined for the analyzed element on the basis of theory and proper experiments. Most often from experiments one gets the decrement of damping δ and coefficient of decay γ (coefficient of loss of energy). For a Kelvin–Voigt viscoelastic material and for parametric excitation one usually adopts a damping matrix $\mathbf{D}$ [13] which is proportional to the matrix of elasticity $\mathbf{S}$; $\mathbf{D}=\gamma(\theta)\mathbf{S}/\theta$, where the coefficient of loss of energy $\gamma(\theta)$ is some function of frequency of the external excitation θ. In the present example of parametric resonance $\theta=2\omega$ and $\gamma=\theta\eta/E$. The coefficient

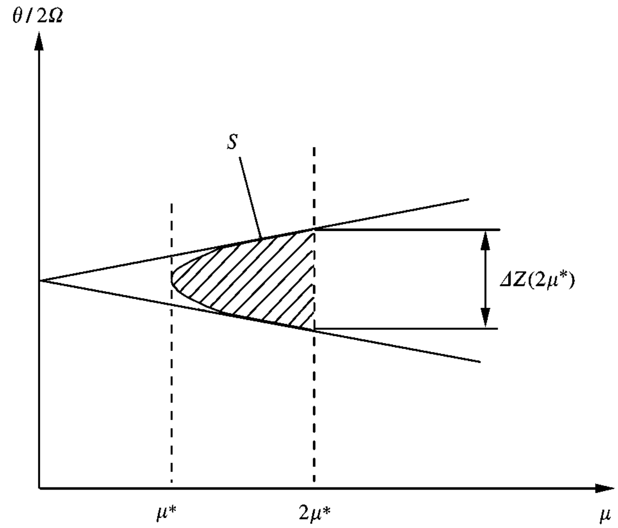

Figure 1. Instability region in the $(\mu, \theta/2\Omega)$ plane, where μ^* is the critical value of the excitation parameter, and $S(h)$ the area of the part of the instability region enclosed between μ^* and $2\mu^*$.

γ may depend also on temperature and some other parameters of the material. These dependences are in the form of graphs. The adopted model is useful when the mentioned parameters are constant. In these examples $\mathbf{D} = \eta\mathbf{S}/E$ and $2\mathscr{E} = \mathbf{M}^{-1}\mathbf{D} = \eta\mathbf{M}^{-1}\mathbf{S}/E = \eta\mathbf{C}^{-1}/E$. So for Kelvin–Voigt viscoelastic material ($2\varepsilon = \tau\omega^2$), one has

$$\mu^*(\mathbf{h}) = \frac{\Delta(\mathbf{h})}{\pi} = \frac{\tau J_2}{(J_1(J_2 + \beta_0 J_3))^{1/2}} = F_1(J_1, J_2, J_3), \tag{25}$$

where $J_1^{(11)} = J_1$, $J_2^{(11)} = J_2$, $J_3^{(11)} = J_3$ (cf. equation (4)). If one analyzes the transverse vibrations of non-prismatic beams the operators $\hat{M}(\mathbf{h})$, $\hat{S}(\mathbf{h})$ take form (22) and the operator $\hat{P}_\beta(\mathbf{h}) = \partial^2/\partial x^2$. The square of the critical value of the excitation parameter as the objective function is introduced. If additionally $\beta_0 = 0$ one has

$$(\mu^*(\mathbf{h}))^2 = \tau^2 R(J_1, J_2) = R_1(J_1, J_2), \tag{26}$$

where $R(J_1, J_2)$ is defined in equation (20).

Now a parametrically excited system will be analyzed in the (β_t, $\theta/2\Omega$) plane. On the basis of equation (12) the amplitude of the oscillating part of parametric excitation is $\beta_t(\mathbf{h}) = 2\mu(\beta_{\mathrm{cr}} - \beta_0)$. So one looks for such a value of the amplitde of the oscillating part of the excitation $\beta_t(\mathbf{h})$ for such a value of the amplitude of the oscillating part of the excitation $\beta_t(\mathbf{h})$ for which the unstable solution occurs. If $\mu = \mu^*$, β_t equals the critical value of amplitude of the oscillating part of the excitation,

$$\beta_t^*(\mathbf{h}) = 2\mu^*(\mathbf{h})(\beta_{\mathrm{cr}}(\mathbf{h}) - \beta_0) = F_2(J_1, J_2, J_3), \tag{27}$$

where

$$\beta_{\mathrm{cr}} \cong -\frac{1}{B_1} = -\frac{(\phi_1, \hat{S}(\mathbf{h})[\phi_1])}{(\phi_1, \hat{P}_\beta[\phi_1])} = -\frac{J_2}{J_3}.$$

If $2\varepsilon_k = \tau\omega^2$ the critical value of the amplitude of the oscillating part of the parametric excitation takes the form

$$\beta_t^*(\mathbf{h}) = -2\tau\frac{J_2}{J_3}\left(\frac{J_2 + \beta_0 J_3}{J_1}\right)^{1/2}. \tag{28}$$

For $\mu > \mu^*$ or for $\beta_t > \beta_t^*$ the region of instability appears. If one analyzes the transverse vibrations of non-prismatic beams the operators $\hat{M}(\mathbf{h})$, $\hat{S}(\mathbf{h})$ take form (22) and the operator $\hat{P}_\beta(\mathbf{h}) = \partial^2/\partial x^2$.

