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ALMOST-SURE STABILITY OF LINEAR GYROSCOPIC SYSTEMS

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This paper studies the stability behaviour of a linear gyroscopic system parametrically perturbed by a (multiplicative) real noise of small intensity. To this end, its maximal Lyapunov exponent is calculated using the method of Sri Namachchivaya *et al.* [1]. The results derived are suitable for cases where the response frequencies ω_1, ω_2 are non-commensurable and the infinitesimal generator associated with the noise process, $\xi(t)$ has a simple zero eigenvalue. These results are then employed to determine the almost-sure stability boundaries of a rotating shaft subjected to random axial loading.

1. INTRODUCTION

Operational efficiency, reliability and safety are primary requirements of engineering systems. Each of these is influenced by the system response, the stability of this response and its rate of convergence toward a steady state. These, in turn, are dependent on the type, intensity and duration of the excitation applied to the system. Clearly, any instability of the response will lead to stress fluctuations, reduced performance, noise and vibration resulting in fatigue problems and, in extreme cases, catastrophic failure.

The behaviour of engineering systems often reflect the influence of some random component which may arise from several possible sources. These may include coupling with high-dimensional or non-linear external systems or loadings which are poorly defined spatially or temporally. The influence of these random excitations will be most profound under operational conditions close to the onset of deterministic instability; however it may not be clear *a priori* whether they will expand or reduce the region of stable operation. Thus, it is evident that the application of stochastic stability theory can yield results relevant to practical engineering systems.

The strongest definition of stochastic stability, that of almost-sure stability, guarantees that all sample functions, except for those of measure zero, converge

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asymptotically to the trivial solution with probability one. Consider a d-dimensional linear system subjected to a real noise $\xi(t)$.

$$\dot{x} = \mathbf{A}(\xi(t))x, \quad x(0) = x_0 \neq 0.$$

The exponential growth rate of its solution is given by the Lyapunov exponent

$$\lambda(x) = \lim_{t \to \infty} \frac{1}{t} \log \| x(t; x_0) \|,$$

with suitable norm $\| \cdot \|$. By Oseledec's multiplicative ergodic theorem this system possesses at most d such Lyapunov exponents. The largest of these can be regarded as being the stochastic analogue of the largest real part of the eigenvalue of a deterministic system. Hence, a negative value of this quantity provides a necessary and sufficient condition for asymptotic sample stability of the system; a positive value indicates sample instability with probability one. The top Lyapunov exponent, therefore, provides a means of delineating those regions of parameter space corresponding to stable and unstable behaviour.

Although asymptotically stable, the rate of convergence of the system response towards its steady state value may be undesirably low. By itself, the top Lyapunov exponent provides no insight into either the rate of convergence of the system response towards its steady state value or the stability of its moments. This information is provided by the moment Lyapunov exponent, denoted by $g(p)$,

$$g(p) = \lim_{t \to \infty} \frac{1}{t} \log \mathbf{E}[\, \| x(t; x_0) \|^p],$$

where $\mathbf{E}[\cdot]$ denotes the expectation operator. It is noted that $g(p)$ is a convex function with $g(p)/p$ increasing and $g(0) = 0$. It was found, first by Molčanov [2] for two-dimensional systems, and later generalized by Arnold [3] to include d-dimensional systems, that the two are related by

$$\lambda(x) = g'(0).$$

By using this result and the convexity properties of $g(p)$ various researchers [1, 4] have produced asymptotic expansions representing the moment Lyapunov exponent of a linear stochastic system as a function of its maximal Lyapunov exponent. The relationship between $\lambda(x)$ and $g(p)$ is illustrated in Figure 1.

Gyroscopic systems find wide usage in engineering applications. They can be major system components themselves or sub-components of larger, more complicated systems. Common examples include rotating shafts when treated in a rotating co-ordinate frame, pipes conveying fluid, and elastic strips moving in an axial direction. When modelling their transverse vibrations, each of these simple systems is formulated as a gyroscopic system. Such systems are widely known to exhibit interesting stability properties whose analysis is non-trivial. Due to this and their common usage in industrial applications, where there is a desire to increase mechanical efficiency and operational safety and to minimize noise and vibration, further study of the stability of this class of systems would be beneficial.

Currently available results pertaining to maximal Lyapunov exponents of linear systems include those for real and white noise cases for systems of two and four

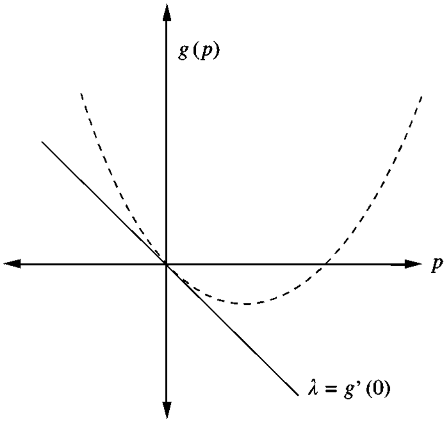

Figure 1. Moment and maximal Lyapunov exponent.

dimensions. These have been obtained through a variety of methods including the method of stochastic averaging [5, 6] and the forward perturbative method [7, 8]. In addition, Sri Namachchivaya *et al.* [1] used a backward perturbative method to obtain analytic expressions for the top Lyapunov exponent and moment Lyapunov exponent for a four-dimensional linear system.

