



NON-LINEAR MODAL INTERACTIONS IN SHALLOW SUSPENDED CABLES

V. N. PILIPCHUK AND R. A. IBRAHIM

Department of Mechanical Engineering, Wayne State University, Detroit, MI 48202, U.S.A.

(Received 20 April 1997, and in final form 7 April 1999)

This paper examines different regimes of non-linear modal interactions of shallow suspended cables. In a high-energy level, the equations of motion in terms of in-plane and out-of-plane co-ordinates are strongly coupled and cannot be linearized. For this type of problem, a special co-ordinate transformation is introduced to reduce the number of strongly non-linear differential equations by one. The resulting equations of motion are written in terms of stretching, transverse (geometrical bending), and swinging co-ordinates, and are suitable for analysis using standard quantitative and qualitative techniques. Both free and forced vibrations of the cable are considered for in-plane and out-of-plane motions. The cable stretching free vibrations results in parametric excitation to the cable transverse motion. Under in-plane forced excitation the stretching motion is directly excited while the transverse motion is parametrically excited.

© 1999 Academic Press

1. INTRODUCTION

Within the framework of the linear theory of suspended cables, one can determine the natural frequencies and mode shapes of in-plane and out-of-plane motions. For small sag-to-span ratio, the symmetric in-plane modes are governed by a parameter which is a function of cable geometry and elasticity [1, 2]. The influence of tension and deflection induced by dynamically exciting a suspended cable was examined by Irvine and Griffin [3]. This influence is mainly found due to a resonant-like phenomenon. The effect of weak non-linearities on the normal-mode natural frequencies was found to produce considerable variation in the cable natural frequencies [4]. The non-linear free vibration analysis was then further studied [4–7] and it was shown that the dynamical behavior is either hardening or softening, depending on cable properties. In addition to regular periodic motion, complex chaotic motion was predicted for certain excitation and cable parameters; see for example references [8, 9].

The non-linear modal interaction between in-plane and out-of-plane modes has been examined by several authors [5, 10–12]. Under external excitation of the first in-plane mode, the out-of-plane mode can be indirectly excited if the natural frequencies of the in-plane and out-of-plane modes are close to the ratio 2:1. The

analysis has been extended to include three- and four-mode interactions by Lee and Perkins [13]. This type of modelling can result in the existence of simultaneous internal resonance conditions among the interacting modes. The presence of simultaneous internal resonances can result in quasi-periodic and chaotic motions. The analytical models of single- or multi-modal interaction in the previous investigations were obtained using Galerkin's method for discretizing the original partial differential equations of the continuum. However, this type of discretization has been questioned by Pakdemirli *et al.* [14] for cables with non-zero sag.

Rega *et al.* [7] have shown that one-mode approximation for a shallow cable generally yields a strongly non-linear oscillator with more than one fixed point, i.e., an oscillator with a non-local structure of the phase portraits. For many cases of a reduced single-degree-of-freedom system, the phase plane representation and the energy integral give a sufficiently complete description of the essentially non-linear behavior. Simple numerical experiments showed that a qualitatively reasonable description of the non-linear dynamical regimes of a shallow cable can be provided based on at least a two-mode approximation. Moreover, the modal co-ordinates are *strongly coupled*. In such multi-dimensional cases, phase-plane representation is not applicable and the energy integral does not give a complete description of the cable motion.

This paper introduces a special type of co-ordinate transformation which provides a description of the in-plane and out-of-plane motions. The new co-ordinates are constructed along and perpendicular to the unstretched manifold, i.e., the manifold of cable positions with identical centerline lengths. The new co-ordinates have a certain physical meaning, and the analytical modelling has the advantage of reducing the number of non-linear equations of motion by one equation. Specific examples are given to demonstrate the technique. The present form of the transformation was proposed originally for different models of shallow elastic shells [15]. However, the classic cable model possesses some special features, such as absolute flexibility of its centerline, and the possibility of significant amplitudes of both in-plane and out-of-plane (swinging) motions. As a rule, the cable transverse motion is accompanied by small oscillation of the centerline length about its pre-stretched state; however, the cable model generally does not admit any compression. This means that the centerline oscillation of sufficiently large amplitude should become unstable and non-realizable practically. From the mathematical point of view, the negative tension changes type of the cable partial differential equations of motion (from hyperbolic to elliptic) and results in the ill-posed problem reported by Triantafyllou and Howell [16]. With reference to this work, it will be shown (see section 6) that the problem becomes well-posed if a small bending rigidity is included into the model.

2. EQUATIONS OF MOTION

The system under consideration is shown in Figure 1. In the absence of a gravitational field, the cable lies in the xz plane, and an arbitrary point P of the cable has the co-ordinates $(x, 0, w_0)$, where $w_0 = w_0(x)$ describes the undeformed elastic line of the cable. Under gravitational field and environmental excitation the

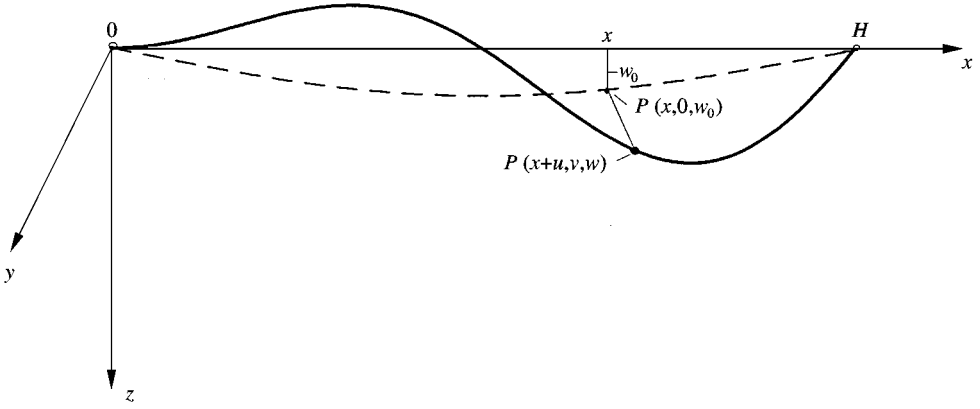


Figure 1. Schematic diagram of a shallow cable showing the co-ordinate system.

co-ordinates of P are changed from $(x, 0, w_0)$ to $(x + u, v, w)$, where $u = u(x, t)$ and $v = v(x, t)$ are displacements of the point along the x and y directions, respectively, and $w = w(x, t)$ is the z co-ordinate of the same point at the deformed position. It is assumed that a longitudinal component of the particle velocity is negligible compared to transverse ones. Note that this assumption is frequently used in the dynamical theory of elastic shallow structures, starting from Kirchhoff's work [17]. Regarding the cable, this assumption was introduced and justified by Irvine and Caughey [2]. The elimination of the longitudinal velocity component does not completely eliminate the stretching rigidity influence on a time scale of transverse (in-plane or out-of-plane) motions associated with cable stretching.

Under this assumption, the Lagrangian function of the cable can be written as

$$L = \int_0^l \left\{ \frac{\rho A}{2} (v_{,t}^2 + w_{,t}^2) - \frac{EA}{2} e^2 + [\rho g A + f(x, t)] w \right\} ds, \quad (1)$$

where l is the original length of the cable; subscripts preceded by a comma denote differentiation with respect to the subscripted variable; ρ is the density (mass/unit volume) of the cable; A is the original cross-sectional area, which is assumed to be constant; E is Young's modulus of elasticity; g is the gravitational acceleration; $f(x, t)$ is a distributed external in-plane force acting upon the cable in addition to the cable weight; and

$$e = \frac{[(1 + u_{,x})^2 + v_{,x}^2 + w_{,x}^2]^{1/2}}{[1 + w_{0,x}^2]^{1/2}} - 1 \quad (2)$$

is the longitudinal strain of the cable.

Under the assumption that the cable is shallow (i.e., the sag-to-length ratio is very small and the cable has a small slope), the following relationships hold:

$$|v_{,x}| \ll 1, \quad |w_{,x}| \ll 1, \quad |u| = O(v^2, w^2) \quad (3)$$

The last equality justifies the elimination of the longitudinal component of velocity in the kinetic energy expression. Expanding relation (2) and keeping only terms up

to second order gives

$$e = u_{,x} + \frac{1}{2}(v_{,x}^2 + w_{,x}^2 - w_{0,x}^2). \quad (4)$$

Note that although this expression has a weakly non-linear form, however, it will result in strong non-linear equations of motion due to the presence of more than one fixed point, as will be shown later.

The partial differential equations of motions can be obtained by means of the Hamilton principle:

$$\begin{aligned} \delta I = & \int_{t_1}^{t_2} \int_0^H \{ \rho A (v_{,t} \delta v_{,t} + w_{,t} \delta w_{,t}) \\ & - EAe \left(\frac{\partial e}{\partial u_{,x}} \delta u_{,x} + \frac{\partial e}{\partial v_{,x}} \delta v_{,x} + \frac{\partial e}{\partial w_{,x}^2} \delta w_{,x} \right) \\ & + [\rho g A + f(x, t)] \delta w \} dx dt = 0, \end{aligned}$$

where I is the action, δ denotes variation, ds has been replaced by dx , and l by the span of the cable H , based on relations (3).

Using relation (4) for evaluating the partial derivatives of e , integrating by parts with respect to x and t , and taking into account the boundary conditions

$$u(0, t) = u(H, t) = 0, \quad v(0, t) = v(H, t) = 0, \quad w(0, t) = w(H, t) = 0, \quad (5)$$

one obtains

$$\begin{aligned} \delta I = & \int_{t_1}^{t_2} \int_0^H \{ EAe_{,x} \delta u + [- \rho A v_{,tt} + EA(ev_{,x})_{,x}] \delta v \\ & + [- \rho A w_{,tt} + EA(ew_{,x})_{,x} - \rho g A - f(x, t)] \delta w \} dx dt = 0. \end{aligned}$$

This gives the following equations of motion:

$$\begin{aligned} e_{,x} = 0, \quad v_{,tt} - \frac{E}{\rho} (ev_{,x})_{,x} = 0, \\ w_{,tt} - \frac{E}{\rho} (ew_{,x})_{,x} = g(1 + p(x, t)), \end{aligned} \quad (6)$$

where $p(x, t) \equiv f(x, t)/(\rho g A)$.