The optimization problem against the loss of dynamic stability of the parametrically excited beam consists of determining the area h of the cross-section, which extremizes the critical value of excitation parameter μ^* or the critical value of amplitude of oscillating part of excitation β_t^*. The vibrating parametrically excited system will be most stable if μ^* attains a maximum in the ($\theta/2\Omega$, μ) plane or β_t^* attains a maximum in the ($\theta/2\Omega$, β_t) plane. One can see that critical parameters, the critical value of the excitation parameter or the critical value of the amplitude of the oscillating part of the excitation separate stable and unstable solutions (cf.

Figure 1). Maximization of the values of the critical parameters β_t^*, μ^* also allows one to move away from unstable solution regions. For such an objective function one can control not only the geometrical parameters of optimization (parameters of the shape) but then one may also influence non-stability regions through a change of the coefficient of damping (cf. equation (26)) and through the change of the parameter β_0 of the external parametric load (cf equation (27).

Therefore, one has two proper measures of periodic parametric resonance, the critical parameters: the critical value of the excitation parameter, see equations (24–26), or the critical value of the amplitude of the oscillating part of the harmonic excitation, see equations (27, 28). The objective functions, e.g. μ^*, β_t^* are of the form of non-additive functionals.

If through the optimization it is not possible to move away from an unstable solution region, the phenomenon of parametric resonance occurs (there exists a non-stable solution of equation of motion), and the resonance amplitude grows to infinity. The non-linearities limit the growth, and the amplitudes of parametric resonance are finite in the region of instability ((cf. references [1, 5, 6]). In such examples the proper objective function is the amplitude of the steady state of the parametric resonance. Now in the optimization procedure one looks for the minimum value of the resonance amplitude. The resonance amplitude can be obtained on the basis of the non-linear equation of motion.

$$\ddot{f}_k + 2\varepsilon_k(\mathbf{h})\dot{f}_k + \Omega_k^2(\mathbf{h})(1 - 2\mu_k(\mathbf{h})\cos\theta t)f_k + \varphi_k(f, \dot{f}, \ddot{f}) = 0, \tag{29}$$

where the function $\varphi_k(f, \dot{f}, \ddot{f})$ includes non-linear effects, e.g. geometrical non-linearities like non-linear damping, non-linear elasticity or non-linear inertia: μ_k, Ω_k were introduced in equation (12).

For one mode the amplitude equals (cf. reference [1])

$$\mathscr{A}(\mu(\mathbf{h}),\quad \theta/2\Omega(\mathbf{h}),\quad \mu^*(\mathbf{h})) = \sqrt{a^2 + b^2} = F_1^N(J_1, J_2, J_3), \tag{30}$$

where a and b are coefficients in equation (13). For example, for non-linear elasticity one has

$$\mathscr{A} = \frac{2\Omega(\mathbf{h})}{\sqrt{3\gamma}}\sqrt{n^2(\mathbf{h}) - 1 \pm \sqrt{(\mu^2(\mathbf{h}) - \left(\frac{n\Delta(\mathbf{h})}{\pi}\right)^2}} = F_1^N(J_1, J_2, J_3). \tag{31}$$

Now in the optimization procedure one looks for the minimum value of the resonance amplitude determined by equation (30) or (31).

In the resonance state the other objective functions may be introduced. They are some measures of the instability region. One of them is associated with the area $S(\mathbf{h})$ of a part of the instability region, e.g. that enclosed between μ^* and $2\mu^*$, cf. references [10, 12]. The second of them is the interval of excitation frequency, $\Delta\theta(\mu = 2\mu^*) = 2 \cdot \Omega \cdot \Delta Z$ for example; see Figure 1. Now in the optimization procedure one looks for the minimum value of this measure of instability region: cf. reference [12]. References [10–12] were devoted to the optimization of the parametrically excited system with respect to the minimization of some measure of the instability region.

5. VARIATIONAL OPTIMIZATION

The optimization problem against the loss of dynamic stability consists of determining the vector of control function **h** (e.g. the area of the cross-section or the thickness of the plate or some physical parameters of vibrating elements or its surrounding) which extremizes one of the functionals (24)–(28) or functional (30) under some constraints — e.g. under a constant volume constraint. Variational calculus is used to find the solution. The necessary conditions for extreme values of objective function can be derived by setting to zero the first variation of one of the non-additive functionals with constraints: (cf. reference [14]). The resulting equations are called optimality conditions. To solve these actual structural optimization problems, the equation of natural vibration with the proper boundary conditions must be employed in addition to the optimality conditions.