For gyroscopic systems, mean-square stability results are available [9, 10]. These results were based on the method of stochastic averaging, and were derived for small intensity real noise processes. The general results of Doyle and Sri Namachchivaya [8] are suitable for application to gyroscopic systems but are not sufficiently transparent to be readily applicable. In this work, a gyroscopic system is considered from the outset with a view towards providing simpler, but equivalent, stability results to those available from above. The results presented in this paper represent the first application of the concept of the top Lyapunov exponent to the determination of almost-sure stability of gyroscopic systems.

To this end, the backward perturbative method developed by Sri Namachchivaya *et al.* [1] is employed to obtain maximal Lyapunov exponents for a general four-dimensional linear gyroscopic systems driven by a multiplicative real noise. It is assumed that the response frequencies ω_1, ω_2 are non-commensurable and the generator describing the noise process has a simple zero eigenvalue. The analytical results obtained are then applied to the example of an axially loaded rotating shaft system in a rotating reference frame.

2. STATEMENT OF THE PROBLEM AND FORMULATION

A linear gyroscopic system has equations of motion of the form

$$\mathbf{M}\ddot{q} + 2\mathbf{G}\dot{q} + \mathbf{K}q = 0, \tag{1}$$

where $\mathbf{M} = \mathbf{M}^{\mathrm{T}} > 0$, $\mathbf{K} = \mathbf{K}^{\mathrm{T}}$, $\mathbf{G}^{\mathrm{T}} = -\mathbf{G}$ and the superscript T denotes the transpose operator. The vector q denotes a generalized co-ordinate. Generally, the matrix $\mathbf{K}$ is composed of potential terms and terms depending on the gyroscopic parameter Ω.

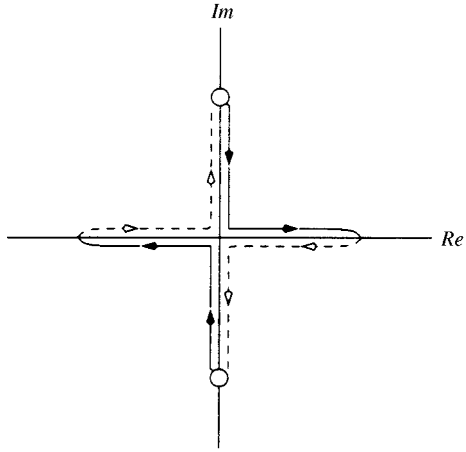

Figure 2. Trajectory of system eigenvalues.

It is widely known that the gyroscopic parameter characterizes stability for this class of system, and this will be described for the deterministic, conservative case. If gyroscopic coupling is initially absent (i.e., $\Omega = 0$), the system will have two pairs of pure imaginary eigenvalues. As shown in Figure 2, as Ω is increased, one pair of eigenvalues (the pair closer to the origin) coalesces at the origin and splits along the real axis resulting in divergence instability. However, due to gyroscopic effects, the eigenvalues once again coalesce at the origin near the second critical value of Ω. Beyond this critical value the system is then said to be restabilized.

Naturally, as dissipation is introduced into the system its behaviour is further complicated. The relationship between stability, damping and gyroscopic forces is described by the Kelvin–Tait–Chetaev theorem (for details see Chetaev [11]); from this it follows that if a system is stable in the absence of gyroscopic and dissipative forces, their addition to the system will lead to asymptotic stability. Conversely, if an unstable equilibrium can be stabilized by the addition of purely gyroscopic forces then, depending on whether or not it exhibits complete dissipation, the addition of damping may enhance or destroy this stability. Depending on the coefficients of a particular gyroscopic system, internal damping forces may give rise to complete dissipation. A fuller account of gyroscopic stability is widely available (see, e.g., references [12, 13]).

In this work, explicit results are derived for the maximal Lyapunov exponent of a four-dimensional linear gyroscopic system perturbed by a multiplicative real noise of small intensity. This system is described with equations of motion

$$\ddot{q}_1 - 2\Omega\dot{q}_2 + \kappa_1 q_1 + \varepsilon f(\xi(t)) \sum_{j=1}^{2} k_{1j} q_j + \varepsilon^2 \zeta_1 \dot{q}_1 = 0,$$

$$\ddot{q}_2 - 2\Omega\dot{q}_1 + \kappa_2 q_2 + \varepsilon f(\xi(t)) \sum_{j=1}^{2} k_{2j} q_j + \varepsilon^2 \zeta_2 \dot{q}_2 = 0, \quad (2)$$

where stochastic coupling of modes is represented by the terms k_{ij}. Again, Ω denotes the gyroscopic parameter and the q_i's represent generalized co-ordinates.

Potential and internal damping forces are proportional to the κ_i and $\varepsilon^2\zeta_i$ terms respectively. Here the magnitudes of the stochastic perturbation and the dissipation have been scaled appropriately in order to make dominant the effects of the stochastic perturbation. The stochastic term $\varepsilon\xi(t)$ is a small-intensity, real noise process on a smooth, connected Riemannian manifold, M (with or without boundary). This is assumed to admit representation by an infinitesimal generator $G(\xi)$ with a simple zero eigenvalue. The function $f(\xi(t)), f: M \to R$, is smooth and time varying with zero mean.

