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FREE AND FORCED VIBRATIONS OF STEPPED RODS AND COUPLED SYSTEMS

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Analytical and closed-form frequency responses of the longitudinal vibration of N-stepped rods as well as the coupled system of multi-stepped rods and lumped elements with non-classical boundary conditions are proposed in this paper. The closed-form frequency equation for these systems is also obtained. In this article, a new two-way state-flow model based on the dynamics of the rod is developed. In addition to the state-flow model developed for lumped elements, concentrated mass, and spring and damper, the exact frequency response of both stepped rod and the coupled system can be directly written out according to the graph theory. Since the results are analytical and closed-form, they can be directly implemented for analytical and numerical analysis without resorting to complex operations, such as inversion, recursive calculation or iteration computation. Moreover, the derived equations are feasible not only for the multi-stepped and multi-stage rods but also for the coupled system of multi-stepped rods and lumped elements. Finally, the results were checked using multi-stepped rods with classical and non-classical boundary conditions as well as a combination of stepped rods and lumped elements to demonstrate the efficiency of this method.

1. INTRODUCTION

The study of the longitudinal vibration of rods has been applied in many engineering fields [1–3]. The free and forced vibration for a single uniform rod with a fixed boundary or one coupled with lumped elements, such as mass, spring and damper, has been investigated [3, 4]. Many studies on deriving the frequency equation and natural frequency prediction for free vibrations have already been proposed. Particularly, the analytical frequency equation and exact natural frequency of the non-uniform rods with typical cross-section variations, such as polynomial [5, 6], sinusoidal [7] and exponential [8] variations, associated with different boundary conditions have been obtained, but without the forced response. For the stepped rods with non-typical cross-section variations, the dynamic stiffness matrix method [9, 10] and transfer matrix method [11, 12] were applied to the free and forced vibration analysis. The classical method is, however, superbly suited for multi-segment rods by numerical computation. Based on the transfer matrix method, an algebraic algorithm was also developed for the evaluation of the

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velocity ratio of a general linear dynamic system [13]. Recently, a recursive form frequency equation for the free vibration of a torsional and rod system was derived [14]. The closed-form frequency equation for a stepped rod with one to four segments was also presented. However, expressing the results in analytical form for general stepped rods using these classical methods is quite complex. Moreover, it is also difficult to represent the forced response in analytical form. It is especially very useful to represent the frequency equation as an analytical and closed-form type for the analysis of vibration, acoustic and buckling [1, 3, 9].

In this paper, both an analytical and closed-form frequency equation and a forced response for N-stepped rods are derived. A new two-way state-flow model is developed for this analysis based on the wave equation of rod segments. The results were obtained according to the graph analysis method [15, 16]. The coupled systems of multiple stepped rods and lumped elements, such as concentrated mass, spring and damper, are also investigated. Finally, some examples for various configurations of stepped rods and lumped elements with different boundary conditions are provided to demonstrate the efficiency of this method.

2. STATE-FLOW MODEL FOR ROD SEGMENT

The longitudinal vibration of a long rod consisting of N uniform stepped segments is firstly considered. It is assumed that the force acting on the plane face of an element segment of the rod is uniform for an applied axial excitation. Let $u_i(x_i, t)$ be the longitudinal displacement at the point x_i of the ith segment. The differential equation of the ith segment for a small longitudinal displacement can be expressed as

$$\frac{\partial^2 u_i(x_i, t)}{\partial x_i^2} - \left(\frac{\rho_i}{E_i^*}\right)\frac{\partial^2 u_i(x_i, t)}{\partial t^2} = 0, \tag{1}$$

where ρ_i is the mass density of the ith segment of the rod and E_i^* is the complex Young's modulus of the ith segment defined as $E_i(1 + j\delta_i)$, with δ_i representing the internal damping factor associated with the dynamic Young's modulus E_i of the ith segment and j equal to $\sqrt{-1}$. When one end of the rod, at $x_i = l_i$ of the Nth segment, is fixed and subjected to sinusoidal excitation at the other end, the general solution of the displacement is given as