These equations describe the horizontal, $u(x, t)$, and $v(x, t)$ and the vertical $w(x, t)$ co-ordinates of point P. The first relation reveals that the strain e is only a function of time. In this case, one can express the displacement u in terms of the other displacements using relation (4), i.e.,

$$u(x, t) = ex - \frac{1}{2} \int_0^x (v_{,x}^2 + w_{,x}^2 - w_{0,x}^2) dx.$$

Using the boundary condition for $x = H$, $u(H, t) = 0$, one defines the value of the strain as

$$e = \frac{1}{2H} \int_0^H (v_{,x}^2 + w_{,x}^2 - w_{0,x}^2) dx. \quad (7)$$

The additional equations for the co-ordinates $v(x, t)$ and $w(x, t)$ can be rewritten as

$$v_{,tt} - \frac{E}{\rho} e v_{,xx} = 0, \quad w_{,tt} - \frac{E}{\rho} e w_{,xx} = g(1 + p(x, t)). \quad (8)$$

The same differential equations were considered by Lee and Perkins [12, 13]. The only difference is that in the present analysis the value $w = w(x, t)$ is the *co-ordinate* measured from the horizontal level, but it was considered as the *displacement* measured from the static equilibrium position in references [12, 13]. A special feature of equations (8) is that the co-ordinate values $w = v = u = 0$ do not correspond to the cable's equilibrium, and hence a linearization around $w = v = u = 0$ is not physically reasonable. However, this feature is not related to the purpose of the present work, because the linearization will be done around the manifold of unstretched centerline positions ($e = 0$), and not around the original equilibrium. From the standpoint of the proposed transformations, the chosen co-ordinate systems gives an essential advantage since it brings an expression for the strain e into its simplest form (7).

The *undeformed* centerline length will be state by a configuration at which

$$u = 0, \quad v = 0, \quad w = w_0(x) = D_0 \sin(\pi x/H), \quad (9)$$

where D_0 is constant.

The selected configuration (9) is suitable because it is close to the stable static equilibrium, and the coefficient D_0 gives a good estimate for the cable sag in the static equilibrium position. Note that configuration (9) is not an exact equilibrium position since the function $w_0(x)$ does not satisfy the static equation, $-(E/\rho)ew_{,xx} = g$.

The original (unstretched) length of the cable under the assumption of small sag is expressed by parameters H, D_0 as

$$l = \int_0^H (1 + \frac{1}{2} w_{0,x}^2) dx = H[1 + \frac{1}{4}(\pi D_0/H)^2] = H[1 + O(D_0^2/H^2)]. \quad (10)$$

Finally, introducing the dimensionless parameters

$$\eta = \frac{x}{H}, \quad V = \frac{v}{D_0}, \quad W = \frac{w}{D_0}, \quad W_0 = \frac{w_0}{D_0}, \quad \tau = t / \left(\frac{H^2}{D_0} \sqrt{\frac{\rho}{E}} \right), \quad (11)$$

$$\varepsilon = (H/D_0)^2 e \equiv \varepsilon \{W, V\},$$

the equation of motion and the boundary conditions take the form

$$W_{,\tau\tau} - \varepsilon[W, V]W_{,\eta\eta} = \mu(1 + p), \quad V_{,\tau\tau} - \varepsilon[W, V]V_{,\eta\eta} = 0, \quad (12)$$

$$W(0, \tau) = W(1, \tau) \quad \text{and} \quad V(0, \tau) = V(1, \tau), \quad (13)$$

where

$$\varepsilon[W, V] = \frac{1}{2} \int_0^1 (W^2_{,\eta} + V^2_{,\eta} - W_{0,\eta}^2) d\eta \quad (14)$$

is a strain functional of the cable, and

$$\mu = \rho g H^4 / (ED_0^3) \quad (15)$$

is a dimensionless gravity parameter.

We consider the cable dynamics when the cable centerline displacement can be of order of the sag, i.e., in terms of the original co-ordinates the following expressions hold:

$$v(x, t) \sim w(x, t) \sim w_0(x) \quad (16)$$

This means that both linear and non-linear terms in equations (12) can have the same order of magnitude. This assumption, however, does not contradict the weakly non-linear approximation of the cable strain (4). In fact, the approximation for strain has been obtained under the condition that $u_{,x}(x, t)$, $v_{,x}(x, t)$, $w_{,x}(x, t)$, $w_{0,x}(x, t)$, are small enough compared with 1, but this does not contradict relationship (16).

3. FREE VIBRATION OF THE FIRST TWO IN-PLANE MODES

In this section, physical treatment of the simple case of two in-plane modes motion and some numerical simulations will be demonstrated in order to justify the need for a special co-ordinate transformation. For example, in the absence of gravity, the cable equilibrium positions are obtained by eliminating the inertia terms and setting $\mu = 0$ in equations (12). In this case, one obtains the trivial solution $W = 0$, $V = 0$, which corresponds to an extremely compressed cable. This equilibrium is unstable and hardly realizable physically. Other roots of the static equilibrium are given by the relationship

$$\varepsilon[W, V] = 0. \quad (17)$$

This equation describes a manifold of the cable configuration with no stretching deformation, i.e., the cable preserves its length on the manifold. Under physically realizable conditions the length of the cable will not be significantly changed during the motion. This implies that at any time instant the cable is sufficiently close to the manifold described by equation (17). However, it is not necessarily close to the original equilibrium position.

To demonstrate this, consider the case of free in-plane vibration with two-mode approximation

$$W = W_1(\tau) \sin \pi \eta + W_2(\tau) \sin 2\pi \eta, \quad W_0 = \sin \pi \eta,$$

$$V \equiv 0, \quad p \equiv 0.$$

The related trigonometric expansion for constant gravitational force in equation (12) includes only odd harmonics, and its “two-modes” representation is $\mu \sim (4\mu/\pi) \sin \pi \eta$. Substituting these relationships into equations (12), and equating coefficients of $\sin \pi \eta$, $\sin 2\pi \eta$ to zero gives

$$W_{1,\tau\tau} + \pi^2 \varepsilon(W_1, W_2) W_1 = 4\mu/\pi, \quad W_{2,\tau\tau} + 4\pi^2 \varepsilon(W_1, W_2) W_2 = 0, \quad (18)$$

where

$$\varepsilon(W_1, W_2) = (\pi^2/4)(W_1^2 + 4W_2^2 - 1).$$

In terms of W_1 and W_2 co-ordinates the functional ε becomes a function of W_1 and W_2 . In this case, equations (18) describe two-mode free vibration of the cable. It should be noted that the set $\{\sin \pi\eta, \sin 2\pi\eta\}$ represents *the exact normal modes of the non-linear system* under consideration in the absence of gravity (see the structure of the left-hand parts of equations (18)). Here gravity leads to some changes in the odd-order modes; however, it can be shown that for the shallow system these changes are not significant.

It is suitable to represent the motion of the system by its path in the configuration plane W_1W_2 . Figures 2(a), 3(a), and 4(a) give examples of the path which has been obtained by numerical simulations of equations (18) for different total energy levels in comparison with the saddle-point level E^* (see below). Figures 2(b), 3(b), and 4(b) represent the corresponding time-history records.

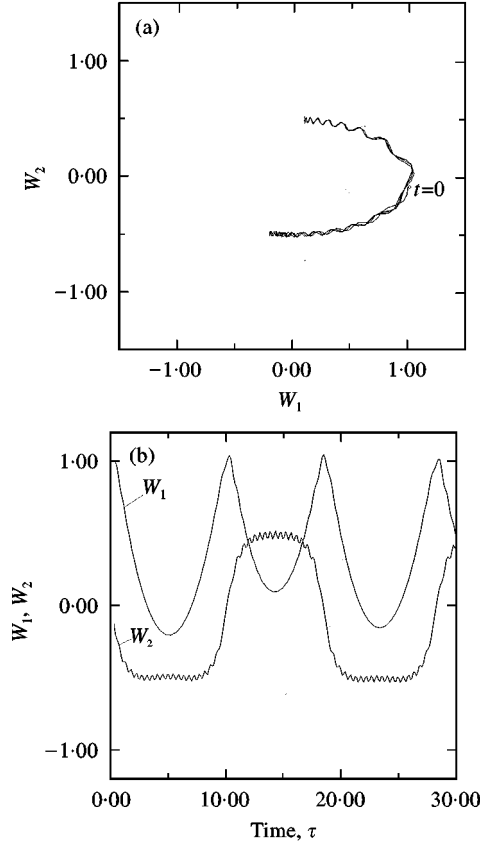


Figure 2. (a) Trajectory in the configuration space of the cable motion described by two in-plane modes for low total energy $E = 0.0653 < E^*$ and $\mu = 0.09$; (b) time-history records of two in-plane modes for $E = 0.0653 < E^*$, $\mu = 0.09$, and initial conditions $W_1(0) = 1.00059$, $W_2(0) = 0.0$, $\dot{W}_1(0) = 0.0$, $\dot{W}_2(0) = -0.6$.

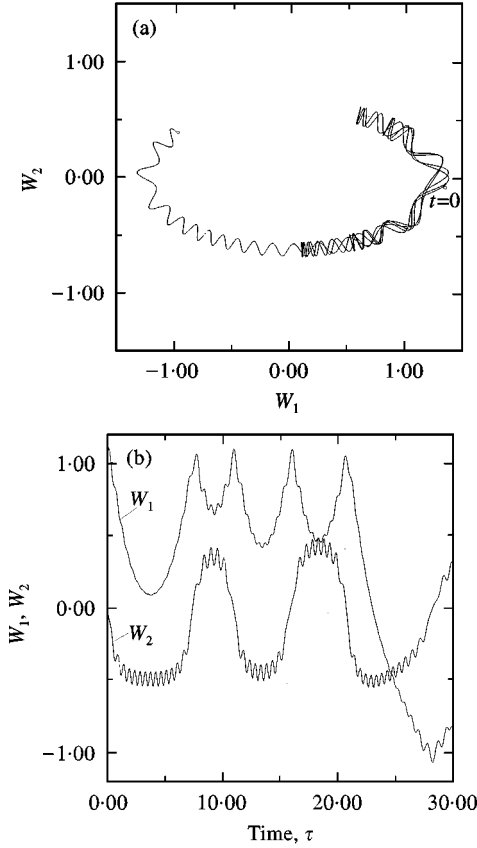


Figure 3. (a) Trajectory in the configuration space of the cable motion described by two in-plane modes for higher total energy $E = 0.2437 > E^*$ and $\mu = 0.09$; (b) time-history records of two in-plane modes for $E = 0.2437 > E^*$, $\mu = 0.09$, and with initial conditions $W_1(0) = 1.1$, $W_2(0) = 0.0$, $\dot{W}_1(0) = 0.0$, $\dot{W}_2(0) = -0.45$.