After defining the objective function is parametric resonance (cf. section 4), one formulates the optimization problem against the loss of dynamic stability as follows: look for a control function **h** that minimizes or maximizes the functional denoted

$$J = F(J_1, \dots, J_r), \tag{32}$$

where

$$J_i = \int_D f_i(\mathbf{x}, \mathbf{h}, \boldsymbol{\phi}, \boldsymbol{\phi}'')\,\mathrm{d}\tau, \tag{33}$$

where f_i are the known functions of spatial variables $\mathbf{x} = \{x_1, \dots, x_s)$, control function $\mathbf{h} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ and function of state $\Phi\{\phi_1, \dots, \phi_s\}$. In the examples

$$J_i = \int_D \phi_k \hat{O}_i(\mathbf{h})[\phi_k]\,\mathrm{d}\tau = (\phi_k, \hat{O}_i(\mathbf{h})[\phi_k]), \quad i = 1, 2, 3, \tag{34}$$

where $\hat{O}_1 = \hat{M}$, $\hat{O}_2 = \hat{S}$, $\hat{O}_3 = \hat{P}_\beta$ are the known operators on **h**; cf. equation (4). The non-additive functionals J are

$$J = F(J_1, \dots, \mathrm{J}_r) \in (R, F_1, R_1, F_2, F_1^N\}. \tag{35}$$

The constraints are

$$F_i(J_1, \dots, J_n) = c_i = \mathrm{const}, \quad i = 1, 2, \dots, k. \tag{36}$$

In the present case one looks for an approximate solution of the equation of motion (1) in the form of a series of unknown eigenfunctions of equation (2). The eigenfunctions—functions of state ϕ_n—satisfy the equation of natural transverse vibrations of non-prismatic elements without damping, and proper boundary conditions which are the additional constraints

$$L[\mathbf{h}(x)]\phi(x) = [\hat{S}(\mathbf{h}) - \omega^2 \hat{M}(\mathbf{h})]\phi = 0, \quad [N(\mathbf{h}(x)\phi(x)]_\Gamma = 0. \tag{37}$$

To receive the necessary conditions of optimality for non-additive functionals one introduces the augmented Lagrange functional $\mathscr{F}$ defined by

$$\mathscr{F} = F + \sum_{l=1}^{k} \lambda_l (F_l - c_l), \tag{38}$$

where F_l are constraints (36), and λ_l are Lagrange's multipliers. Calculate the first variation of the augmented Lagrange's functional

$$\delta\mathscr{F} = \sum_{i=1}^{r} \frac{\partial \mathscr{F}}{\partial J_i} \delta J_i, \tag{39}$$

where

$$\frac{\partial \mathscr{F}}{\partial J_i} = \frac{\partial F}{\partial J_i}(J_1, \ldots, J_r) + \sum_l \lambda_l \frac{\partial F_1}{\partial J_i}(J_1, \ldots, J_r),$$

$$\delta J_i = \delta \int_D f_i(\mathbf{x}, \mathbf{h}, \boldsymbol{\phi}, \boldsymbol{\phi}'')\, \mathrm{d}\tau = \int_D \delta f_i(\mathbf{x}, \mathbf{h}, \boldsymbol{\phi}, \boldsymbol{\phi}'')\, \mathrm{d}\tau. \tag{40}$$

The equation of natural transverse vibrations of non-prismatic elements without damping and proper boundary conditions are the additional constraints. In order to include constraints (37) one takes into account the variational form of equations (37):

$$L(\mathbf{h})\delta\phi + M(\phi, \mathbf{h})\delta\mathbf{h} = 0, \quad N(\mathbf{h})\delta\phi + T(\phi, \mathbf{h})\delta\mathbf{h} = 0. \tag{41}$$

The operator $M(\phi, h) = (\partial L/\partial h)\phi$ stands at the variation $\delta\mathbf{h}$ in the variational equation of state. Introducing the adjoint variables of state v expresses the first variation of the functional through the variation $\delta\mathbf{h}$ only. On the basis of equation (37) one has (see e.g. reference [14])

$$\int_D v[L(\mathbf{h})\delta\phi + M(\phi, \mathbf{h})\delta h]\, \mathrm{d}\tau = \int_D [\delta\phi L^*(\mathbf{h})v + \delta h M^*(\phi, \mathbf{h})v]\, \mathrm{d}\tau, \tag{42}$$

where the operators L^*, M^*, are the adjoint operators to L, M. Setting the first variation of one of the non-additive functionals with constraints to zero [14–16] gives a general form of the equation of state for the adjoint variable and the necessary conditions of optimality for non-additive functionals in the form