$$u_i(x_i, t) = (a_{i,1}\mathrm{e}^{-\mathrm{j}k_i x_i} + a_{i,2}\mathrm{e}^{\mathrm{j}k_i x_i})\mathrm{e}^{\mathrm{j}\omega t}, \tag{2}$$

where k_i is the wave number of the ith segment, defined by $k_i = \omega/c_i$ with c_i being the phase speed of wave propagation $c_i = (E_i^*/\rho_i)^{1/2}$. $a_{i,1}$ and $a_{i,2}$ are dependent on the end conditions of the segment i given as

$$a_{i,1} = \frac{(U_{i-1}\mathrm{e}^{\mathrm{j}k_i l_i} - U_i)}{2j\sin(k_i l_i)}, \tag{3}$$

$$a_{i,2} = \frac{(U_i - U_{i-1}\mathrm{e}^{-\mathrm{j}k_i l_i})}{2j\sin(k_i l_i)}, \tag{4}$$

where U_{i-1} and U_i are the complex amplitudes of the longitudinal displacement at both ends of the ith segment for $x_i = 0$ and $x_i = l_i$. The force $f_i(x_i, t)$ that acts on the plane faces of the segment is dependent on the displacement $u_i(x_i, t)$ of the segment in the rod expressed as

$$\frac{\partial u_i(x_i, t)}{\partial x} - \frac{f_i(x_i, t)}{A_i E_i^*} = 0, \tag{5}$$

where A_i is the cross-segment area of the ith segment of the rod and $f_i(x_i, t)$ is defined as positive for tension force. Substituting equation (2) into equation (5), the force at the location x_i of the segment i can be expressed as

$$f_i(x_i, t) = (a_{i,3}\mathrm{e}^{-\mathrm{j}k_i x_i} + a_{i,4}\mathrm{e}^{\mathrm{j}k_i x_i})\mathrm{e}^{\mathrm{j}\omega t}, \tag{6}$$

where

$$a_{i,3} = -ja_{i,1}A_i E_i^* k_i, \tag{7}$$

$$a_{i,4} = ja_{i,2}A_i E_i^* k_i. \tag{8}$$

From equation (6), we see that $a_{i,3}$ and $a_{i,4}$ can also be expressed as the complex amplitudes of the forces that act on the ends of the ith segment, F_{i-1} for $x_i = 0$ and F_i for $x_i = l_i$. Thus, the relationship between each complex amplitude of the force and displacement at both ends can be expressed as

$$F_i = C_i F_{i-1} - A_i E_i^* k_i S_i U_{i-1}, \tag{9}$$

$$U_{i-1} = C_i U_i - \frac{S_i F_i}{A_i E_i^* k_i}, \tag{10}$$

where C_i and S_i are the symbols of $\cos(k_i l_i)$ and $\sin(k_i l_i)$. According to the above equations (9) and (10), these relationships can be expressed as a two-way state-flow model as shown in Figure 1, in which T_i is denoted as $\tan(k_i l_i)$. For the graph model of each segment, two variables correspond to one force and one

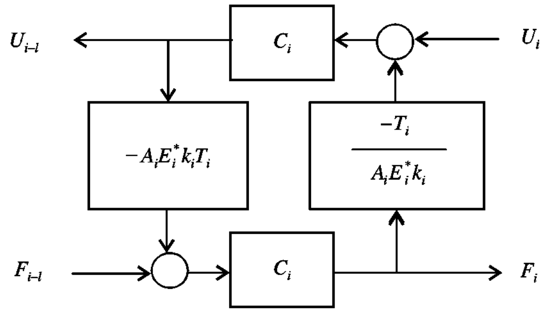

Figure 1. Two-way state-flow model for the ith rod segment.