Crucial to calculation of the top Lyapunov exponent is transformation of the equations of motion into Khas'minskii's logarithmic-polar form in which the equation for the angles is independent of the norm. In addition, the equation governing the noise $\xi(t)$ is written as a Stratonovich stochastic differential equation

$$\dot{\rho} = q(\xi, s), \quad \dot{s} = h(\xi, s), \quad \mathrm{d}\xi = \mu_\xi(\xi)\,\mathrm{d}t + \sigma_\xi(\xi) \circ \mathrm{d}W_t, \quad \xi \in M.$$

This representation is obtained following a procedure employed by Sri Namachchivaya and Tien [14]. The solutions of the unperturbed equations of motion (equation (2), where $\varepsilon = 0$) are given by

$$q_1(t) = Q_{11}\mathrm{e}^{\mathrm{i}\omega_1 t} + Q_{12}\mathrm{e}^{-\mathrm{i}\omega_1 t} + Q_{13}\mathrm{e}^{\mathrm{i}\omega_2 t} + Q_{14}\mathrm{e}^{-\mathrm{i}\omega_2 t},$$
$$q_2(t) = \tilde{\alpha}_1 Q_{11}\mathrm{e}^{\mathrm{i}\omega_1 t} - \tilde{\alpha}_1 Q_{12}\mathrm{e}^{-\mathrm{i}\omega_1 t} + \tilde{\alpha}_2 Q_{13}\mathrm{e}^{\mathrm{i}\omega_2 t} - \tilde{\alpha}_2 Q_{14}\mathrm{e}^{-\mathrm{i}\omega_2 t},$$
$$\tilde{\alpha}_i = (\omega_i^2 - \kappa_1)/2\Omega\omega_i, \quad i = 1, 2,$$

where Q_{ij} represents the contribution of the jth eigenmode to the ith degree of freedom. The response frequencies ω_1, ω_2 are given by

$$\omega_{1,2} = \frac{1}{\sqrt{2}}[\kappa_1 + \kappa_2 + 4\Omega^2 \pm \sqrt{(\kappa_1 + \kappa_2 + 4\Omega^2)^2 - 4\kappa_1\kappa_2}]^{1/2}, \omega_1 > \omega_2.$$

Introducing polar co-ordinates ($\rho = \log \| q \|$, θ, ϕ_1, ϕ_2),

$$q_1(t) = \mathrm{e}^\rho \cos\theta \cos\phi_1 + \mathrm{e}^\rho \sin\theta \cos\phi_2,$$
$$q_2(t) = -\tilde{\alpha}_1 \mathrm{e}^\rho \cos\theta \sin\phi_1 - \tilde{\alpha}_2 \mathrm{e}^\rho \sin\theta \sin\phi_2, \quad \phi_i = \omega_i t + \varphi_i.$$

These are transformed to first order form by applying the method of variation of constants to the variables (ρ, θ, ϕ_1, ϕ_2). Thus, in Khas'minskii form, the dynamical system is represented as

$$\dot{\rho} = \sum_{j=0}^{2} \varepsilon^j q^j(\phi_1, \phi_2, \theta, \xi) = q(\xi, s),$$
$$\dot{s} = \begin{bmatrix} \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{2} \varepsilon^j s^j(\phi_1, \phi_2, \theta, \xi) \\ \omega_1 + \sum_{j=0}^{2} \varepsilon^j h_1^j(\phi_1, \phi_2, \theta, \xi) \\ \omega_2 + \sum_{j=0}^{2} \varepsilon^j h_2^j(\phi_1, \phi_2, \theta, \xi) \end{bmatrix} = h(\xi, s),$$
$$\mathrm{d}\xi = \mu_\xi(\xi)\mathrm{d}t + \sigma_\xi(\xi) \circ \mathrm{d}W_t, \quad \xi \in M. \tag{3}$$

These equations represent in explicit form the diffusion for $\log \| q \|$, the angles ϕ_1, ϕ_2, θ and the input noise $\xi(t)$. The coefficients of each of these equations are contained in Appendix A. It is noted that the first order terms of the expansions for each of $q(\xi, s)$ and $h(\xi, s)$ are identically zero. The processes (ϕ_1, ϕ_2, θ) are independent of ρ and they form a diffusive Markov process with unique probability measure μ and infinitesimal generator, $\mathscr{L}^\varepsilon$,

$$\mathscr{L}^{\varepsilon *}\mu = 0.$$

The generator, $\mathscr{L}^\varepsilon$, assumed to be non-degenerate, is given by

$$\mathscr{L}^\varepsilon = \mathscr{L}^0 + \varepsilon\mathscr{L}^1 + \varepsilon^2\mathscr{L}^2, \tag{4}$$

where

$$\mathscr{L}^0 = G(\xi) + \sum_{i=1}^{2} \omega_i \frac{\partial}{\partial \phi_i},$$

$$\mathscr{L}^1 = s^1(\phi_1, \phi_2, \theta, \xi)\frac{\partial}{\partial \theta} + \sum_{i=1}^{2} h_i^1(\phi_1, \phi_2, \theta, \xi)\frac{\partial}{\partial \phi_i},$$

$$\mathscr{L}^2 = s^2(\phi_1, \phi_2, \theta, \xi)\frac{\partial}{\partial \theta} + \sum_{i=1}^{2} h_i^2(\phi_1, \phi_2, \theta, \xi)\frac{\partial}{\partial \phi_i}.$$