To give a physical interpretation of the shape of these trajectories, we establish surface of the potential energy of the system described by equations (18),

$$\Pi = \frac{\pi^4}{16}(W_1^2 + 4W_2^2 - 1)^2 - \frac{4}{\pi}\mu W_1,$$

which is shown by Figure 5(a). Figure 5(b) shows the projection of the potential energy on the $W_1 W_2$ plane. A potential hill at the center corresponds to the cable state under maximum compression. The system moves relatively slowly along a potential channel around the hill with a high vibration frequency in an orthogonal direction to the channel. The slow motion corresponds to the cable's transverse displacement, while the high-frequency oscillation belongs to cable stretching. There are two stationary points at the channel bottom. One is the stable original static position, and the other is an unstable saddle point (the inverted position of the cable). Taking into account the surface symmetry with respect to

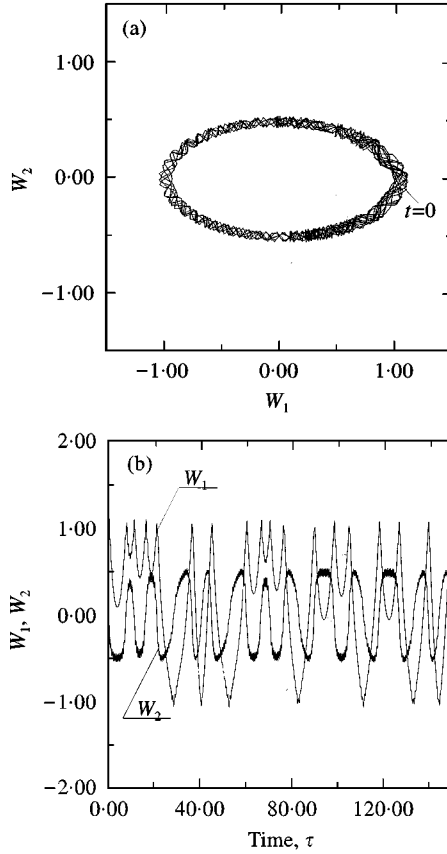


Figure 4. (a) Trajectory in the configuration space for long time motion of two in-plane modes for higher total energy level and same parameters as Figure 3(b); (b) long time-history records of two in-plane modes for $E = 0.2437 > E^*$, $\mu = 0.09$ demonstrating the long-term irregular behavior. Simulation is obtained using the fourth-order Runge-Kutta method with step of integration 0.0001.

axis $W_2 = 0$, the co-ordinates of these points can be found by solving the cubic equation

$$W_1^3 - W_1 - \frac{16}{\pi^5} \mu = 0.$$

This equation has three roots: the stable position $W_1 = 1 + O(\mu)$, and two unstable positions, the inverted position $W_1 = -1 + O(\mu)$ and the compressed almost straight position $W_1 = O(\mu)$. The energy level corresponding to the inverted position can be easily calculated. For the value $\mu = 0.09$, which is used for numerical simulations, this energy is equal to $\Pi^* = 0.229184$. The results of the numerical simulation are obtained for two special cases.

- (1) When the total energy of the system $E = T + \Pi$ satisfies the condition $E \ll \Pi^*$, the motion has a sufficiently regular character, as shown in Figures 2(a) and (b). This is a quasi-linear region. A typical sequence of the elastic

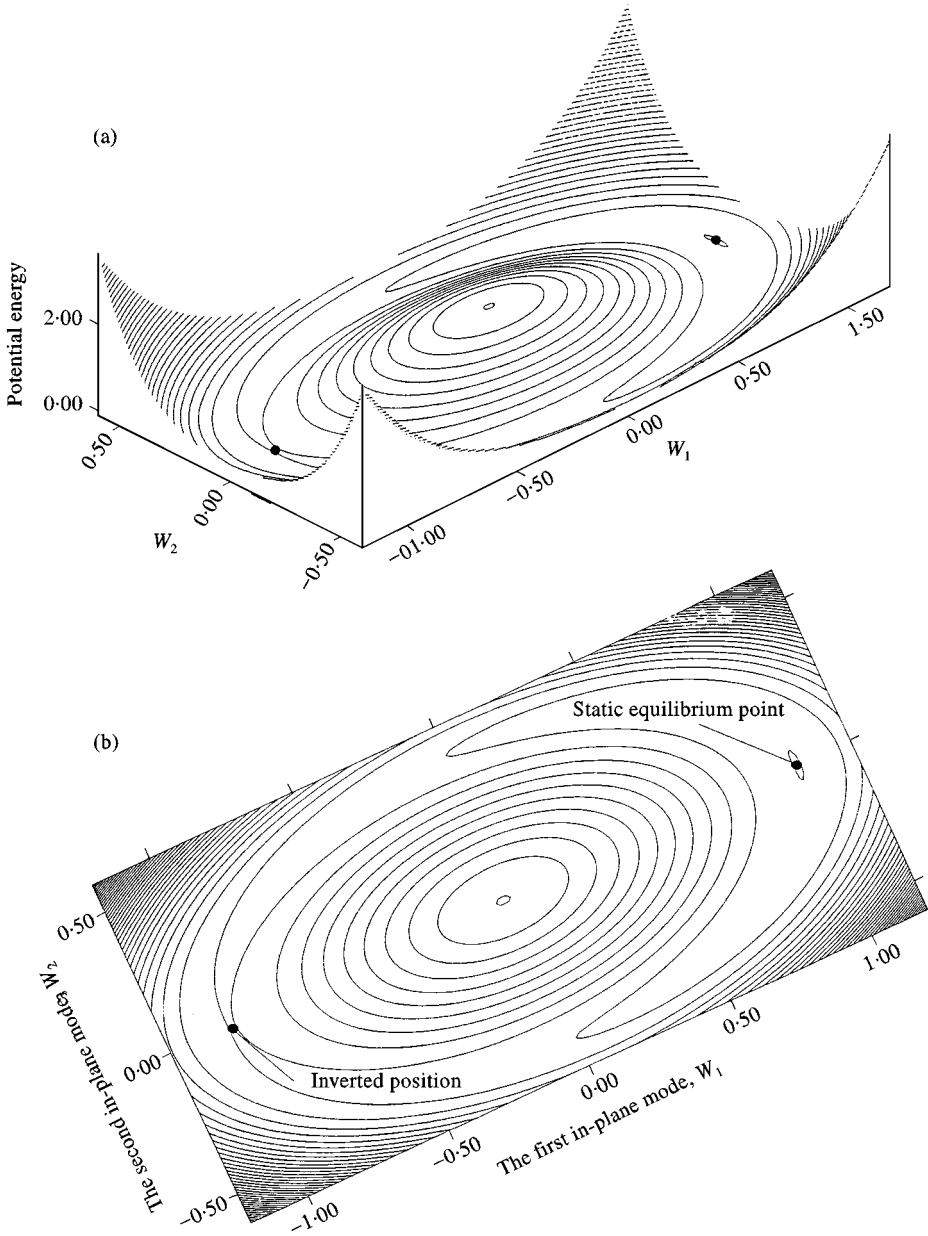


Figure 5. (a) Surface of the potential energy $\bar{\Pi} = (16/\pi^4)\Pi(W_1, W_2) = (W_1^2 + 4W_2^2 - 1)^2 - \bar{\mu}W_1$, for $\bar{\mu} = (64/\pi^5)\mu = 0.1$; (b) projection of the potential energy surface on the W_1, W_2 plane.

lines of the cable for this vibration case is shown in Figure 6(a) for different values of ψ , where ψ is the angle which defines the relationship between the co-ordinates W_1, W_2 , such that $W_2/W_1 \sim (1/2) \tan \psi$.

- (2) When the total energy has an order of Π^* , one has a significant influence of the unstable saddle point. As a result, the system displays very complicated behavior, as shown in Figures 3(a), (b) and 4(a), (b). Note that the motion of

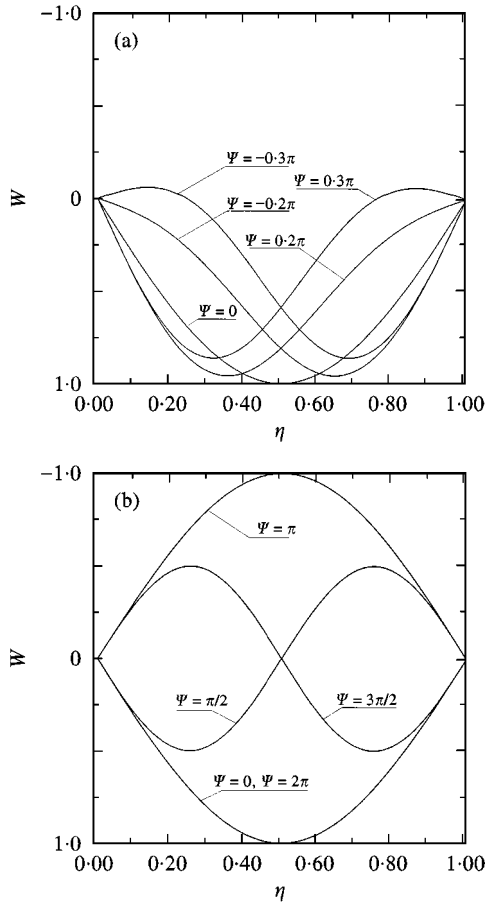


Figure 6. (a) Cable configurations considering two in-plane mode representation for different values of the angle ψ in the neighborhood of the static equilibrium position; (b) global cable configurations considering two in-plane mode representation for different values of the angle ψ .

the system can be qualitatively represented by the motion of a ball rolling on the potential surface. Having the energy, $E \sim \Pi^*$, the ball is able to move on the surface of negative curvature in the neighborhood of the saddle point. (Possible positions of the elastic line of the cable are shown in Figure 6(b)). It is known that a system on a surface of negative curvature can show stochastic-like behavior. A full description of the corresponding, essentially non-linear orbits is beyond the scope of the present work. However, a technique for simplification of the problem will be considered.