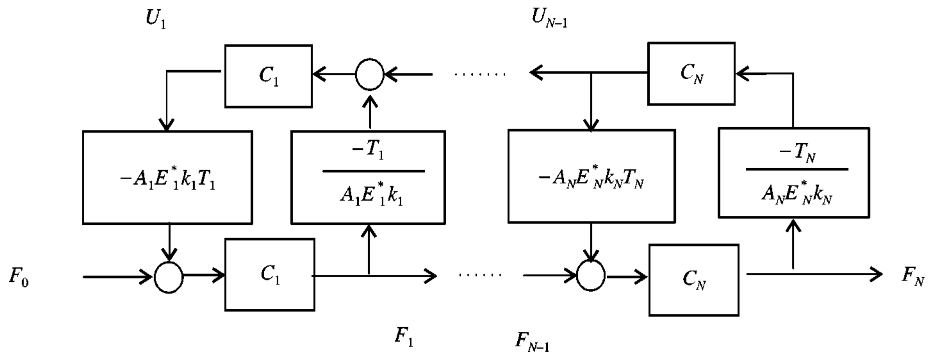

Figure 2. State-flow model for the multiple segment rod.

displacement, with opposite directions at the right or left end. Moreover, the lower-side state-flows at both ends are all forces whose direction is from left to right. The upper-side state-flows at both ends are also in the same direction. Thus, both state-flow models can be connected in a series. This technology can be extended over the entire system to construct the state-flow graph model of the N stepped rods as shown in Figure 2, in which the input state-flow from the left end matches the direction of the excitation.

3. FREQUENCY RESPONSE OF STEPPED ROD

The complex frequency response of any state of the state-flow graph model can be calculated by the gain formula [17, 18] described as

$$H = \frac{\sum_i P_i D_i}{D}, \tag{11}$$

where H is the complex frequency response function, P_i is the path gain of ith forward path, D is the determinant of the graph, and D_i is the cofactor of the ith forward path determinant of the graph with the loops touching the ith forward path removed.

For the graph model of a multiple stage rod, the force transmissibility from $f_0(t)$ to $f_N(t)$ is defined as the magnitude of the complex frequency response ratio of $F_N(\omega)$ to $F_0(\omega)$, denoted as $|H_{F_N}|$. Based on Figure 2, the number of the forward path from the variable F_0 to F_N is only one, in which the forward path gain is

$$P_1 = \prod_{j=1}^{N} C_j. \tag{12}$$

With respect to this forward path, the cofactor of this path is 1. From Figure 2, we see that the path of a state-flow forms a closed-loop when the state-flow passes through any vertical path from the lower to higher horizontal path, through the higher horizontal path from right to left, through any vertical path from the higher to lower horizontal path, and through the lower horizontal path from the left to the starting point. Thus, there are $N(N+1)/2$ loops in the graph model of the system. Thereby, the determinant of the graph can be expressed as

$$D = 1 + \sum_{i_2=1}^{N} \sum_{i_1=1}^{i_2} (-L_{i_1,i_2}) + \sum_{i_4=2}^{N} \sum_{i_3=2}^{i_4} \sum_{i_2=1}^{i_3-1} \sum_{i_1=1}^{i_2} \prod_{j=1}^{2} (-L_{i_{2j-1},i_{2j}}) + \cdots + \prod_{j=1}^{N} (-L_{j,j}), \tag{13}$$

where $L_{i,j}$ is the loop gain of each closed loop of the state-flow graph model defined as

$$L_{i,j} = \frac{A_j E_j^* k_j T_j T_i}{A_i E_i^* k_i} \prod_{k=i}^{j} C_k^2. \tag{14}$$

Substituting equations (12) and (13) into equation (11), the force transmissibility leads to

$$|H_{F_N}| = \left| \frac{\prod_{j=1}^{N} C_j}{1 + \sum_{i_2=1}^{N} \sum_{i_1=1}^{i_2} (-L_{i_1,i_2}) + \sum_{i_4=2}^{N} \sum_{i_3=2}^{i_4} \sum_{i_2=1}^{i_3-1} \sum_{i_1=1}^{i_2} \prod_{j=1}^{2} (-L_{i_{2j-1},i_{2j}}) + \cdots + \prod_{j=1}^{N} (-L_{j,j})} \right|. \tag{15}$$