The superscript ε has been introduced to indicate explicitly the presence of small parameters which will facilitate the subsequent analysis. It was shown by Arnold [3] that the quantity $L(p) = \mathscr{L} + pq(\xi, s)$ has a principal eigenvalue given by $g(p)$ with eigenfunction $\phi(p)$ possessing the property $\|\phi(p)\| = 1$. Additionally, the adjoint operator $L^*(p)$ has a unique eigenfunction, $v(p)$ with the property $\langle \phi(p), v(p)\rangle = 1$. This may be expressed as

$$L(p)\phi(p) = g(p)\phi(p), \quad \langle \phi(p), v(p)\rangle = 1, \quad \forall p \in R.$$

Differentiating both sides and making use of the properties of $g(p)$ yields

$$\mathscr{L}\psi = \lambda - q(\xi, s), \tag{5}$$

where $\psi = \partial\phi/\partial p|_{p=0}$. Taking appropriate scalar products with $v(p)$ yields the solvability condition for the top Lyapunov exponent,

$$\lambda = \langle \mu(\xi, s), q(\xi, s)\rangle,$$

where $\langle \cdot\,, \cdot\rangle$ represents the inner product operator. Since $q(\xi, s)$ and $\mathscr{L}$ are given in powers of ε, as in equations (3) and (4) respectively, by following Sri Namachchivaya *et al.* [1] it can be shown that the asymptotic representation for λ,

$$\lambda^\varepsilon = \sum_{k=0}^{\infty} \lambda_k \varepsilon^k,$$

is valid. Upon substituting for $\mathscr{L}$ and $q(\xi, s)$ and making the formal expansions to order ε^2,

$$\psi^\varepsilon = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2, \qquad \lambda^\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2,$$

a series of Poisson equations is produced when equation (5) is expanded in orders of ε,

$$\mathscr{L}^0\psi_0 = \lambda_0 - q^0, \tag{6}$$

$$\mathscr{L}^0\psi_1 = \lambda_1 - q^1 - \mathscr{L}^1\psi_0, \tag{7}$$

$$\mathscr{L}^0\psi_2 = \lambda_2 - q^2 - \mathscr{L}^1\psi_1 - \mathscr{L}^2\psi_0. \tag{8}$$

These equations define the unknown coefficients λ_k from $\mathcal{O}(1)$ to $\mathcal{O}(\varepsilon^2)$. Coefficients above $\mathcal{O}(\varepsilon^2)$ are ignored in this work.

3. EVALUATION OF SOLUTIONS

The objective of this section is the development of an expression for the first non-vanishing term in the expansion for λ^ε. Each order of the approximation for λ^ε is extracted from the Poisson equations (6)–(8) which, in turn, involves solving for each successive order of ψ^ε. Each of the equations to be solved involves the use of the differential operator $\mathscr{L}^0$. This raises the need for the solution of the corresponding adjoint problem $\mathscr{L}^{0*}\psi_0^* = 0$ with the periodic boundary conditions $\psi_0^*(\phi_1, \phi_2, \theta, \xi) = \psi_0^*(\phi_1, \phi_2 + 2n\pi, \theta, \xi) = \psi_0^*(\phi_1 + 2m\pi, \phi_2, \theta, \xi)$.

3.1. SOLUTION TO $\mathcal{O}(1)$

Employing the solvability condition yields

$$\lambda_0 = 0 \quad \text{and} \quad \mathscr{L}^0\psi_0 = 0.$$

From this it is clear that $\psi_0 = \psi_0(\theta)$. Additionally, from the forward equation, $\mathscr{L}^{0*}\psi_0^* = 0$. A solution to this partial differential equation was found by Sri Namachchivaya and Van Roessel [7] and their result is given as a proposition.

Proposition. *Suppose G has an isolated simple zero eigenvalue and the frequencies ω_1, ω_2 are non-commensurable, then the stationary solution of*

$$\mathscr{L}^{0*}\psi_0^* = 0 \quad \textit{where} \quad \mathscr{L}^{0*} = G^* - \sum_{i=1}^{2}\omega_i\frac{\partial}{\partial\phi_i}$$

is given by

$$\psi_0^*(\theta, \xi) = \frac{v(\xi)\mathscr{F}(\theta)}{4\pi^2},$$

where $v(\xi)$ is the invariant density associated with G and $\mathscr{F}(\theta)$ is an arbitrary function.

3.2. SOLUTION TO $\mathcal{O}(\varepsilon)$

As ψ_0 is a function only of θ, equation (7) reduces to

$$\mathscr{L}^0\psi_1 = \lambda_1 - q^1(\phi_1, \phi_2, \theta, \xi) - s^1(\phi_1, \phi_2, \theta, \xi)\psi_0' \equiv \lambda_1 - f(\xi)R(\phi_1, \phi_2, \theta).$$