$$L^*(h)v + \sum_{l=1}^{r} \frac{\partial \mathscr{F}}{\partial J_l} \left(\frac{\partial f_l}{\partial \phi} - \sum_{j=1}^{s} \frac{\partial}{\partial x_j} \frac{\partial f_l}{\partial \phi_{x_j}} + \frac{1}{2} \sum_{j,d=1}^{s} \frac{\partial^2}{\partial x_j \partial x_d} \frac{\partial f_l}{\partial \phi_{x_j x_d}} \right) = 0,$$

$$M^*(\phi, h)v + \sum_{l=1}^{r} \frac{\partial \mathscr{F}}{\partial J_l} \frac{\partial f_l}{\partial h} = 0, \tag{43}$$

where $f_l = f_l(\mathbf{x}, \mathbf{h}, \boldsymbol{\phi}, \boldsymbol{\phi}', \boldsymbol{\phi}'')$. The first of equation (43) is the equation of state for the adjoint problem; the second one represents the optimality condition.

6. EXAMPLES

In the examples of optimization of parametrically excited systems the critical value of the excitation parameter μ^* determined by equation (24) is the objective function. The parametrically excited system will be most stable if in optimizing with a constant volume constraint and some additional constraints (e.g. geometrical) μ^* attains a maximum. The system will be optimized with respect to μ^{*2}.

As examples, the optimization of parametrically excited beams, with different boundary conditions are considered. The equation of motion of a Kelvin–Voigt viscoelastic beam takes the form, cf. equation (1),

$$\frac{\partial^2}{\partial x^2}\left[K_\alpha h^\alpha \frac{\partial^2 w}{\partial x^2} + \tau K_\alpha h^\alpha \frac{\partial^3 w}{\partial x^2\,\partial t}\right] + \rho h(x)\frac{\partial^2 w}{\partial t^2} + \beta(t)\frac{\partial^2 w}{\partial x^2} = 0, \tag{44}$$

where $w(x, t)$ is a transverse displacement of the cross-section x at the time t, $\beta(t) = \beta_0 + \beta_t \cos\theta t$ is a longitudinal force, $\tau = \eta/E$, and η is the coefficient of internal damping. The remaining parameters are introduced in equation (22).

On the basis of section 2, an approximate solution of the above problem is sought by applying the Galerkin method in the form of a series of unknown eigenfunctions ϕ_i of equation (23) of the non-damped natural vibrations of the "non-prismatic" elements:

$$w = \sum_{k=1}^{N} f_k(t)\phi_k(x). \tag{45}$$

After some transformations one has for one mode (cf. references [11, 12])

$$\ddot{f} + 2\varepsilon\dot{f} + \Omega^2(1 - 2\mu\cos\theta t)f = 0,$$

where the coefficient of damping is

$$2\varepsilon = \tau J_2/J_1,$$

$$\Omega^2 = \omega^2(1 - \beta_0/\beta_{cr}), \qquad \omega^2 = J_2/J_1,$$

$$J_1 = \rho\int_0^l h(x)[\phi_1]^2\,\mathrm{d}x, \qquad J_2 = K_\alpha\int_0^l h^\alpha(x)\left[\frac{\partial^2\phi_1}{\partial x^2}\right]^2\mathrm{d}x.$$

Taking into account the assumption $\beta_0 = 0$ one has $\beta(t) = \beta_t\cos\theta t$. On the basis of the procedure described in section 4, the objective function has form (26). So in the optimization procedure one looks for the maximum of the functional

$$J = F(J_1, J_2) = (\mu^*(\mathbf{h}))^2 = \tau^2\omega^2 = \tau^2\left(K_\alpha\int_0^l h^\alpha\left[\frac{\partial^2\phi_1}{\partial x^2}\right]^2\mathrm{d}x\Big/\left(\rho\int_0^l h(x)[\phi_1]^2\,\mathrm{d}x\right)\right. \tag{46}$$

where ρ, $h(x)$, E, $K_\alpha = EA_\alpha$, $\tau = \eta/E$ and the length of the beam l are the physical and geometrical parameters. The area of cross-section of the beam $h(x)$ is taken as the control function in the optimization procedure.

After defining the objective function (46) (cf. section 4) and the control function, the optimization problem against the loss of dynamic stability is formulated as follows.