When the complex frequency response ratio of the force acting on the right end of the segment i to the force excitation on the left end of the rod, denoted as H_{F_i}, is considered, we see that there is only one forward path with path gain $\prod_{j=1}^{i} C_j$ from the variable F_0 to F_i as shown in Figure 2. Moreover, the cofactor of this forward path which is formed by the part of the graph model on the right side of F_i can be represented as

$$D_i = 1 + \sum_{i_2=i+1}^{N} \sum_{i_1=i+1}^{i_2} (-L_{i_1,i_2}) + \sum_{i_4=i+2}^{N} \sum_{i_3=i+2}^{i_4} \sum_{i_2=i+1}^{i_3-1} \sum_{i_1=i+1}^{i_2} \prod_{j=i+1}^{i+2} (-L_{i_{2j-1},i_{2j}}) + \cdots + \prod_{j=i+1}^{N} (-L_{j,j}). \tag{16}$$

We see that equations (13) and (16) are power series with the same structure. Thus, both equations can be rewritten in general forms as

$$D_i = \sum_{k=0}^{N-i} E_{i+1,N,k}, \tag{17}$$

$$D = \sum_{k=0}^{N} E_{1,N,k}, \tag{18}$$

where

$$E_{i,N,K} = \begin{cases} \sum_{i_{2k}=i+k-1}^{N} \sum_{i_{2k-1}=i+k-1}^{i_{2k}} \cdots \sum_{i_2=i}^{i_3-1} \sum_{i_1=i}^{i_2} \prod_{j=1}^{k} (-L_{i_{2j-1}, i_{2j}}) & \text{for } k \geqslant 1, \\ 1 & \text{for } k = 0. \end{cases} \quad (19)$$

We see that the natural frequency of the system is equal to the eigenvalues of the denominator of H_{F_N}. The frequency equation can be given as

$$1 + \sum_{i_2=1}^{N} \sum_{i_1=1}^{i_{2k}} (-L_{i_1, i_2}) + \sum_{i_4=2}^{N} \sum_{i_3=2}^{i_4} \sum_{i_2=1}^{i_3-1} \sum_{i_1=1}^{i_2} \prod_{j=1}^{2} (-L_{i_{2j-1}, i_{2j}}) + \cdots + \prod_{j=1}^{N} (-L_{j,j}) = 0. \quad (20)$$

Substituting equations (12), (17) and (18) into equation (11), the complex frequency response H_{F_i} leads to

$$H_{F_i} = \frac{\prod_{j=1}^{i} C_j \sum_{k=0}^{N-i} E_{i+1,N,k}}{\sum_{k=0}^{N} E_{1,N,k}}. \quad (21)$$

When the frequency response ratio of the displacement at the right end of the ith segment to the excitation force H_{U_i} is considered, we see N-i forward paths from F_0 to U_i. The path gain for each forward path is

$$\frac{-T_{i+1} C_{i+1}^2}{A_{i+1} E_{i+1}^* k_{i+1}} \prod_{j=1}^{i} C_j, \quad \frac{-T_{i+2} C_{i+1}^2 C_{i+2}^2}{A_{i+2} E_{i+2}^* k_{i+2}} \prod_{j=1}^{i} C_j, \ldots, \quad \frac{-T_N}{A_N E_N^* k_N} \prod_{p=i+1}^{N} C_P^2 \prod_{j=1}^{i} C_j.$$

The cofactor of this forward path passing through the gain of $(-T_n/A_n E_n^* k_n) \prod_{p=i+1}^{n} C_p^2 \prod_{j=1}^{i} C_j$ is formed by the part on the right of segment F_n. Then, the frequency response ratio H_{U_i} leads to

$$H_{U_i} = \frac{\sum_{n=i+1}^{N} (-T_n/A_n E_k^* k_n) \prod_{k=i+1}^{n} C_k \prod_{j=1}^{n} C_j \sum_{p=0}^{N-n} E_{n+1,N,p}}{\sum_{k=0}^{N} E_{1,N,k}}. \quad (22)$$

The displacement and force response at the location x_i of the segment i can be obtained from equations (2) and (6) by substituting the results of equations (22) and (21) into the coefficients $a_{i,1}$, $a_{i,2}$, $a_{i,3}$ and $a_{i,4}$.