Note that the numerical results showed in these figures have been obtained for the two-modes model in order to illustrate better and make clear a mathematical sense of the coordinate transformation. It should be noted that some of the two-mode regimes may become unstable with respect to any small perturbation of higher modes, and hence it is hard to realize practically, as will be shown in section 6.

4. NON-LINEAR CO-ORDINATE TRANSFORMATION

Based on qualitative features of the two-dimensional motion shown in the previous section, the co-ordinates transformation will be introduced first in its simplest form for the two in-plane modes model. The mathematical structure of the transformation can then be generalized. Note that the system trajectory is located around the ellipse $\varepsilon(W_1, W_2) = 0$ which gives the two-dimensional special manifold (17). Figures 3 and 4 show that it is hardly possible to obtain the corresponding solution starting from the linearized system about the original static equilibrium position. However, it is possible to linearize the system on the normal to the ellipse direction. For this reason, we will establish the transformation of variables in the configuration plane of the cable. With reference to Figure 7, one can write the following transformation [15] $\{W_1, W_2\} \rightarrow \{\xi, s\}$:

$$W_1 = W_1^0(s) + n_1(s)\xi, \quad W_2 = W_2^0(s) + n_2(s)\xi, \quad (19)$$

where $(W_1^0, W_2^0) = (W_1^0(s), W_2^0(s))$ is an arbitrary point on the ellipse $W_1^2 + 4W_2^2 = 1$; $s = s(\tau)$ is the arc length of the ellipse measured from the point $(W_1, W_2) = (1, 0)$; $\xi = \xi(\tau)$ is the normal to the ellipse co-ordinate; and n_1 and n_2 are the direction cosines of the normal vector. Note that the new co-ordinates ξ and s possess a clear mechanical sense, where the variable ξ describes stretching, and s defines the in-plane transverse motion of the cable.

The normal vector is defined by the relationships

$$(n_1, n_2) = \frac{\text{grad } \varepsilon(W_1^0, W_2^0)}{|\text{grad } \varepsilon(W_1^0, W_2^0)|_{R^2}}$$

or

$$n_1 = \frac{W_1^0}{(W_1^{02} + 16W_2^{02})^{1/2}}, \quad n_2 = \frac{4W_2^0}{(W_1^{02} + 16W_2^{02})^{1/2}}, \quad (20)$$

where $(|\dots|_{R^2})$ is the norm in R^2 .

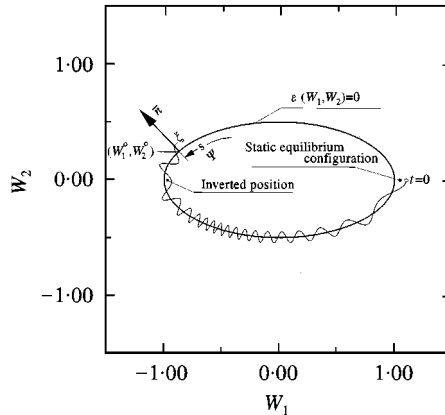


Figure 7. Transformation of co-ordinates in the configuration plane showing the normal ξ and tangential s , ψ on the trajectory in the configuration space W_1, W_2 . The initial conditions for trajectory are: $W_1(0) = 1.1$, $W_2(0) = 0.0$; $\dot{W}_1(0) = 0.0$, $\dot{W}_2(0) = -0.5$.

Substituting expressions (19) into the modal equations of motion (18) and taking projections on the normal (n_1, n_2) and on the tangent ($dW_1^0/ds, dW_2^0/ds$), respectively, gives the equations of motion in terms of the new co-ordinates ξ and s :

$$\begin{aligned} \xi_{,\tau\tau} + [\omega_0^2(s) - k^2(s)s_{,\tau}^2]\xi &= (4\mu/\pi)n_1(s) + k(s)s_{,\tau}^2, \\ [1 + k(s)\xi]s_{,\tau\tau} + 2k(s)s_{,\tau}\xi_{,\tau} &= (4\mu/\pi)W_{1,s}(s) - k_{,s}(s)s_{,\tau}^2\xi, \end{aligned} \quad (21)$$

where

$$\omega_0^2(s) = (\pi^4/2)[W_1^{02}(s) + 16W_2^{02}(s)]$$

defines the square of the stretching natural frequency when s is frozen, and

$$k(s) = n_{1,s}(s)W_{1,s}(s) + n_{2,s}W_{2,s}(s)$$

is the ellipse curvature at the point defined by s .

Note that the system is linearized about $\xi = 0$ as stated earlier. Furthermore, a relationship between the normal and the tangent vectors $n_{i,s}(s) = kW_{i,s}(s)$, $i = 1, 2$, has been used. Equations (21) are more suitable for analysis because they have a linear form with respect to ξ . It is obvious that the effective stretching natural frequency depends implicitly on the time parameter τ and explicitly on the co-ordinate s and its time derivative.

We will not seek a solution for equations (21). We will obtain useful information concerning the cable mechanical properties. For example, Figure 8 presents the dependence of stretching frequency $\omega_0(s)$ on the cable position. The position is defined by the angle ψ (see Figure 6(b)):

$$W_1^0 = \cos \psi, \quad W_2^0 = \frac{1}{2} \sin \psi, \quad ds^2 = \frac{1}{4}(1 + 3 \sin^2 \psi) d\psi^2, \quad 0 \leq \psi \leq 2\pi.$$

Figure 8 reveals that the cable possesses the maximum stretching frequency at the extreme transverse position.

The geometrical treatment of a two-mode in-plane motion on the configuration plane W_1, W_2 enables one to introduce the co-ordinate transformation (19), and to obtain a simplified form of the equations of motion as given by equation (21). It is

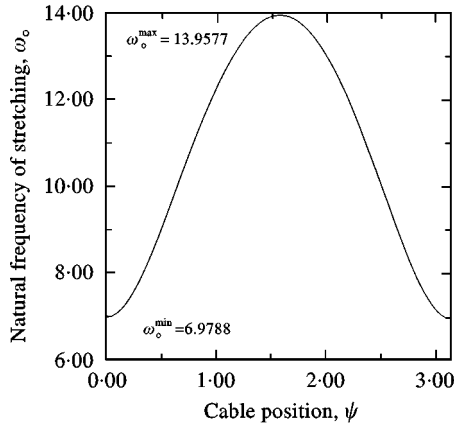


Figure 8. Natural frequency of stretching at various transverse positions of the cable ψ .

also possible to give the corresponding geometrical treatment in a three-dimensional Euclidean space for a three-mode approximation of the cable dynamics. For example, the first three in-plane modes W_1, W_2, W_3 will form an ellipsoid, $W_1^2 + 4W_2^2 + 9W_3^2 = 1$, in the three-dimensional space of configurations R^3 . The normal vector position n will be defined in terms of s_1 and s_2 co-ordinates on the ellipsoid. For more than three modes, it is not easy to visualize the motion by a similar approach. However, a geometrical treatment is possible, in terms of functions' space (see Appendix A). A more formal way of generalization will be presented in this section.

First, one has to obtain a transformation of the type (19) in terms of $W(\eta, \tau)$ for the general case of in-plane motion. This will be done by multiplying the first and second equations of equation (19) by $\sin \pi\eta, \sin 2\pi\eta$, respectively, and adding the two results. Taking into account the expressions for the normal vector components (20), one may write

$$W = W^0(\eta, s) + n(\eta, s)\zeta, \quad (22)$$

where $W = W_1 \sin \pi\eta + W_2 \sin 2\pi\eta$, $W^0 = W_1^0 \sin \pi\eta + W_2^0 \sin 2\pi\eta$, and the "normal vector" is

$$n = n_1 \sin \pi\eta + n_2 \sin 2\pi\eta = \frac{W_1^0 \sin \pi\eta + 4W_2^0 \sin 2\pi\eta}{(W_1^{02} + 16W_2^{02})^{1/2}}$$

or

$$n = \frac{-W_{,\eta\eta}^0}{\sqrt{2\langle W_{,\eta\eta}^{02} \rangle_\eta}},$$

where the definition $\langle \dots \rangle_\eta \equiv \int_0^1 (\dots) d\eta$ has been used. For future transformation, it is convenient to eliminate the numerical factor 2 in the denominator, and to use the representation

$$n = -W_{,\eta\eta}^0/\omega_0, \quad \omega_0 = \sqrt{\langle W_{,\eta\eta}^{02} \rangle_\eta}, \quad (23)$$

which is normalized by the relationship $\langle n^2 \rangle_\eta = 1$.

The function $W^0(\eta, s)$ describes an arbitrary unstretched configuration of the cable, i.e.,

$$\varepsilon[W^0, 0] = \frac{1}{2} \int_0^1 (W_{,\eta}^{02} - W_0^2) d\eta \equiv \frac{1}{2} (\langle W_{,\eta}^{02} \rangle_\eta - \langle W_0^2 \rangle_\eta) \equiv 0. \quad (24)$$

The form of relationships (22)–(24) is suitable for the general case. Indeed one can suppose that the expansion for W and W^0 contain an arbitrary number of terms, N , and the arc length s is replaced by a set of curvilinear co-ordinates, $\{s_1, \dots, s_M\}$, on the M -dimensional ellipsoid, where $M = N - 1$.