Look for such a control function h that maximizes functional (46) with the constraints

(1) isoperimetric condition of constant volume V_0 of the beam,

$$F_1(J_1, J_2) = \int_0^l h(x)\,\mathrm{d}x = V_0, \tag{47}$$

(2) the equation of state in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[K_\alpha h^\alpha(x)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right] - \rho(x)h(x)\omega^2\phi(x) = 0, \tag{48}$$

(3) the boundary conditions, e.g. for a simply supported beam

$$\phi(0) = \left[h^\alpha(x)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right]_{x=0} = 0, \qquad \phi(l) = \left[h^\alpha(x)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right]_{x=l} = 0,$$

for a simply supported–fixed beam

$$\phi(0) = \left[h^\alpha(x)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right]_{x=0} = 0, \qquad \phi(l) = \left[\frac{\mathrm{d}\phi}{\mathrm{d}x}\right]_{x=l} = 0 \tag{49}$$

for a fixed–fixed beam

$$\phi(0) = \left[\frac{\mathrm{d}\phi}{\mathrm{d}x}\right]_{x=0} = 0, \qquad \phi(l) = \left[\frac{\mathrm{d}\phi}{\mathrm{d}x}\right]_{x=l} = 0 \quad \text{(cf. Figure 2)},$$

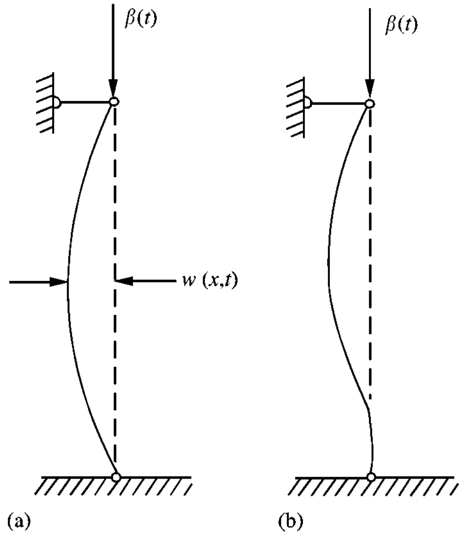

Figure 2. The parametrically excited vibrating beams—examples for two different boundary conditions.

(4) additionally in some examples the area of cross-section $h(x)$ fulfills the inequalities

$$h_1 \leqslant h(x) \leqslant h_2, \tag{50}$$

where h_1, h_2 are constant or generally also some known functions of x. One can adopt the proper values of constraints arising, e.g. from the strength conditions),

(5) the remaining parameters E, A_α, ρ, η, l are constants in the optimization processes, cf. equations (46)–(49); in the examples the beams are cylindrical and have similar cross-sections, the moment of inertia equals $I = ch^2$, where c is a constant.

In the optimization procedure the control function $h(x)$ and the function of state $\phi_1(x)$ are sought.

On the basis of equation (46) the parametrically excited beams are most stable if the value of the natural frequency ω is a maximum (the values of damping η and E, ρ, l, V_0 are fixed).

The examples of optimization are divided into two groups of problems:
I. optimization without geometrical constraints; II. optimization with geometrical constraints.

In optimization without geometrical constraints variational calculus is used to find the solution. On the basis of equation (46) the optimization of parametrically vibrating beams with respect to the maximum of the critical value of μ^* converts itself into optimization problems with respect to the maximum of the natural frequency. By the conclusion of the papers of Banichuk, Niordson, and Olhoff [14, 17, 18] one has the values of the quantity $\omega_{\max}/\omega_{pr}$, where ω_{max} is the maximal (optimal) value of the beam's first frequency, ω_{pr} denotes the first natural frequency of prismatic beam with the same physical parameters: density ρ, Young's modulus E and with the same constant parameters η, V_0, l. On the basis of equation (46) one immediately has the quotient of optimal (maximal) value of excitation parameter to the value of excitation parameter for prismatic beams, μ^*_{max}/μ^*_{pr}, for the first mode of vibration and for the same constant parameters. One also has the relative change of the excitation parameter in per cent, $(\Delta\mu^*/\mu^*_{pr}) \times 100\%$.

Numerical calculations depend on the boundary conditions. For example, for a cylindrical, simply supported beam, the square of the optimal value of the first natural frequency and the square of the prismatic beam equals $\omega^2_{max} = 110{\cdot}66\, G$, $\omega^2_{pr} = \pi^4\, G$, $G = EV/4\pi\rho l^3$ (cf. reference [11]).

So one has (cf reference [18]),

for a simply supported beam $\mu^*_{max}/\mu^*_{pr} = 1{\cdot}066$, $\Delta\mu^*/\mu^*_{pr} = 6{\cdot}6\%$;

for a simply supported–fixed beam $\mu^*_{max}/\mu^*_{pr} = 1{\cdot}57$, $\Delta\mu^*/\mu^*_{pr} = 57\%$; and

for a fixed–fixed beam, $\mu^*_{max}/\mu^*_{pr} = 4{\cdot}32$, $\Delta\mu^*/\mu^*_{pr} = 332\%$.

The optimal shapes and the eigenfunctions (forms of the vibration) of the beams are presented in Figure 3 (cf. reference [18]).