4. RESPONSE FOR RODS WITH IDENTICAL MATERIAL

If each segment of the rod is made by the same material, the structure properties E_i^* and ρ_i of each segment are identical. When the length of each segment is also identical, the gain of each vertical path of the graph model will be dependent on the area of the segment. Then, the loop gain defined as equation (14) becomes

$$L_{i,k} = \frac{A_k}{A_i} S^2 C^{2(k-i)}, \tag{23}$$

where C and S are the functions $\cos(kl)$ and $\sin(kl)$, in which l is the length of each segment of rod.

When the area of each segment is also identical, the loop gain $L_{i,k}$ in the graph model can be reduced to $S^2 C^{2(k-i)}$. Thus, the force frequency response ratio of F_i associated with excitation F_0, H_{F_i}, can be simplified to (see the appendix)

$$H_{F_i} = \frac{\cos((N-i)lk)}{\cos(Nlk)}. \tag{24}$$

Then the force transmissibility $|H_{F_N}|$ can be obtained from equation (24) given as

$$|H_{F_N}| = \frac{1}{|\cos(Nkl)|}. \tag{25}$$

In the same way, the displacement frequency response ratio of U_i associated with excitation F_0, H_{U_i}, leads to

$$H_{U_i} = \frac{-\sin((N-i)lk)}{AE^*k\cos(Nlk)}. \tag{26}$$

Equations (26) and (24) coincide with the displacement and force frequency response ratios at $x = (i/N)l$ of a uniform rod with length l for the conditions of a fixed end at $x = l$ and subjected to a sinusoidal excitation force at the other end.

5. COMBINED SPRING, MASS AND ROD SYSTEM

When a massless spring is connected by a series with $(i-1)$th and $(i+1)$th rod segments, the dynamic of the spring is given as

$$U_{i-1} = U_i - \frac{F_i}{K_i}, \tag{27}$$

where K_i is the stiffness of the spring. Based on equation (27), a two-way state-flow model with the same structure for the rod segment is constructed as shown in

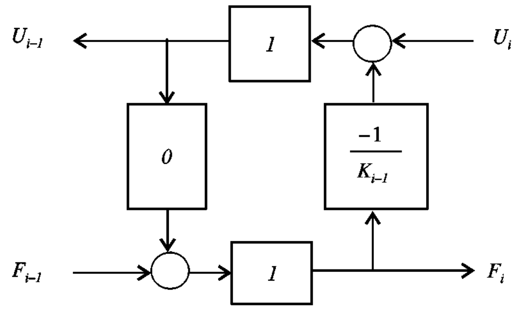

Figure 3. State-flow model for a massless spring.

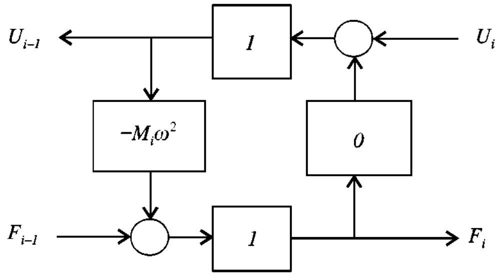

Figure 4. State-flow model for a concentrated mass.

Figure 3. Thus, the derived formulas for force and displacement frequency response in equations (21) and (22) can also be applied. The dynamic equation and the two-way state-flow model of the massless spring can be extended to a massless damper by replacing K_i with $j\omega B_i$, in which B_i is the damping coefficient of the damper. For a parallel connection of a spring and a damper system inserted in series, the same state-flow model as in Figure 3 can also be applied by simply replacing K_i with $K_i + j\omega B_i$. In the same way, this state-flow model can be extended to a connected series of multiple springs and dampers based on the model reduction method [15, 16].

If a concentrated mass M_i inserted into $(i-1)$th and $(i+1)$th rod segments is considered, the dynamic equation of the mass can be given as

$$F_i = F_{i-1} - \omega^2 M_i U_{i-1}. \tag{28}$$

Thus, the two-way state-flow model for the concentrated mass can be expressed as Figure 4, which has the same structure as that of the rods. From Figures 4 and 5, we see that both the state-flow models for the concentrated mass and spring can be combined as a single two-way state-flow model, identical to the state-flow model developed in the previous paper [15]. Thus, the compound lumped mass, spring and damper systems can be combined in this state-flow model developed for the stepped rod. Moreover, the analytical results developed for the stepped rod can be applied for the combined rod and lumped mechanical system.