Applying the solvability condition leads to

$$\lambda_1 = \frac{1}{16\pi^2\Delta}\langle f(\xi)R(\phi_1, \phi_2, \theta), v(\theta)\mathscr{F}(\theta)\rangle, \tag{9}$$

where the inner product is defined as

$$\langle(\cdot),(\cdot\cdot)\rangle = \int_0^2 \int_M \int_0^{2\pi} \int_0^{2\pi} (\cdot)(\cdot\cdot)\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\mathrm{d}\xi\,\mathrm{d}\theta$$

and

$$\begin{aligned} R(\phi_1, \phi_2, \theta) = {} & \psi^{s_1} \sin 2\phi_1 - \psi^{s_2} \sin 2\phi_2 - \psi^{s_+} \sin \phi^+ + \psi^{s_-} \sin \phi^- \\ & + \psi^{c_1} \cos 2\phi_1 - \psi^{c_2} \cos 2\phi_2 - \psi^{c_+} \cos \phi^+ + \psi^{c_-} \cos \phi^- + \psi^{cs_0}, \end{aligned}$$

in which

$$\psi^{cs_0} = \frac{1}{2\Delta} a_{12}^-(-\cos 2\theta + \sin 2\theta\, \psi_0'),$$

$$\psi^{s_1} = \frac{1}{4\Delta} c_{21}^-[1 + \cos 2\theta - \sin 2\theta\, \psi_0'], \quad \psi^{s_2} = \frac{1}{4\Delta} c_{12}^-[1 - \cos 2\theta + \sin 2\theta\, \psi_0'],$$

$$\psi^{s_+} = \frac{1}{4\Delta}[(c_{11}^- - c_{22}^-) \sin 2\theta + [c_{11}^- + c_{22}^- + (c_{11}^- - c_{22}^-) \cos 2\theta]\, \psi_0'],$$

$$\psi^{s_-} = \frac{1}{4\Delta}[(c_{11}^+ + c_{22}^+) \sin 2\theta + [c_{11}^+ - c_{22}^+ + (c_{11}^+ + c_{22}^+) \cos 2\theta]\, \psi_0'],$$

$$\psi^{c_1} = \frac{1}{4\Delta} a_{12}^+[1 + \cos 2\theta - \sin 2\theta\, \psi_0'], \quad \psi^{c_2} = \frac{1}{4\Delta} a_{21}^+[1 - \cos 2\theta + \sin 2\theta\, \psi_0'],$$

$$\psi^{c_+} = \frac{1}{4\Delta}[(a_{11}^+ - a_{22}^+) \sin 2\theta + [a_{11}^+ + a_{22}^+ + (a_{11}^+ - a_{22}^+) \cos 2\theta]\, \psi_0'],$$

$$\psi^{c_-} = \frac{1}{4\Delta}[(a_{11}^- - a_{22}^-) \sin 2\theta + [a_{11}^- + a_{22}^- + (a_{11}^- - a_{22}^-) \cos 2\theta]\, \psi_0'].$$

As each component of $R(\phi_1, \phi_2, \theta)$ is periodic in (ϕ_1, ϕ_2), equation (9) is evaluated to yield

$$\lambda_1 = 0. \tag{10}$$

Consequently equation (7) further simplifies to

$$\mathscr{L}^0 \psi_1 = -q^1(\phi_1, \phi_2, \theta, \xi) - s^1(\phi_1, \phi_2, \theta, \xi)\psi_0',$$

which must be solved for ψ_1. This is achieved by employing the Green's function, $g(\xi, t; \eta, 0)$ for the operator G. This is the solution of

$$\left(\frac{\partial}{\partial t} - G\right) g = 0, \quad g(\xi, 0; \eta, 0) = \delta(\xi - \eta).$$

Thus, by appropriate rescaling of variables it is found that the $\mathcal{O}(\varepsilon)$ term in the asymptotic approximation for the stationary density, ψ^ε, is given by

$$\psi_1(\phi_1, \phi_2, \theta, \xi) = \int_0^\infty -K(\xi, \tau) R(\tau, \phi_1, \phi_2, \theta)\,\mathrm{d}\tau, \tag{11}$$

where

$$K(\xi, \tau) = \int_M f(\eta) g(\eta, \tau; \xi, 0)\, \mathrm{d}\eta.$$

3.3. SOLUTION TO $\mathcal{O}(\varepsilon^2)$

In order to solve for λ_2, equation (8) is required, i.e.,

$$\mathscr{L}^0 \psi_2 = \lambda_2 - q^2(\phi_1, \phi_2, \theta, \xi) - \mathscr{L}^1 \psi_1(\phi_1, \phi_2, \theta, \xi) - \mathscr{L}^2 \psi_0(\theta).$$

The solvability condition for this is

$$\langle \lambda_2 - q^2 - \mathscr{L}^1 \psi_1 - \mathscr{L}^2 \psi_0, \psi_0^* \rangle = 0,$$

which, in turn, implies

$$\int_0^{\pi/2} \left\{ \lambda_2 - \frac{1}{4\pi^2} \int_M \int_0^{2\pi} \int_0^{2\pi} v(\xi) q^2 \,\mathrm{d}\phi_1 \,\mathrm{d}\phi_2 \,\mathrm{d}\xi \right.$$
$$\left. - \frac{1}{4\pi^2} \int_M \int_0^{2\pi} \int_0^{2\pi} v(\xi) [- \mathscr{L}^1 \psi_1 + \mathscr{L}^2 \psi_0] \,\mathrm{d}\phi_1 \,\mathrm{d}\phi_2 \,\mathrm{d}\xi \right\} \mathscr{F}(\theta)\, \mathrm{d}\theta = 0.$$