Under some assumptions the optimization of the parametrically vibrating beams with respect to the maximum of the critical value μ^* is resolved into an optimization problem with respect to the maximum of natural frequency and the

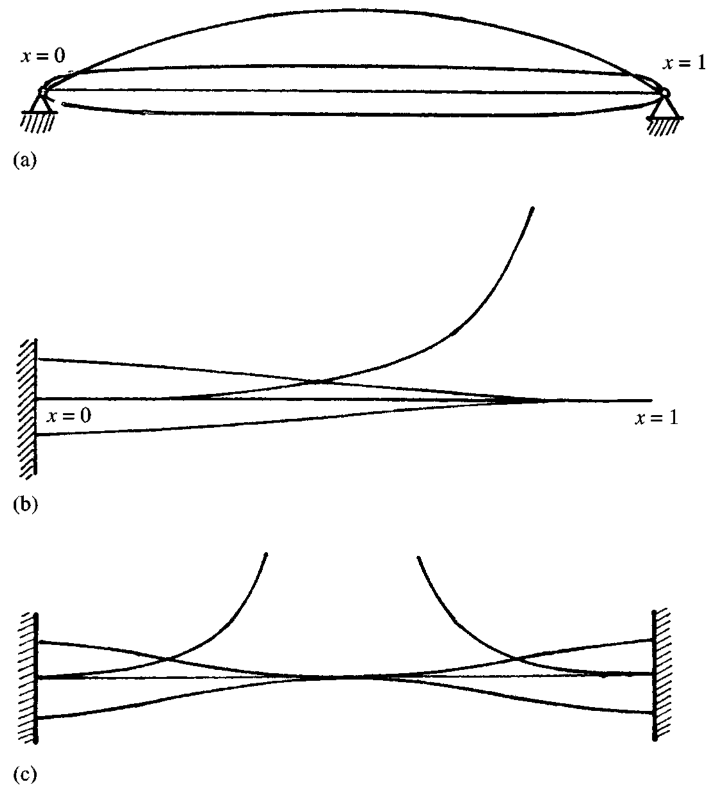

Figure 3. The shape of the beams and the forms of vibrations of optimal elements: (a) for a simply supported beam; (b) for a simply supported–fixed beam; (c) for a fixed–fixed beam; cf. reference [18].

equation of natural transverse vibrations of a non-prismatic rod is the important constraint in the optimization. This equation,

$$L[h(x)]\phi(x) = 0 \Leftarrow [\hat{S}(\mathbf{h}) - \omega^2 \hat{M}(\mathbf{h})]\phi = 0 \tag{51}$$

describes the Euler–Bernoulli beam (the shear deformations and rotary inertia are neglected). It is assumed that the transverse dimension of the beam is small in comparison with the length of the beam. If the transverse dimensions are comparable with the length, the more exact Timoshenko beam theory is suitable (cf. reference [1]). This theory is particularly important in optimization with respect to higher natural frequency.

In the next group of examples — in optimization problems of parametrically excited beams with geometrical constraints — Pontryagin's maximum principle is used. The optimization of a parametrically excited non-prismatic beam with additional geometrical constraints with respect to the maximum of the square of critical value of exciting parameter μ^* is considered. Under the above-mentioned assumptions this problem is converted into an optimization problem with respect to the maximum also of the natural frequency. Griniew and Filippow [19] presented the extreme values of the natural frequencies of beams for different boundary conditions and for different values of constant volume. In Figure 4, taken from the book by Griniew and Filippow [19], the results for beams with a circular cross-section are presented. In this case the constraints are that the length and volume of the beam are constant and additionally the area of cross-section $h(x)$ fulfills the inequalities $h_1 \leqslant h(x) \leqslant h_2$, where h_1, h_2 are constant. One can adopt the proper values of constraints arising, e.g. from the strength conditions. In our examples the parameters of the beams are $l = 1{\cdot}2$ m, $h_1 = 4 \times 10^{-4} m^2$, $h_2 = 2h_1$, $E = 1{\cdot}96 \times 10^{11}$ N/m^2, $\rho = 7{\cdot}8 \times 10^3$ kg/m^3. In Figure 4, the quotient of dimensionless optimal (maximal or minimal) circular frequency $\omega_{max(min)}$, to circular frequency ω_1 of prismatic beam with the area of cross-section h_1, versus the