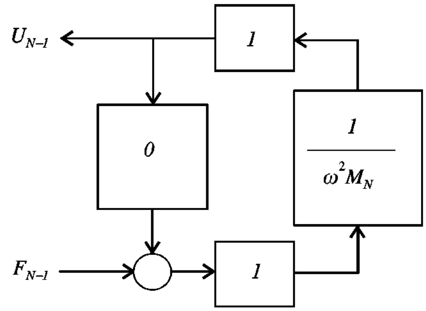

Figure 5. State-flow model for a lumped mass at the boundary.

If a concentrated mass M_N is located at the right end of the rod to replace the fixed end boundary, the dynamics of the equation of this concentrated mass can be given as

$$U_{N-1} = \frac{F_{N-1}}{\omega^2 M_N}. \tag{29}$$

Based on equation (29), the state-flow model for this ending concentrated mass is shown in Figure 5.

Since Figures 3–5 have the same structure as Figure 1, these state-flow models can be replaced by a general model, which has the same structure as Figure 1 as well as the gains $G_{1,i}$, $G_{2,i}$ and $G_{3,i}$ in the locations of $-A_iE_i^*k_iT_i$, C_i and $-T_i/A_iE_i^*k_i$ of Figure 1. The loop gain of each closed-loop of the state-flow graph model is redefined as

$$L_{i,j} = G_{1,i}G_{3,j}\prod_{k=i}^{j} G_{2,k}^2. \tag{30}$$

Thus, the previously derived formulas for the force transmissibility, force and displacement frequency response, and frequency equation can be used without modification.

6. EXAMPLES

A four-segment stepped free-fixed rod with an equal length, l, subjected to a force excitation at the free end as shown in Figure 6 is first considered. It is assumed that the structural properties, E^*, and ρ of each segment are identical. The area of each segment is based on the relationship $A_1:A_2:A_3:A_4 = 1:2:3:2$. If the force transmissibility is to be calculated, the two-way state-flow graph of the system can be obtained using four state-flow graph models of rod segment, as shown in Figure 2, connected by a series. We see that one forward path exits from the excitation to F_b with a gain C^4 and ten loops in the graph model. These loop

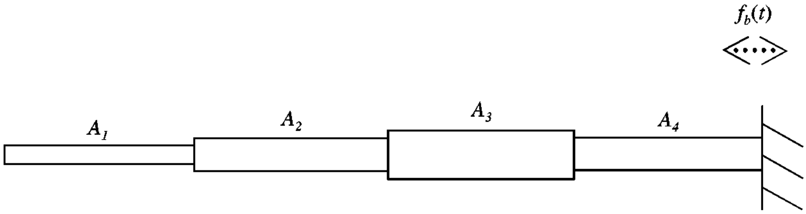

Figure 6. Stepped rod with identical structural properties.

gains are

$$L_{1,1} = L_{2,2} = L_{3,3} = L_{4,4} = S^2, \quad L_{1,2} = \tfrac{1}{2}C^2S^2, \quad L_{1,3} = \tfrac{1}{3}C^4S^2,$$

$$L_{1,4} = \tfrac{1}{2}C^6S^2, \quad L_{2,3} = \tfrac{2}{3}C^2S^2, \quad L_{2,4} = C^4S^2, \quad L_{3,4} = \tfrac{3}{2}C^2S^2. \tag{31}$$

By the derived formula, equation (15), the force transmissibility to the fixed boundary can be obtained as

$$|H_{F_b}| = \frac{1}{|\frac{25}{4}C^4 - 6C^2 + \frac{3}{4}|}. \tag{32}$$

Therefore, the frequency equation of the stepped rod is

$$\tfrac{25}{4}C^4 - 6C^2 + \tfrac{3}{4} = 0. \tag{33}$$

The natural frequency of the rod is equal to the eigenvalues of equation (33):

$$\omega = \frac{\sqrt{\rho}}{l\sqrt{E^*}}\left(\cos^{-1}\sqrt{3 \pm \frac{\sqrt{69}}{4}} \pm 2n\pi\right) \quad \text{for } n = 0, 1, 2, \ldots \tag{34}$$