The next step is to generalize relationships (22) and (23) for the case when both the in-plane and out-of-plane components of the motion take place. This can be done by means of introducing the two-component vector function

$$U(\eta, \tau) = \begin{pmatrix} W(\eta, \tau) \\ V(\eta, \tau) \end{pmatrix},$$

where the original cable position is

$$U_0(\eta) = \begin{pmatrix} W_0(\eta) \\ 0 \end{pmatrix}.$$

Relations (22)–(24) can be formally expanded in terms of the vector functions (see the proof in Appendix A)

$$U = U^0(\eta; s_1, \dots, s_M) + n(\eta; s_1, \dots, s_M)\xi, \quad (25)$$

$$n = \frac{-U_{,\eta\eta}^0}{\omega_0} \equiv -\frac{1}{\omega_0} \begin{pmatrix} W_{,\eta\eta}^0 \\ V_{,\eta\eta}^0 \end{pmatrix}, \quad \omega_0 = \sqrt{\langle U_{,\eta\eta}^{0T} U_{,\eta\eta}^0 \rangle_\eta} \equiv |U_{,\eta\eta}^0|, \quad (26)$$

where the vector function U^0 defines the arbitrary position of the cable with undeformed length, that is the following relationship holds:

$$\varepsilon[W^0, V^0] = \frac{1}{2} \int_0^1 (W_{,\eta}^{02} + V_{,\eta}^{02} - W_{0,\eta}^2) d\eta \equiv \frac{1}{2} (\langle W_{,\eta}^{02} + V_{,\eta}^{02} \rangle_\eta - \langle W_{0,\eta}^2 \rangle_\eta) \equiv 0$$

or

$$\varepsilon[U^0] = \frac{1}{2} (|U_{,\eta}^0|^2 - |U_{0,\eta}|^2) \equiv 0. \quad (27)$$

The symbol $|X| \equiv \sqrt{\langle X^T X \rangle_\eta}$ denotes the norm of the vector function, $X(\eta)$, in the space considered (see Appendix A). Accordingly, $\langle X^T X \rangle_\eta$ should be understood as a scalar product. For example, the norm of the normal vector is

$$|n| = \sqrt{\langle n^T n \rangle_\eta} = \frac{1}{\omega_0} \sqrt{\langle U_{,\eta\eta}^{0T} U_{,\eta\eta}^0 \rangle_\eta} = \frac{\omega_0}{\omega_0} = 1.$$

The set $\{s_1, \dots, s_M\}$ represents curvilinear orthogonal co-ordinates on the M -dimensional ellipsoid. This ellipsoid is described in terms of vector functions U by the equation

$$\varepsilon[U] = 0. \quad (28)$$

The dimension M is given by the expression $M = N + K - 1$, where N and K are the number of in-plane and out-of-plane modes, respectively. Thus $N + K$ is the complete dimension of the configuration space considered. An arbitrary point on the ellipsoid, U^0 , corresponds to the arbitrary position of the cable with undeformed length (27). An explicit dependence $U^0(\eta; s_1, \dots, s_M)$ on (s_1, \dots, s_M) can be established by using generalized spherical co-ordinates. This is not necessarily for general transformations; however, it can be easily carried out for specific cases, as will be shown later. To this end one must define those functions which will play the role of tangent vectors to the ellipsoid. These functions are

$$v_j(\eta; s_1, \dots, s_M) = \frac{\partial U^0(\eta; s_1, \dots, s_M)}{\partial s_j}, \quad j = 1, \dots, M. \quad (29)$$

All co-ordinates s_j are assumed to be normalized so that the conditions

$$\langle v_i^T v_j \rangle_\eta = \delta_{ij}, \quad i, j = 1, \dots, M \quad (30)$$

hold, where δ_{ij} is the Kronecker symbol.

It can be shown (see Appendix A) that

$$\langle n^T v_j \rangle_\eta = 0 \quad j = 1, \dots, M. \quad (31)$$

To this end we write the cable equations of motion (12) in terms of the co-ordinate vector $U(\eta, \tau)$ as

$$U_{,\tau\tau} - \varepsilon[U]U_{,\eta\eta} = \mu P, \quad (32)$$

where

$$\varepsilon[U] = \frac{1}{2}(|U_{,\eta}|^2 - |U_{0,\eta}|^2) \quad (33)$$

is dimensionless strain;

$$P = \begin{pmatrix} 1 + p(\eta, \tau) \\ 0 \end{pmatrix}$$

is a column-vector of the external force, including gravity.

We introduce the co-ordinate transformation

$$U(\eta, \tau) \rightarrow \{\xi(\tau); s_1(\tau), \dots, s_M(\tau)\}, \quad (34)$$

where ξ corresponds to the normal vector n and $s_j, j = 1, \dots, M$, define the position of the origin of the normal vector n on the ellipsoid $U^0 = U^0(\eta; s_1(\tau), \dots, s_M(\tau))$. Thus, the shape of the cable at any moment τ will be defined by the set of co-ordinates $\{\xi(\tau); s_1(\tau), \dots, s_M(\tau)\}$. For example, for a two-mode approximation, there are two new co-ordinates $\{\xi(\tau); s(\tau)\}$ as shown in Figure 7. The transformation (34) is given by equation (25),

$$U = U^0 + n\xi. \quad (35)$$

When the $\xi = 0$ expression (35) gives an arbitrary configuration of zero strain, $\varepsilon[U^0] = 0$. For $\xi \neq 0$ we substitute equation (35) into equation (33), taking into account equation (26), and integrating by parts gives

$$\varepsilon[U^0 + n\xi] = \omega_0 \xi + \frac{1}{2} |n_{,\eta}|^2 \xi^2. \quad (36)$$

The last expression clearly shows that the normal co-ordinate, ξ , is associated with the centerline strain.

Substituting equation (35) into the equation of motion (32) and taking into account equation (36) one obtains

$$n\xi_{,\tau\tau} + (n\omega_0^2 + n_{,\tau\tau})\xi + 2n_{,\tau}\xi_{,\tau} + \omega_0(\frac{1}{2}|n_{,\eta}|^2 n - n_{,\eta\eta})\xi^2 - \frac{1}{2}|n_{,\eta}|^2 n_{,\eta\eta}\xi^3 = \mu P - U_{,\tau\tau}^0. \quad (37)$$

Taking projections of this equation on the normal vector n and on the j th tangent vector $v_j = \partial U^0 / \partial s_j$ in the sense of equations (30) and (31), and using the relationship $\langle n^T n_{,\eta\eta} \rangle_\eta = -|n_{,\eta}|^2$ (this is a result of integration by parts), one finally obtains the set of ordinary differential equations,

$$\xi_{,\tau\tau} + (\omega_0^2 + \langle n^T n_{,\tau\tau} \rangle_\eta)\xi + \frac{3}{2}\omega_0 |n_{,\eta}|^2 \xi^2 + \frac{1}{2}|n_{,\eta}|^4 \xi^3 = \mu \langle n^T P \rangle_\eta - \langle n^T U_{,\tau\tau}^0 \rangle_\eta, \quad (38)$$

$$\langle v_j^T U_{,\tau\tau}^0 \rangle_\eta + \langle v_j^T n_{,\tau\tau} \rangle_\eta \xi + 2 \langle v_j^T n_{,\tau} \rangle_\eta \xi_{,\tau} - \langle v_j^T n_{,\eta\eta} \rangle (\omega_0 \xi^2 + \frac{1}{2}|n_{,\eta}|^2 \xi^3) = \mu \langle v_j^T P \rangle_\eta, \quad (39)$$

where

$$j = 1, \dots, M.$$

These are $1 + M$ non-linear ordinary differential equations. If the value M equals infinity, the equations are equivalent to the original set of partial differential equations (32). Furthermore, these equations are written explicitly in terms of ζ and implicitly in terms of the curvilinear co-ordinates s_j on the ellipsoid. The explicit form in terms of s_j will be presented for specific cases. However, this set of equations has the advantage that one can linearize the system with respect to the co-ordinate ζ . Alternatively, one can consider quasi-linear system of equations with respect to ζ .

Now the dimension of the original non-linear problem $N + K$ has been reduced to $M = N + K - 1$. Generally, the linear and non-linear parts of the system are coupled. One possibility of decoupling these components is obtained under the assumption that the cable weight is small compared to its elastic strength, i.e.,

$$\mu = \frac{\rho g H^4}{ED_0^3} \ll 1, \quad (40)$$

Under this assumption, one can asymptotically split the motion into two components, *slow* and *fast*. The slow component belongs mainly to the swinging-transverse motion and lies along the tangent with respect to the manifold direction. The fast component, on the other hand, is due to cable extension and coincides with the normal to the manifold.

5. ANALYSIS OF THE TRANSFORMED SYSTEM

5.1. EXAMPLE OF SWINGING-STRETCHING COUPLING

Consider the case of non-linear coupling between the first in-plane and first out-of-plane modes,

$$U = \begin{pmatrix} W \\ V \end{pmatrix} = \begin{pmatrix} W_1 \\ V_1 \end{pmatrix} \sin \pi \eta, \quad W_1 = W_1(\tau), \quad V_1 = V_1(\tau). \quad (41)$$

Substituting this vector into the manifold equation (28) gives

$$\varepsilon[U] = \varepsilon(W_1, V_1) \equiv (\pi^2/4)(W_1^2 + V_1^2 - 1) = 0. \quad (42)$$

This manifold has the form of a circle in the configuration plane $W_1 V_1$. In terms of the polar angle $\varphi(\tau)$, one obtains the expression for an arbitrary point on the circle $U^0 = U^0(\eta; \varphi)$,

$$U^0 = \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} \sin \pi \eta, \quad (43)$$

with stretching frequency and the normal given by the expressions

$$\omega_0 = \sqrt{2}\pi^2/2, \quad n = \sqrt{2}U_{,\eta}^0. \quad (44)$$

For this case there is one co-ordinate on the manifold, $s_1 \equiv s$, which can be expressed in terms of the angle φ by means of the relationship

$$ds^2 = |\partial U^0 / \partial \varphi|^2 d\varphi^2 = (1/2)d\varphi^2.$$

The factor $1/2$ appears because the length is taken according to the definition $|X| \equiv \sqrt{\langle X^T X \rangle_\eta}$. For the present case, the angle φ defines an angle between the plane of the cable and the original vertical plane during swinging oscillations.