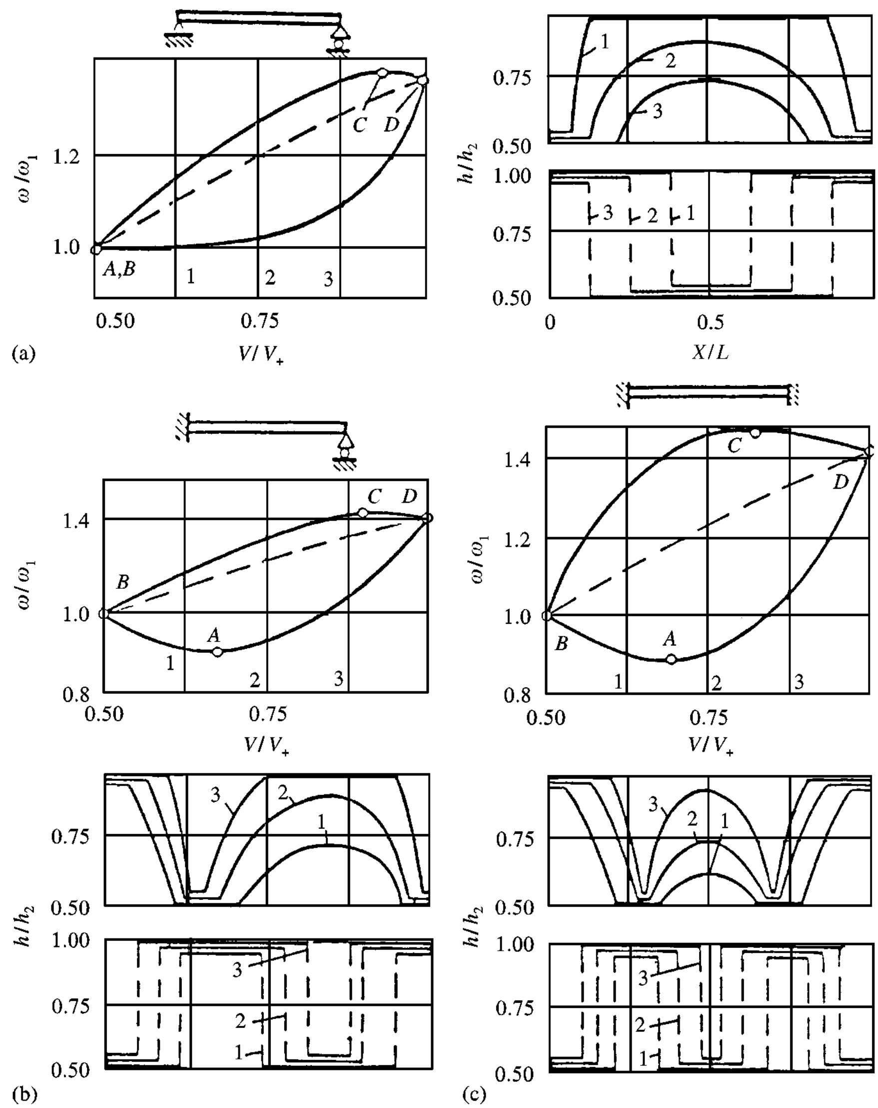

Figure 4. The quotient of $\omega_{max(min)}/\omega_1$ versus the quotient of the volume V/V_+ and the shapes are presented for circular cross-section beams; (a) simply supported; (b) simply supported–fixed; (c) fixed–fixed; cf. reference [19]. The parameters of circular beams are $l = 1{\cdot}2$ m, $h_1 = 4 \times 10^{-4}$ m^2, $h_2 = 2h_1$, $E = 1{\cdot}96 \times 10^{11}$ N/m^2, $\rho = 7{\cdot}8 \times 10^3$ kg/m^3.

quotient of the volume V, to the maximum value of volume $V^+ = \int_0^1 h_2 \, dx = h_2 l$ is presented in the form of closed curves. The upper parts of the curves (BCD) present the maximal values of $\omega_{max}/\omega_1(\omega_1 = \omega(h_1))$ in optimization with different values of constant volume. In the lower part of the graph (DAB) the minimum values of ω_{min}/ω_1 in optimization with different values of constant volume are presented. The numerical results depend on the boundary conditions. According to the conclusion of Griniew and Filippow [19], one has the values of ω_{max}/ω_1 and after some simple calculations obtains ω_{max}/ω_{pr} where ω_{max} is the maximal (optimal) value, $\omega_{pr}(h)$ denotes the first natural circular frequency of a prismatic beam with the same constant parameters (V, l, density ρ, and Young's modulus E). The cross-sectional areas are similar, the moment of inertia $I = ch^2$. Next, on the basis of equation (50) the ratio μ^*_{max}/μ^*_{pr} of optimal (maximal) value of the excitation parameter to the value of the excitation parameter for "prismatic" beams for the first mode and for the same constant parameters is derived. One also has the relative change of the excitation parameter in percent, $(\Delta\mu^*/\mu^*_{pr}) \times 100\%$. Numerical calculations depend on the boundary conditions.