After making use of the fact that $\psi_0 = \psi_0(\theta)$ and substituting for ψ_1 from equation (11), this can be written more concisely as

$$\int_0^{\pi/2} \{ \lambda_2 - I_1(\theta) - I_2(\theta) \} \mathscr{F}(\theta)\, \mathrm{d}\theta = 0, \tag{12}$$

$$I_1(\theta) = \frac{1}{4\pi^2} \int_M \int_0^{2\pi} \int_0^{2\pi} v(\xi) q^2 \,\mathrm{d}\phi_1 \,\mathrm{d}\phi_2 \,\mathrm{d}\xi,$$

$$I_2(\theta) = \frac{1}{4\pi^2} \int_M \int_0^{2\pi} \int_0^{2\pi} \left[\int_0^{\infty} K(\xi, T) \left\{ s^1 \frac{\partial R}{\partial \theta} + \sum_{i=1}^{2} h_i^1 \frac{\partial R}{\partial \phi_i} \right\} \mathrm{d}T + s^2 \frac{\partial \psi_0}{\partial \theta} \right] \mathrm{d}\phi_1 \,\mathrm{d}\phi_2 \,\mathrm{d}\xi.$$

At this point, the sine spectrum, $\Gamma(\omega)$, and cosine spectrum $S(\omega)$, are introduced according to the usual relations,

$$\Gamma(\omega) = 2 \int_0^{\infty} \mathscr{R}(\tau) \sin \omega\tau \,\mathrm{d}\tau, \quad S(\omega) = 2 \int_0^{\infty} \mathscr{R}(\tau) \cos \omega\tau \,\mathrm{d}\tau,$$

and the correlation function $\mathscr{R}(\tau)$ is defined as

$$\mathscr{R}(\tau) = \int_M f(\xi) K(\xi, \tau)\, \mathrm{d}\tau.$$

The solvability condition, equation (12), reduces to

$$\int_0^{\pi/2} \left\{ \lambda_2 - Q(\theta) - \mu(\theta) \psi_0'(\theta) - \frac{1}{2} \sigma^2(\theta) \psi_0''(\theta) \right\} \mathscr{F}(\theta)\, \mathrm{d}\theta = 0, \tag{13}$$

with

$$\sigma^2(\theta) = A\cos^2 2\theta + B\cos 2\theta + C, \quad \mu(\theta) = \sigma^2(\theta)\cot 2\theta - \frac{\Lambda}{2}\sin 2\theta,$$

$$Q(\theta) = J(\theta) + D = \sigma^2(\theta) + \frac{\Lambda}{2}\cos 2\theta + D.$$

The functions A, B, C, D and Λ are given by

$$A = -\gamma, \quad B = \frac{\alpha^-}{2} = \frac{1}{2}(\alpha_{11} - \alpha_{22}),$$

$$C = \gamma + \frac{\alpha^+}{2} = \gamma + \frac{1}{2}(\alpha_{11} + \alpha_{22}),$$

$$D = \frac{1}{2}(\Lambda_1 + \Lambda_2) - \tilde{\mu}, \; \Lambda = \Lambda_1 - \Lambda_2,$$

where

$$\gamma = \frac{a_{12}^{-2}}{4\Delta^2}S(0) + \frac{1}{4}(\beta_{12} + \beta_{21}) - \frac{1}{4}(\alpha_{11} + \alpha_{22}) + \frac{1}{2}\tilde{\mu},$$

$$\alpha_{ii} = \frac{1}{8\Delta^2}[(c_{ii}^{+2} + a_{ii}^{-2})S(\omega^-) + (c_{ii}^{-2} + a_{ii}^{+2})S(\omega^+)],$$

$$\beta_{ij} = \frac{1}{8\Delta^2}[c_{ji}^{-2} + a_{ji}^{+2}]S(2\omega_i),$$

$$\tilde{\mu} = \frac{1}{8\Delta^2}[(-c_{11}^+c_{22}^+ + a_{11}^-a_{22}^-)S(\omega^-) + (c_{11}^-c_{22}^- + a_{11}^+a_{22}^+)S(\omega^+)],$$

$$\Lambda_1 = \beta_{12} + (\gamma_1 + \gamma_2) - \frac{b_{21}^{1+}}{2\Delta}, \quad \gamma_1 = \frac{\Gamma(\omega^+)}{8\Delta^2}[c_{11}^-a_{22}^+ - c_{22}^-a_{11}^+],$$

$$\Lambda_2 = \beta_{21} - (\gamma_1 + \gamma_2) + \frac{b_{12}^{2+}}{2\Delta}, \quad \gamma_2 = \frac{\Gamma(\omega^-)}{8\Delta^2}[c_{11}^+a_{22}^- + c_{22}^+a_{11}^-].$$

Equation (13) must hold for arbitrary $\mathscr{F}(\theta)$; thus

$$\lambda_2 - Q(\theta) = \mu(\theta)\psi_0'(\theta) + \tfrac{1}{2}\sigma^2(\theta)\psi_0''(\theta). \tag{14}$$

In order to solve this, the corresponding adjoint equation for $p_0(\theta)$ is used. A solution to this is provided using the concepts of scale and speed measures, $p_0(\theta) = Nm(\theta)$; thus

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}[\sigma^2(\theta)m(\theta)] - \frac{\mathrm{d}}{\mathrm{d}\theta}[\mu(\theta)m(\theta)] = 0, \tag{15}$$

where p_0 is normalized by the condition

$$N\int_M m(\theta)\,\mathrm{d}\theta = 1.$$

The solution of equation (15) is given by

$$m(\theta) = \beta'(\theta)\,\mathrm{e}^{-A\beta(\theta)},$$

where

$$\beta(\theta) = \int_0^\theta \frac{\sin 2z}{\sigma^2(z)}\,\mathrm{d}z = \frac{1}{2}\int_{\cos 2\theta}^{1} \frac{\mathrm{d}x}{Ax^2 + Bx + C}.$$