The equations for $\xi(\tau)$ and $\varphi(\tau)$ follow from equations (38) and (39). Keeping only linear terms in ξ , one obtains the following coupled set of two ordinary differential equations:

$$\xi_{,\tau\tau} + \left(\frac{\pi^4}{2} - \varphi_{,\tau}^2 \right) \xi = \frac{2\sqrt{2}\mu}{\pi} (1+p) \cos \varphi + \frac{\sqrt{2}}{2} \varphi_{,\tau}^2, \quad (45)$$

$$\frac{1}{2} (1 + \sqrt{2}\xi) \varphi_{,\tau\tau} + \sqrt{2} \varphi_{,\tau} \xi_{,\tau} = -\frac{2\mu}{\pi} (1+p) \sin \varphi. \quad (46)$$

The original variables are expressed through stretching and swinging co-ordinates $\xi(\tau)$ and $\varphi(\tau)$, respectively, by means of relationship (35), and can be written as

$$w = D_0 (1 + \sqrt{2}\xi) \cos \varphi \sin \pi\eta, \quad v = D_0 (1 + \sqrt{2}\xi) \sin \varphi \sin \pi\eta. \quad (47)$$

Note that equations (45) and (46) are analogous to the equations of motion of an elastic pendulum (see for example reference [18]). In order to apply the corresponding results to the cable system, one must satisfy the condition of internal resonance between the linear frequency of stretching, $\omega_{st} = \sqrt{2}\pi^2/2$, and the frequency of swinging, $\omega_{sw} = 2\sqrt{\mu/\pi}$. For example, when $\omega_{st} = 2\omega_{sw}$ the physical parameter has the special value, $\mu = \pi^5/32$, and the unperturbed system experiences energy transfer from the stretching mode to the swinging mode. For large amplitudes the situation becomes much more complicated, because of the essential influence of the unstable inverted equilibrium $\varphi = \pi$.

To this end two types of asymptotic approximation will be introduced. These approximation will decouple the system such that the stretching co-ordinate can be independently expressed as a function of swinging and transverse co-ordinates.

5.2. ASYMPTOTIC DECOUPLING OF THE SYSTEM

Under the condition of small gravity (40), one can introduce slow time, $t^0 = \sqrt{\mu}\tau$, and start from transformation (25), with

$$\xi = \mu \bar{\xi}(\tau) + o(\mu), \quad U^0 = \bar{U}^0(\eta; s_1(t^0), \dots, s_1(t^0)) + o(1), \quad p = \bar{p}(\eta, t^0) + o(1), \quad (48)$$

where the symbol $o(\mu)$ denotes higher order terms compared with μ .

Substituting equation (48) into equations (38) and (39), and keeping the leading order terms, one obtains the set of equations

$$\bar{\xi}''_{,\tau\tau} + \omega_0^2 \bar{\xi} = \langle n^T \bar{P} \rangle_\eta - \left\langle n^T \frac{\partial^2 \bar{U}^0}{\partial t^{02}} \right\rangle_\eta, \quad (49)$$

$$\left\langle \frac{\partial \bar{U}^{0T}}{\partial s_j} \frac{\partial^2 \bar{U}^0}{\partial t^{02}} \right\rangle_\eta = \left\langle \frac{\partial \bar{U}^{0T}}{\partial s_j} \bar{P} \right\rangle_\eta, \quad j = 1, \dots, M, \quad (50)$$

where the function \bar{U}^0 satisfies the equation for the special manifold,

$$\left| \frac{\partial \bar{U}^0}{\partial \eta} \right|^2 = \left| \frac{\partial U_0}{\partial \eta} \right|^2, \quad (51)$$

and

$$\bar{P} = \begin{pmatrix} 1 + \bar{p}(\eta, t^0) \\ 0 \end{pmatrix}$$

is the vector-column of the external excitation.

The Lagrange function corresponding to equation (50) can be written in the form

$$L = \frac{1}{2} \left| \frac{\partial \bar{U}^0}{\partial t^0} \right|^2 + \langle \bar{U}^{0T} \bar{P} \rangle_\eta. \quad (52)$$

Now the system is asymptotically decoupled. The slow component of the motion is described by the Lagrangian (52) on the manifold (51) independently of the fast component. Within the framework of a given formulation, the fast component has no influence on the slow one, but the parameters of the fast motion will slowly change during the process of the slow motion. In order to promote our understanding of this method, we will study the problem of modal interaction of the first two in-plane modes of the cable.

For example, let us consider the case of the first two in-plane modes. The transformation (25) takes the form

$$U = \bar{U}^0 + \mu n \bar{\xi},$$

where

$$U = \begin{pmatrix} W_1 \sin \pi \eta + W_2 \sin 2\pi \eta \\ 0 \end{pmatrix}, \quad \bar{U}^0 = \begin{pmatrix} \bar{W}_1(t^0) \sin \pi \eta + \bar{W}_2(t^0) \sin 2\pi \eta \\ 0 \end{pmatrix}, \quad (53)$$

$$n = \begin{pmatrix} n_w \\ 0 \end{pmatrix}, \quad n_w = -\sqrt{2} \frac{\bar{W}_1(t^0) \sin \pi \eta + 4\bar{W}_2(t^0) \sin 2\pi \eta}{\sqrt{\bar{W}_1^2(t^0) + 16\bar{W}_2^2(t^0)}}.$$

The original undeformed position of the cable and the external force are

$$U_0 = \begin{pmatrix} \sin \pi \eta \\ 0 \end{pmatrix}, \quad \bar{P} = \begin{pmatrix} 1 + p^0(t^0) \\ 0 \end{pmatrix}.$$

Here the first mode represents the stretching effect while the second in-plane mode is due to transverse motion. The expression for the special manifold (51) takes the form

$$\bar{W}_1^2 + 4\bar{W}_2^2 = 1. \quad (54)$$

Introducing the angular co-ordinate $\psi(t^0)$ on the ellipse (54) as $\bar{W}_1 = \cos \psi(t^0)$, $\bar{W}_2 = (1/2) \sin \psi(t^0)$ one writes

$$\bar{U}^0 = \begin{pmatrix} \cos \psi \sin \pi \eta + \frac{1}{2} \sin \psi \sin 2\pi \eta \\ 0 \end{pmatrix}. \quad (55)$$

Substituting this expression into equation (49) and using equation (23) one obtains the equation for the stretching variable $\bar{\xi}$:

$$\begin{aligned} \bar{\xi}''_{,\tau\tau} + \omega_0^2 \bar{\xi} &= \frac{2\pi}{\omega_0} [1 + p^o(t^0)] \cos \psi + \frac{\pi^2}{2\omega_0} \left(\frac{d\psi}{dt^0} \right)^2, \\ \omega_0^2 &= \frac{\pi^4}{2} (1 + 3 \sin^2 \psi). \end{aligned} \quad (56)$$

In terms of the angle $\psi(t^0)$ the Lagrange function (52) takes the form

$$L = \frac{\omega_0^2(\psi)}{8\pi^4} \left(\frac{d\psi}{dt^0} \right)^2 + \frac{2}{\pi} [1 + p^o(t^0)] \cos \psi \quad (57)$$

and the corresponding differential equation of motion is

$$\frac{\omega_0}{4\pi^4} \frac{d}{dt^0} \left(\omega_0 \frac{d\psi}{dt^0} \right) + \frac{2}{\pi} [1 + p^o(t^0)] \sin \psi = 0. \quad (58)$$

This equation does not depend on the stretching amplitude governed by equation (56). It is obvious that the original coupled co-ordinates W_1 and W_2 are transformed into the new co-ordinates ξ and ψ with one way coupling as indicated by equations (56) and (58). Equation (58) describes a parametrically excited pendulum with a variable ‘‘mass parameter’’, $\omega_0^2/(4\pi^4)$, which depends on the angle ψ . In the case of free vibration, $p^o(t^0) \equiv 0$, this equation possesses the first integral and can be analytically solved in terms of quadratures as

$$\pm \frac{1}{4\pi^{3/2}} \int_{\psi(0)}^{\psi(t^0)} \frac{\omega_0(\psi) d\psi}{(C + \cos \psi)^{1/2}} = t^0, \quad C, \psi(0) = \text{const}. \quad (59)$$

Thus in the absence of external excitation the strongly non-linear coupled two inplane modes (18) are described in an explicit analytical form. Indeed, solving for the function $\psi(t^0)$ we can solve the linear equation for stretching (56). The solution of equation (56) is easily obtained because the right-hand side of this equation is slowly dependent on time t^0 . In order to describe the cable behavior in terms of the original modal co-ordinates, transformation (35) should be used. It is hardly possible to obtain any analytical solution directly without assuming weak non-linearity if one starts from a set of ordinary differential equations for the modal co-ordinates W_1 and W_2 as described by equations (18). In fact, the co-ordinates

W_1 and W_2 are coupled, and any direct decoupling needs elimination of non-linear terms which couple the two modes. But this is not correct because both co-ordinates possess the same order of magnitude, $O(1)$ for the case under consideration.

Under external excitation equation (58) should be solved numerically. For equation (56), the stretching mode is excited directly, and as ψ increases the contribution of the first expression on the right-hand side of equation (56) vanishes when $\psi = \pi/2$. In this case the stretching amplitude reaches its minimum value.

Remark 1. The case of non-linear interaction between the first in-plane mode and second out-of-plane mode,

$$\bar{U}^0 = \begin{pmatrix} \bar{W}_1(t^0) \sin \pi\eta \\ \bar{V}_2(t^0) \sin 2\pi\eta \end{pmatrix} \quad (60)$$

will lead to a similar set of equations, when the corresponding manifold is $\bar{W}_1^2 + 4\bar{V}_2^2 = 1$.

6. VALIDITY OF THE PROPOSED TREATMENT

Some qualitative remarks will be made here based on N in-plane and N out-of-plane modes for the cable's centerline. The purpose of this analysis is to clarify how the higher modes of the transverse motion may influence the centerlines length oscillation. Specifically, the reduction of number of degrees of freedom (modes) becomes quite questionable if the higher modes are unstable and their amplitudes are increasing in time. The equations of motion will be written in terms of the modal co-ordinates; however, the group of terms related to the centerlines length oscillation will be expressed in terms of new co-ordinates on the special manifold in order to simplify the problem. This "mixed" description is required by formulation of the problem because one is going to investigate an influence of the stretching oscillation on the modal co-ordinates.