Since it may not be easy to compare the computed results of the four-segment rod by the classical method, a two-segment rod with the same dimension and properties as those of the first two segments of the rod in the previous example is considered. In the same way, the frequency equation and the natural frequency of the two-segment rod can be obtained as

$$3C^2 - 1 = 0, \tag{35}$$

$$\omega = \frac{\sqrt{\rho}}{l\sqrt{E^*}}\left(\cos^{-1}\frac{1}{\sqrt{3}} \pm 2n\pi\right) \quad \text{for } n = 0, 1, 2, \ldots \tag{36}$$

Both results are identical to those calculated by the classical method in common textbooks [3, 9].

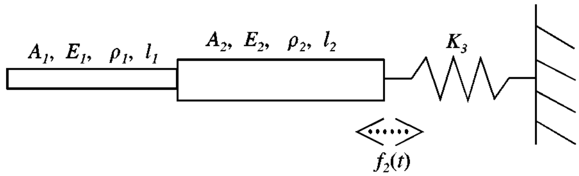

Figure 7. Multi-segment rod partially embedded in an elastic foundation.

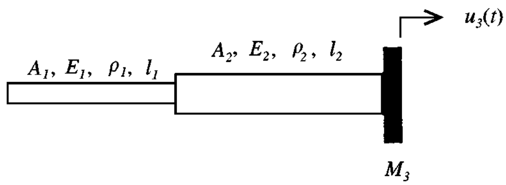

Figure 8. Concentrated mass on the end of multi-segment rod.

The third example is a stepped rod partially embedded in an elastic foundation as shown in Figure 7. The frequency response ratio of the force acting on the spring F_2 to the force excitation from the free end is considered. The state-flow graph model can be configured using two graph models for the rod segment as shown in Figure 1 and one graph model for the spring as shown in Figure 3 connected by a series. According to the graph model, we see that there are five loops and one forward path from excitation to F_2, in which the forward path gain is C^2 and the loop gains are

$$L_{1,1} = S_1^2, \quad L_{2,2} = S_2^2, \quad L_{3,3} = 0, \quad L_{1,2} = \frac{A_1}{A_2} S_1 S_2 C_1 C_2,$$

$$L_{1,3} = \frac{A_1 E_1 k_1}{K_3} S_1 C_1 C_2^2, \quad L_{2,3} = \frac{A_2 E_2 k_2}{K_3} C_2 S_2. \tag{37}$$

The frequency response ratio can be calculated by equation (21) expressed as

$$H_{F_2} = \frac{1}{C_1 C_2 - (1/K_3)(A_2 E_2 k_2 S_2 C_1 + A_1 E_1 k_1 S_1 C_2) - ((A_1 E_1 k_1 / A_2 E_2 k_2) S_1 S_2)}. \tag{38}$$

If the spring and the fixed boundary of the third example are replaced by a concentrated mass M_3 as shown in Figure 8, the state-flow graph model for the spring can be replaced by that for a concentrated mass located at the end as shown in Figure 5. When the purpose is to determine the frequency response of the displacement of the mass subjected to force excitation F_0, there will also be one forward path and six loops. The forward path gain and these loop gains are similar to those for the third example except that $-K_3$ is replaced by $\omega^2 M_3$. Thus, the

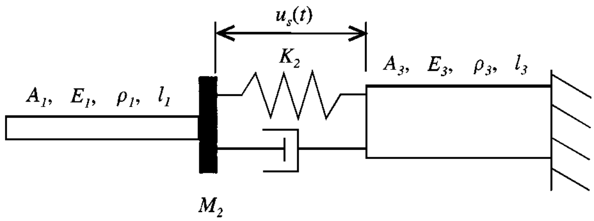

Figure 9. Compound rod and concentrated mass and spring system.

results can be calculated by equation (22) based on the graph model as

$$U_3 = \frac{F_0}{A_1 E_1 k_1 S_1 C_2 + A_2 E_2 k_2 C_1 S_2 + M_3 \omega^2 (C_1 C_2 - A_1 E_1 k_1 S_1 S_2 / A_2 E_2^* k_2)}. \tag{39}$$