The solvability condition for equation (14) can be written as

$$\int_0^{\pi/2} (\lambda_2 - Q(\theta))m(\theta)\,\mathrm{d}\theta = 0.$$

From this,

$$\lambda_2 = \langle Q(\theta), p_0(\theta)\rangle = \langle J(\theta) + D, Nm(\theta)\rangle = D + \tilde{\lambda}_2, \quad \tilde{\lambda}_2 = N\langle J(\theta), m(\theta)\rangle. \tag{16}$$

Equation (16) provides a general expression for λ_2, which is thefirst non-vanishing term in the asymptotic expansion for the top Lyapunov exponent. Recalling that higher order terms are ignored the maximal Lyapunov exponent is given by

$$\lambda^\varepsilon = \varepsilon^2 \lambda_2. \tag{17}$$

4. CLASSIFICATION OF SINGULARITIES

From equation (16), λ_2 is a function of the probability density function of the θ-process. As singularities in the θ-process define its probability distribution in phase space, their influence upon the expression for λ_2 is next investigated. In general, a singularity will occur when either of the following conditions is satisfied,

$$\mu(\theta) = \infty, \quad \sigma^2(\theta) = 0.$$

As the drift and diffusion terms are defined by

$$\mu(\theta) = \sigma^2(\theta)\cot 2\theta - \tfrac{4}{2}\sin 2\theta, \quad \sigma^2(\theta) = A\cos^2 2\theta + B\cos 2\theta + C,$$

the θ-process has entrance boundaries at $\theta = 0$, $\pi/2$ for all possible values of A, B and C. The direction of traverse of the process across a singular point, θ_s is indicated by the drift coefficient, i.e.,

$$\mu(\theta_s) > 0, \quad \theta_s \text{ a forward shunt;}$$

$$\mu(\theta_s) < 0, \quad \theta_s \text{ a backward shunt;}$$

$$\mu(\theta_s) = 0, \quad \theta_s \text{ a trap point.}$$

Employing the concepts of Feller [15], the boundary can be further classified (see references [16] or [17]). It is necessary to consider individually the singularities in each of the six possible singular cases distinguished by A, B and C as follows:

1. $A, B, C \neq 0$;
2. (a) $B = 0$; A, $C \neq 0$,
   (b) A, $B = 0$; $C \neq 0$,
   (c) B, $C = 0$; $A \neq 0$;

3. $C = 0$; $A, B \neq 0$;

4. $A = 0$; $B, C \neq 0$;

5. $A, C = 0$, $B \neq 0$;

6. $A, B, C = 0$.

Which of these parameter combinations that occurs depends on the excitation applied to the system and the operational parameters of the system itself. It is noted that singular case 1, $A, B, C \neq 0$ is most germane to engineering applications. Also, if no additional singularities exist in the interval $\theta \in (0, \pi/2)$ then the top Lyapunov exponent is given simply by

$$\lambda^{\varepsilon} = \varepsilon^2 \left(D - \frac{\Lambda}{2} \left[\frac{M(\pi/2) + M(0)}{M(\pi/2) - M(0)} \right] \right), \quad M(\theta) = -\frac{1}{\Lambda} \mathrm{e}^{-\Lambda \beta(\theta)}. \tag{18}$$

4.1. CASE 1: $A, B, C \neq 0$

In this singular case, probability mass enters the region $\theta \in (0, \pi/2)$ as shown in Figure 3. The boundaries at $\theta = (0, \pi/2)$ are both entrances; consequently an invariant measure exists over the interval $\theta \in (0, \pi/2)$. From equation (18), the $\mathcal{O}(\varepsilon^2)$ approximation to the maximal Lyapunov exponent is given by

$$\lambda_2 = D - \frac{\Lambda}{2} \left| \frac{M(\pi/2) + M(0)}{M(\pi/2) - M(0)} \right|. \tag{19}$$

Defining $4AC - B^2 = \delta$, the expressions for the speed function, $M(\theta)$ corresponding to possible values of δ are given explicitly below:

$$\delta = 0, \; M(\theta) = -\frac{1}{\Lambda} \exp \left\{ \frac{-\Lambda}{2A \cos 2\theta + B} \right\},$$

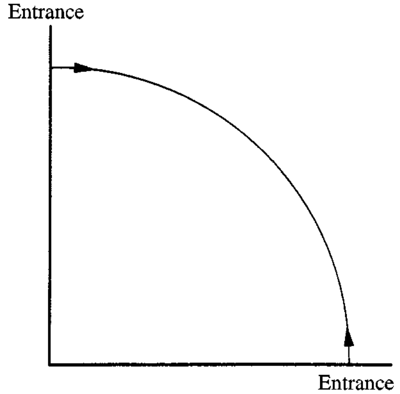

Figure 3. Boundary behaviour for Cases 1, 2(b) and 4.

$$\delta < 0,\ M(\theta) = -\frac{1}{\Lambda}\left|\frac{2A\cos 2\theta + B - \sqrt{-\delta}}{2A\cos 2\theta + B + \sqrt{-\delta}}\right|^{\Lambda/(2\sqrt{-\delta})},$$

$$\delta > 0,\ M(\theta) = -\frac{1}{\Lambda}\exp\left\{\frac{\Lambda}{\sqrt{\delta}}\tan^{-1}\left(\frac{2A\cos 2\theta + B)}{\sqrt{\delta}}\right)\right\}.$$