Suppose that

$$U = \begin{pmatrix} W \\ V \end{pmatrix} = \sum_{k=1}^N \begin{pmatrix} W_k(\tau) \\ V_k(\tau) \end{pmatrix} \sin k\pi\eta, \quad U_0 = \begin{pmatrix} \sin \pi\eta \\ 0 \end{pmatrix} \quad (61)$$

and

$$P = \sum_{k=1}^N \begin{pmatrix} P_k(\tau) \\ 0 \end{pmatrix} \sin k\pi\eta. \quad (62)$$

Substituting these expansions into equation (32) and taking into account that the cable's strain $\varepsilon[U]$ does not depend on η , one obtains

$$W_{k,\tau\tau} + (k\pi)^2 \varepsilon[U] W_k = \mu P_k, \quad V_{k,\tau\tau} + (k\pi)^2 \varepsilon[U] V_k = 0, \quad k = 1, \dots, N, \quad (63)$$

where the strain is given by substituting equation (61) into equation (33) as

$$\varepsilon[U] = \frac{\pi^2}{4} \left[\sum_{k=1}^N k^2 (W_k^2 + V_k^2) - 1 \right]. \quad (64)$$

Equations (63) show that the modal interaction owes its origin to the parametric influence of the strain on the modes. This means that the time-history records of various modes are affected by the time dependence of the strain. If one had an explicit periodic dependence on time for the strain $\varepsilon[U]$, the problem of higher modes' dynamical stability would be treated using the Floquet theory. Unfortunately, the strain depends on time implicitly through all modal coefficients and one has a strongly coupled non-linear problem. However, the proposed co-ordinates transformation simplifies the problem significantly. Namely, one can express the strain in terms of the new co-ordinates by means of relation (36). In line with the idea of transformation, the quadratic term of the stretching co-ordinate, $\xi(\tau)$, can be dropped and relation (36) takes the form

$$\varepsilon[U] \approx \omega_0 \xi(\tau).$$

The linearized equation (38) for $\xi(\tau)$ is

$$\xi_{,\tau\tau} + (\omega_0^2 + \langle n^T n_{,\tau\tau} \rangle_\eta) \xi = \mu \langle n^T P \rangle_\eta - \langle n^T U_{,\tau\tau}^0 \rangle_\eta, \quad (65)$$

where the frequency ω_0 and the normal vector n depend on the vector function

$$U^0 = \begin{pmatrix} W^0 \\ V^0 \end{pmatrix} = \sum_{k=1}^N \begin{pmatrix} W_k^0(\tau) \\ V_k^0(\tau) \end{pmatrix} \sin k\pi\eta$$

as

$$\omega_0^2 = \langle U_{,\eta\eta}^{0T} U_{,\eta\eta}^0 \rangle_\eta = \frac{\pi^4}{2} \sum_{k=1}^N k^4 (W_k^{02} + V_k^{02}), \quad n = \frac{\pi^2}{\omega_0} \sum_{k=1}^N k^2 \begin{pmatrix} W_k^0 \\ V_k^0 \end{pmatrix} \sin k\pi\eta.$$

Recall that U^0 indicates any unstretched configuration of the cable, i.e., at any time τ the following condition holds (see equation (64)):

$$\varepsilon[U^0] = 0 \Rightarrow \sum_{k=1}^N k^2 (W_k^{02} + V_k^{02}) = 1. \quad (66)$$

As indicated earlier, the point U^0 is slowly moving on the manifold (66). Hence, in order to estimate ξ in a leading-order approach, one can suppose that the cable configuration is frozen at an arbitrary point U^0 of the manifold. Under this assumption, one has the equation

$$\xi_{,\tau\tau} + \omega_0^2 \xi = \frac{\mu\pi^2}{2\omega_0} \sum_{k=1}^N k^2 W_k^0 P_k. \quad (67)$$

Under static external force, a single-parameter family of solutions can be written as

$$\xi = \frac{\mu\pi^2}{2\omega_0^3} \sum_{k=1}^N k^2 W_k^0 P_k + A \cos \omega_0 \tau, \quad A = \text{const}. \quad (68)$$

Substituting this expression into equations (64) and (63), and introducing a new time parameter $z = \omega_0 \tau / 2$, one obtains for the k th mode the Mathieu equation with

constant right-hand side

$$\frac{d^2 W_k}{dz^2} + (a + 2b \cos 2z) W_k = c P_k, \quad \frac{d^2 V_k}{dz^2} + (a + 2b \cos 2z) V_k = 0, \quad (69)$$

where

$$a = \frac{2\mu\pi^4 k^2}{\omega_0^4} \sum_{k=1}^N k^2 W_k^0 P_k, \quad b = \frac{2A\pi^2 k^2}{\omega_0}, \quad c = \frac{4\mu}{\omega_0^2}. \quad (70)$$

In case of gravitational force, one has $P_k = (2/k\pi)[1 - (-1)^k]$.

Equation (69) enables one to estimate the future modes behavior around the arbitrary configuration of the cable close to the unstretched manifold. Note that the constant right-hand side of the equation does not have any influence on the problem of parametric instability.

We assume that $a > 0$ during the motion. This condition physically means that the cable's centerline is pre-stretched by the gravitational force, otherwise one should expect the higher modes ($k \gg 1$) to grow very fast after any small initial perturbation. In terms of partial differential equations it results in an ill-posed problem [16].

It is known that the Mathieu equation has a series of instability regions represented on the plane of parameters a and b (the Ince-Strutt diagram), of which the most dangerous goes throughout point $(a, b) = (1, 0)$ and is bounded by the couple of curves [19]:

$$a = 1 \pm b + \text{terms of order } b^2. \quad (71)$$

For example, let us consider the configuration given by

$$W_1^0 = 1, W_2^0 = 0, \dots, W_N^0 = 0; \quad V_1^0 = 0, V_2^0 = 0, \dots, V_N^0 = 0. \quad (72)$$

In this case, $\omega_0^2 = \pi^4/2$ and

$$a = \frac{32\mu k^2}{\pi^5}, \quad b = 2\sqrt{2}Ak^2, \quad c = \frac{8\mu}{\pi^4}. \quad (73)$$

Substituting a and b from these expressions into equation (71), one can obtain borders of the instability region and hence the region itself in terms, for instance, A and k for fixed μ . The instability region can be expressed as

$$|A| > \frac{\sqrt{2}}{4} \left| \frac{1}{k^2} - \frac{32\mu}{\pi^5} \right|, \quad (74)$$

provided that the condition $|b| = 2\sqrt{2}|A|k^2 \ll 1$ holds.

For a certain value of the gravity parameter μ , one can estimate the stability boundaries of higher modes in terms of the stretching amplitude, A . Or *vice versa*, one can estimate which of the modes are unstable for a given amplitude. Around the considered equilibrium position, the amplitude and initial value of the first

mode are coupled through the relationship

$$W_1(0) = 1 + \frac{8\mu}{\pi^5} + \sqrt{2}A$$

when the initial velocity is zero.

From expression (74), it is seen that higher modes ($k = O(\sqrt{\pi^5/(32\mu)})$) may become unstable even for very small values of the stretching amplitude, A . After one or several of the modes became unstable, the system may leave the neighborhood of point (72) and will change parameters (73) accordingly to equation (70). On the other hand, one should bear in mind that the cable's model itself becomes unreliable when the modes number is large, because the model requires the cable centerline be shallow and absolutely flexible (section 2). Practically, the cable always possesses a relatively small but non-zero bending rigidity. This rigidity is negligible for low modes of a small curvature, but it becomes observable when the centerline is highly bent as $\sin k\pi\eta$, ($k \gg 1$).

A possible generalization may take into account different factors, such as bending and torsion deformations. However, in order to remain close to the classic cable theory and at the same time to provide the model with desirable properties one can include bending stiffness only, using simple beam theory. Taking into account the shallowness of the cable and the related expression for curvatures, the equations of motion (8) can be modified as

$$\begin{aligned} v_{,tt} - \frac{E}{\rho} ev_{,xx} + \frac{EI}{\rho A} v_{,xxxx} &= 0, \\ w_{,tt} - \frac{E}{\rho} ew_{,xx} + \frac{EI}{\rho A} w_{,xxxx} &= g(1 + p(x, t)), \end{aligned} \quad (75)$$

where a cross-section of the cable is supposed to be a circle and hence EI is the bending rigidity with respect to any diameter of the section, and the boundary conditions must be reformulated according to the beam theory.

In this case the vector equation of motion (32) will include an additional (bending) term and it takes the form

$$U_{,\tau\tau} - \varepsilon[U]U_{,\eta\eta} + \lambda U_{,\eta\eta\eta\eta} = \mu P, \quad (76)$$

where the non-dimensional bending rigidity parameter $\lambda = EI/(EAD_0^2)$ is assumed to be very small.

Accordingly, one can modify the transformed equations (38) and (39) by taking projections of the new expression $\lambda(U^0 + n\xi)_{,\eta\eta\eta\eta}$ on the normal vector n and on the j th tangent vector $v_j = \partial U^0/\partial s_j$ to the special manifold. From the viewpoint of transformed system, a role of the bending energy was investigated in reference [15] based on a model of shallow elastic systems, including a non-linear beam with initial imperfection (a shallow arch model) that may exhibit a snap-through phenomenon.

To this end, let us show that the bending rigidity improves the stability properties and can make the problem well-posed even under the negative tension of the cable. According to the modified equation (76), equations (63) for the modal co-ordinates

have to be replaced by equations including the related bending terms, $\lambda (k\pi)^4$,

$$\begin{aligned} W_{k,\tau\tau} + [\lambda(k\pi)^4 + (k\pi)^2 \varepsilon[U]] W_k &= \mu P_k, \\ V_{k,\tau\tau} + [\lambda(k\pi)^4 + (k\pi)^2 \varepsilon[U]] V_k &= 0, \quad k = 1, \dots, N. \end{aligned} \quad (77)$$

In case of negative strain $\varepsilon[U] < 0$, the bending rigidity for a certain value of k will result in a positive stiffness coefficient and above that value of k the coefficient is always positive. This means that any small bending stiffness λ makes the problem well-posed. The system becomes more stable dynamically as well, because the bending term $\lambda (k\pi)^4$ shifts the system into a more stable region of the Ince–Strutt diagram of the related Mathieu equation (69).

7. CONCLUSIONS

The non-linear modal interaction of a shallow cable describing large in-plane and out-of-plane motions is treated using a special co-ordinate transformation. Under certain conditions, the technique transforms n strongly non-linear coupled differential equations into $(n - 1)$ coupled equations plus one equation (describing cable stretching) whose output acts as an excitation to the other $(n - 1)$ equations. The method has been demonstrated for different cases of cable dynamics, including in-plane non-linear motion plane and out-of-plane interaction in the presence and in the absence of external excitation. In particular, an explicit analytical solution for the free vibration of the first two in-plane modes has been obtained.