For example, the maximal value of ω_{max}/ω_1 $(\omega_1 = \omega(h_1))$ for a simply supported beam is $(\omega_{max}/\omega_1)_{max} = 1{\cdot}375$ (Figure 4(a)). Because for this value of ω_{max}/ω_1, $V/V_+ = 0{\cdot}9375$, the constant volume $V = V_+ 0{\cdot}9375 = 2h_1 l \times 0{\cdot}9375 = hl$. where $h = 2h_1 \times 0{\cdot}9375$. After simple calculations one has $\omega_h = \omega_{h_1}\sqrt{h/h_1}$, where ω_h and ω_{h_1} are the natural frequencies of prismatic beams with constant area of cross-sections h and h_1 respectively and

$$\omega_{max}/\omega_{pr}(h) = \omega_{max}/\omega_1(h_1)\sqrt{h_1/h} = 1{\cdot}375\sqrt{h_1/h}. \quad (52)$$

Therefore on the basis of Figure 4, one has for maximal values of ω_{max}/ω_1,

for a simply supported beam $\mu^*_{max}/\mu^*_{pr} = 1{\cdot}0045$ and $\Delta\mu^*/\mu^*_{pr} = 0{\cdot}45\%$,

for a simply supported–fixed beam $\mu^*_{max}/\mu^*_{pr} = 1{\cdot}045$ and $\Delta\mu^*/\mu^*_{pr} = 4{\cdot}5\%$, and

for a fixed–fixed beam $\mu^*_{max}/\mu^*_{pr} = 1{\cdot}18$ and $\Delta\mu^*/\mu^*_{pr} = 18\%$.

The strength constraints can be connected with the geometrical constraints. The appropriate shaping of beams increases the critical value of the excitation parameter. The examples of optimizations of parametrically excited systems (beams) for $\beta_0 \neq 0$ may be analyzed in a numerical way. In this case the optimization with respect to critical value of μ^* does not resolve into an optimization problem with respect to maximum of natural frequency.

The present paper is devoted to the optimization problems of mechanical systems described by the equations of motion in modal form with damping proportional to df_k/dt, cf. equation (5). For one mode the equation of motion takes the form (11). The purpose before the optimization procedure is to determine the form of the coefficient 2ε. On the basis of considerations of section 4 one can adopt in the above examples $2\mathscr{E} = M^{-1}D = \eta M^{-1}S/E = \eta C^{-1}/E$ (for a beam see page 883).

So for a Kelvin–Voigt viscoelastic material $2\varepsilon = \tau\omega^2$, cf reference [13]. One may also adopt other models of damping: e.g., the proportional model $2\varepsilon = \alpha + \beta\omega^2$, where α, β are constants, cf. references [13, 21]. References [1, 20, 21] are devoted to the detailed analysis of the damping model of elements of construction. Because of the model of vibration adopted in the present paper the method of optimization of parametrically excited systems does not apply immediately to an integral model of damping, cf. reference [20]. Adoption of an integral model needs some additional consideration and calculation. Apart from the model adopted in the paper the examples of optimization with other models of damping may be analyzed only in a numerical way—the optimization with respect to critical value of μ^* does not convert into optimization problems with respect to the maximum of natural frequency.

7. CONCLUSIONS

The variational optimization against the loss of dynamic stability may be carried out with account taken of a few aspects of the problem and various cost functions. In this paper, only one of the objective functions, most important in parametrically excited systems, namely the critical value of the excitation parameter μ^*, determined by equation (26), is taken into account.

In this optimization, the following assumptions have been accepted. The length and volume are constant and the linear theory of vibration is valid. The equation of state and boundary conditions are the additional constraints. The material characteristics—density, Young's modulus and coefficient connected with damping—are fixed. On the basis of these assumptions, for the suitable shaping of the beam (for the optimal value of cross-section area) the critical value of the excitation parameter μ^* attains a maximum, and the parametrically excited system is most stable (cf. Figure 1). For the case $\mu > \mu^*$ a region of instability appears. The objective function μ^* is of the form of non-additive functionals.

In some cases, when the constant component of excitation equals zero ($\beta_0 = 0$) and for the Kelvin–Voigt viscoelastic material, the functional μ^* is proportional to the square of the natural frequency. So the necessary optimality conditions for the element with constant damping are the same as the optimality conditions for the square of natural frequency. The optimization of the system with respect to the square of natural frequency is a well-known problem and it enables one to immediately obtain numerical results for the problems. Optimization is a way of stabilizing a parametrically excited object. One can increase the excitation parameter several times or 10 times in optimization with geometrical constraints and 10 times or hundreds of times in optimization without geometrical constraints. In the next publication the numerical results of considerations concerning beams of Kelvin–Voigt model and $\beta_0 \neq 0$ will be given. In this case the optimization with respect to the critical value of μ^* does not convert into an optimization problem with respect to the maximum of natural frequency.

In optimization problems of parametrically excited systems one can also look for such values of the amplitude of the oscillating part of the parametric excitation β_t^* for which the unstable solution occurs. The optimization of parametrically excited systems with respect to a maximum of β_t^* is physically most important and may be the subject of next consideration.

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