A compound rod and lumped mass and a spring are considered in the fifth example as shown in Figure 9. The state-flow graph model can be constructed in the same way. Moreover, the parts for the lumped mass and spring mass can be reduced to a single two-way state-flow model similar to that used for a rod segment. Thus, the graph model of the system has six loops. If the frequency response of the extension displacement of the spring U_s is to be calculated, equation (22) can be firstly applied to the calculation of the response of U_2 and U_3. Then, the results will be obtained as

$$U_s = \frac{A_3 E_3 k_3 F_0}{C_3 (A_3 E_3 k_3 C_1 C_3 (K_2 + \mathrm{j} B_2 \omega) - (A_1 E_1 k_1 S_1 + M_2 C_1 \omega^2)(A_3 E_3 k_3 C_3 + S_3 (K_2 + \mathrm{j} B_2 \omega)))}. \tag{40}$$

7. CONCLUSIONS

The analytical and closed-form frequency response of a longitudinal force and displacement of multi-stepped rods and the coupled system of the stepped rods and lumped elements with non-classical boundary conditions subjected to force excitation at the free end has been derived. In this paper, a two-way state-flow model of the rods was firstly developed according to the longitudinal dynamics of rods. Based on the developed model, the frequency response of the force and displacement of each segment of stepped rods was derived. Moreover, the coupled system of multiple stepped rods and lumped elements, spring, damper and concentrated mass system was also investigated. Finally, some examples of the force and displacement response of multi-stepped rods with identical and different properties, stepped rods with non-classical boundary conditions, coupled stepped rods and lumped elements have been examined to show the feasibility of this method.

Although only the longitudinal vibration of rods and lumped elements have been investigated in this article, this method and the derived results can be extended to other dynamic systems governed by the one-dimensional wave equation.

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APPENDIX

The force frequency response ratio of F_i associated with excitation F_0, H_{F_i}, is given as equation (21). Since each property of the rods is identical, the loop gain $L_{i,k}$

in the graph model is reduced to $S^2C^{2(k-i)}$. Thus, the denominator of H_{F_i} as shown in Equation (21) can be expressed as

$$\sum_{k=0}^{N} E_{1,N,K} = 1 - S^2(N + (N-1)C^2 + (N-2)C^4 + \cdots + C^{2(N-1)}) + S^4\left(\frac{N(N-1)}{2!} + \frac{(N-1)(N-2)(N-4)}{2!}C^2 + \cdots + (N-1)C^{2(N-2)}\right)$$
$$- S^6\left(\frac{N(N-1)(N-2)}{3!} + \frac{(N-1)(N-2)(N-3)(N-3)}{3!}C^2 + \cdots + \frac{(N-1)(N-2)}{2!}(C^{2(N-3)}\right)$$
$$+ \cdots + S^{2N-2}(N + (N-1)C^2) + S^{2N}$$
$$= C^{2N} - \binom{N}{2}C^{2N-2}S^2 + \binom{N}{4}C^{2N-4}S^4 - \binom{N}{6}C^{2N-6}S^6 + \cdots$$
$$= C^N \cos(Nkl). \tag{A1}$$

For the numerator of H_{F_i} as shown in equation (21), we see that

$$\prod_{j=1}^{i} C_j = C^i. \tag{A2}$$

Since each property of the rods is identical, the relationship between each loop gain of the graph model is given as

$$L_{i,j} = L_{i+k,j+k}, \quad \text{for } k = 1, 2, \ldots, N \tag{A3}$$

Then,

$$\sum_{k=0}^{N-i} E_{i+1,N,k} = \sum_{k=0}^{N-i} E_{1,N-i,k}. \tag{A4}$$

Equation (A4) has a similar form as equation (A1) except the subscripts N replaced by $N - i$. Thus, equation (A4) can be derived in the same way. Substituting the results of equations (A3) and (A4) into the numerator of Equation (21) will give us

$$\prod_{j=1}^{i} C_j \sum_{k=0}^{N-i} E_{i+1,N,k} = C^N \cos((N-i)kl). \tag{A5}$$

Based on equations (A1) and (A5), the frequency response of force ratio H_{F_i} can be represented as equation (24).