For a number of dynamical regimes of cable motion, the original set of partial differential equations has been reduced to a set of ordinary differential equations similar to those describing an elastic pendulum with internal resonance. The corresponding mass parameter can be a constant or a variable depending on the regime considered. For example, the stretching–swinging motion described by equations (45) and (46) correspond to an elastic pendulum with a constant mass. For the case of in-plane transverse motion the cable behavior resembles the dynamics of a simple pendulum with a variable mass which depends on the transverse motion of the cable (see equation (58)).

ACKNOWLEDGMENT

This research was supported by Office of Naval Research grant number N000149310936, and by NSF grant number CMS-9634223. Additional support was also provided from the Institute for Manufacturing Research at Wayne State University. The authors would like to thank one of the reviewers who drew their attention to the ill-posed problem reported originally in reference [16].

REFERENCES

1. A. SIMPSON 1966 *Proceedings of the Institute of Electrical Engineering* **114**, 870–878. Determination of the in-plane natural frequencies of multi-span transmission lines by a transfer matrix method.

2. M. IRVINE and T. K. CAUGHEY 1974 *Royal Soc. London, Series A* **341**, 299–315. The linear theory of free vibrations of a suspended cable.
3. H. M. IRVINE and J. H. GRIFFIN 1976 *Earthquake Engineering and Structural Dynamics* **4**, 389–402. On the dynamic response of a suspended cable.
4. P. HAGEDORN and B. SCHAFER 1980 *International Journal of Nonlinear Mechanics* **15**, 333–340. On nonlinear free vibrations of an elastic cable.
5. A. LUONGO, G. REGA and F. VESTRONI 1982 *Journal of Sound and Vibration* **82**, 247–259. Monofrequent oscillations of a nonlinear model of a suspended cable.
6. A. LUONGO, G. REGA and F. VESTRONI 1984 *International Journal of Nonlinear Mechanics* **19**, 39–52. Planar nonlinear free vibrations of an elastic cable.
7. G. REGA, F. VESTRONI and F. BENEDETTINI 1983 *International Journal of Solids and Structures* **20**, 95–105. Parametric analysis of large amplitude free vibrations of a suspended cable.
8. G. REGA, F. BENEDETTINI and A. SALVATORI 1991 *Chaos, Solitons and Fractals* **1**, 39–54. Periodic and chaotic motions of an unsymmetrical oscillator in nonlinear structural dynamics.
9. F. BENEDETTINI, A. REGA and F. SALVATORI 1992 *Chaos, Solitons and Fractals* **2**, 303–321. Prediction of bifurcation and chaos for an asymmetric elastic oscillator.
10. F. BENEDETTINI, G. REGA and F. VESTRONI 1986 *Meccanica* **21**, 38–46. Modal coupling in the free non-planar finite motion of an elastic cable.
11. K. TAKAHASHI and Y. KONISHI 1987 *Journal of Sound and Vibration* **118**, 85–97. Nonlinear vibrations of cables in three-dimensions, Part II: out-of-plane vibration under in-plane sinusoidally time-varying load.
12. C. L. LEE and N. C. PERKINS 1992 *Nonlinear Dynamics* **3**, 465–490. Nonlinear oscillations of suspended cables containing a two-to-one internal resonance.
13. C. L. LEE and N. C. PERKINS 1995 *Nonlinear Dynamics* **8**, 45–63. Three-dimensional oscillations of suspended cables involving simultaneous internal resonances.
14. M. PAKDEMIRLI, S. A. NAYFEH and A. H. NAYFEH 1995 *Nonlinear Dynamics* **8**, 65–63. Analysis of one-to-one autoparametric resonance in cables—discretization vs. direct treatment.
15. V. N. PILIPCHUK 1986 *Soviet Applied Mechanics* **22**, 162–168. Method of investigating nonlinear dynamics problems of rectangular plates with initial imperfections.
16. M. S. TRIANTAFYLLOU and C. T. HOWEL 1994 *Journal of Sound and Vibration* **173**(4). Dynamic response of cables under negative tension: an ill-posed problem.
17. H. KAUDERER 1958 *Nichtlinear Mechanik*. Berlin: Springer-Verlag.
18. T. R. KANE and M. E. KAHN 1968 *Journal of Applied Mechanics* **35**, 547–552. On a class of two-degree-of-freedom oscillations.
19. M. ABRAMOWITZ and I. A. STEGEN 1972 *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. New York: Dover Publications.

APPENDIX A: ORTHOGONALITY CONDITION (31)

The purpose of this appendix is to establish the orthogonality condition given by relation (31). The configuration space involves functions consisting of sufficiently smooth vector functions $\{U(\eta): U(0) = U(1) = 0\}$, and a norm $|U|$, which are defined as, respectively,

$$U = \begin{pmatrix} W(\eta) \\ V(\eta) \end{pmatrix}, \quad |U| = \sqrt{\langle U^T U \rangle_\eta} \langle \dots \rangle_\eta = \int_0^1 (\dots) d\eta, \quad (\text{A.1})$$

where T denotes transpose.

Remark 2. The length of a vector in the sense of an Euclidean norm, for example, $|\cdots|_{R^2}$ in the plane W_1, W_2 , and the length of the vector in the sense of the functional configuration subspace of the first two in-plane modes according to definition (78), will differ by a numerical factor

$$\begin{aligned} |(W_1, W_2)|_{R^2} &= (W_1^2 + W_2^2)^{1/2}, \\ |W_1 \sin \pi\eta + W_2 \sin 2\pi\eta| &= \frac{\sqrt{2}}{2} (W_1^2 + W_2^2)^{1/2}. \end{aligned}$$

For the general case, an arbitrary point of the configuration space is expressed by the expansion

$$U = \begin{pmatrix} W(\eta) \\ V(\eta) \end{pmatrix} = \begin{pmatrix} W_1 \sin \pi\eta + W_2 \sin 2\pi\eta + \cdots \\ V_1 \sin \pi\eta + V_2 \sin 2\pi\eta + \cdots \end{pmatrix}$$

and its norm is given by

$$|U| = \frac{\sqrt{2}}{2} (W_1^2 + W_2^2 + \cdots + V_1^2 + V_2^2 + \cdots)^{1/2}.$$

In order to establish the co-ordinate transformation, from strongly non-linear co-ordinates to mixed linear–non-linear co-ordinates, we need to develop an expression for the normal vector to the manifold (28). The problem is to determine a function which will play the role of the gradient of the functional (28), $\varepsilon[U]$, in the functional configuration space. For example in the R^2 geometry, let (W_1^0, W_2^0) be an arbitrary point on the curve $\varepsilon(W_1, W_2) = 0$, and $(W_1^0 + \delta W_1, W_2^0 + \delta W_2)$ be a close point on the curve. One has the following truncated Taylor series:

$$\varepsilon(W_1^0 + \delta W_1, W_2^0 + \delta W_2) = \varepsilon(W_1^0, W_2^0) + \varepsilon_{,W_1^0}(W_1^0, W_2^0)\delta W_1 + \varepsilon_{,W_2^0}(W_1^0, W_2^0)\delta W_2,$$

where higher order terms in $\delta W_1, \delta W_2$ have been ignored. Taking into account that both the original and new points belong to the curve, one writes

$$\varepsilon_{,W_1^0}(W_1^0, W_2^0)\delta W_1 + \varepsilon_{,W_2^0}(W_1^0, W_2^0)\delta W_2 = 0.$$

This is an orthogonality condition of the two vectors. One of them, $(\delta W_1, \delta W_2)$, is tangent to the curve $\varepsilon(W_1, W_2) = 0$, and the other $(\varepsilon_{,W_1^0}(W_1^0, W_2^0), \varepsilon_{,W_2^0}(W_1^0, W_2^0))$ is normal to the curve (gradient). Now, for the functional space, let U^0 be an arbitrary point on the manifold, $\varepsilon[U^0] \equiv 0$, and $U^0 + \delta U$ a close point. In this case, one writes

$$\varepsilon[U^0 + \delta U] = \varepsilon[U^0] + \langle U_{,\eta}^{0T} \delta U_{,\eta} \rangle_{\eta}.$$

The orthogonality condition in the terms of scalar product in the functional space is

$$\langle U_{,\eta}^{0T} \delta U_{,\eta} \rangle_{\eta} = - \langle U_{,\eta\eta}^{0T} \delta U \rangle_{\eta},$$

where integrating by parts has been used. Here δU is a tangent vector to the manifold (28) at the point U^0 , and $-U_{,\eta\eta}^0$ is the normal vector, or the gradient. The corresponding unit normal vector to the manifold is defined as

$$n = \frac{\text{grad } \varepsilon[U^0]}{|\text{grad } \varepsilon[U^0]|} = \frac{-U_{,\eta\eta}^0}{\omega_0}, \quad \omega_0 = |\text{grad } \varepsilon[U^0]|. \quad (\text{A.2})$$

Let $\{s_1, \dots, s_M\}$ be a set of orthogonal curvilinear co-ordinates at the manifold. Generally, the number M is infinite, but practically we always consider a finite number of in-plane modes N , and out-of-plane modes K . For this case one can write $M = N + K - 1$, where $N + K$ is the complete dimension of the configuration space. Note that $\{\partial U^0/\partial s_1, \dots, \partial U^0/\partial s_M\}$ are tangent vectors with respect to the manifold, and, hence one can write

$$\left\langle n^\Gamma \frac{\partial U^0}{\partial s_i} \right\rangle_\eta = 0, \quad \left\langle \frac{\partial U^{0\Gamma}}{\partial s_i} \frac{\partial U^0}{\partial s_j} \right\rangle_\eta = \delta_{ij}, \quad i, j = 1, \dots, M, \quad (\text{A.3})$$

where the second relationship corresponds to our definition of the set $\{s_1, \dots, s_M\}$, while the first can be easily verified. In fact, using integration by parts, and expression (28) for special manifold, gives

$$\left\langle n^\Gamma \frac{\partial U^0}{\partial s_i} \right\rangle_\eta = -\frac{1}{\omega_0} \left\langle U^{0\Gamma} \frac{\partial U^0}{\partial s_i} \right\rangle_\eta = \frac{1}{2\omega_0} \frac{\partial |U_{,\eta}^0|^2}{\partial s_i} = \frac{1}{2\omega_0} \frac{\partial |U_{0,\eta}|^2}{\partial s_i} \equiv 0.$$

This completes the proof of the orthogonality condition.