For each discriminant, the corresponding maximal Lyapunov exponent is evaluated using equation (18),

$$\delta = 0, \quad \lambda^{\varepsilon} = \varepsilon^2\left\{D - \frac{\Lambda}{2}\left[\frac{\exp\left(\dfrac{-4A\Lambda}{B^2 - 4A^2}\right) + 1}{\exp\left(\dfrac{-4A\Lambda}{B^2 - 4A^2}\right) - 1}\right]\right\},$$

$$\delta < 0, \quad \lambda^{\varepsilon} = \varepsilon^2\left\{D - \frac{\Lambda}{2}\left[\frac{\left|\dfrac{B - 2A - \sqrt{-\delta}}{B - 2A + \sqrt{-\delta}}\right|^{\Lambda/(2\sqrt{-\delta})} + \left|\dfrac{B + 2A - \sqrt{-\delta}}{B + 2A + \sqrt{-\delta}}\right|^{\Lambda/(2\sqrt{-\delta})}}{\left|\dfrac{B - 2A - \sqrt{-\delta}}{B - 2A + \sqrt{-\delta}}\right|^{\Lambda/(2\sqrt{-\delta})} - \left|\dfrac{B + 2A - \sqrt{-\delta}}{B + 2A + \sqrt{-\delta}}\right|^{\Lambda/(2\sqrt{-\delta})}}\right]\right\},$$

$$\delta > 0, \quad \lambda^{\varepsilon} = \varepsilon^2\left\{D - \frac{\Lambda}{2}\left[\frac{\exp\left(\dfrac{\Lambda}{\sqrt{\delta}}\tan^{-1}\dfrac{B - 2A}{\sqrt{\delta}}\right) + \exp\left(\dfrac{\Lambda}{\sqrt{\delta}}\tan^{-1}\dfrac{B + 2A}{\sqrt{\delta}}\right)}{\exp\left(\dfrac{\Lambda}{\sqrt{\delta}}\tan^{-1}\dfrac{B - 2A}{\sqrt{\delta}}\right) - \exp\left(\dfrac{\Lambda}{\sqrt{\delta}}\tan^{-1}\dfrac{B + 2A}{\sqrt{\delta}}\right)}\right]\right\}.$$

4.2. CASE 2: B = 0

In addition to the condition that $B = 0$, the analysis of this singular case admits the possibilities of A and C taking zero and non-zero values, i.e.,

(a) $A \neq 0$, $B = 0$, $C \neq 0$; (b) $A = 0$, $B = 0$, $C \neq 0$; (c) $A \neq 0$, $B = 0$, $C = 0$.

Again the classification of each of these sub-cases must be performed individually as follows.

4.2.1. *Case* 2(*a*): $A \neq 0$, $C \neq 0$

When $AC \geqslant 0$, no singularities occur in $\theta \in (0, \pi/2)$ and the resulting boundary behaviour is similar to that of Case 1. The speed density is found to be

$$m(\theta) = \frac{\sin 2\theta}{A\cos^2 2\theta + C}\exp\left\{\frac{\Lambda}{2\sqrt{AC}}\tan^{-1}\left(\frac{\sqrt{A}\cos 2\theta}{\sqrt{C}}\right)\right\}.$$

Employing this in equation (18) yields the Lyapunov exponent

$$\lambda^{\varepsilon} = \varepsilon^2\left\{D - \frac{\Lambda}{2}\coth\left[\frac{\Lambda}{2\sqrt{AC}}\tan^{-1}\frac{-A}{\sqrt{AC}}\right]\right\}.$$

If $AC < 0$ singularities exist in $\theta \in (0, \pi/2)$ at

$$\theta_{s_1} = \tfrac{1}{2}\cos^{-1}\sqrt{-C/A}, \quad \theta_{s_2} = \tfrac{1}{2}\cos^{-1}(-\sqrt{-C/A}).$$

As shown in Figure 4, the direction of traverse of the process across these points depends on the sign of Λ. Thus, separate probability density functions, $p_0(\theta)$, exist accordingly:

$$m(\theta) = \frac{\sin 2\theta}{A\cos^2 2\theta + C}\left|\frac{A\cos 2\theta + \sqrt{-AC}}{A\cos 2\theta - \sqrt{-AC}}\right|^{\Lambda/4\sqrt{-AC}},$$

$$N_1 = [M(\theta_{s_1}) - M(0)]^{-1}, \quad N_2 = [M(\pi/2) - M(\theta_{s_2})]^{-1},$$

$$p_0(\theta) = \begin{cases} N_1 m(\theta), & \Lambda > 0, \\ N_2 m(\theta), & \Lambda < 0. \end{cases}$$

However, the resulting expression for the Lyapunov exponent is quite simple,

$$\lambda^{\varepsilon} = \varepsilon^2\left\{D + \frac{|\Lambda|}{2}\right\}.$$

4.2.2. *Case* 2(*b*): $A = 0,\ C \neq 0$

In this case, no additional singularities occur within the interval $\theta \in (0, \pi/2)$. The behaviour of the process is similar to Case 1 and the speed density is given by

$$m(\theta) = \frac{\sin 2\theta}{C}\exp\left\{\frac{\Lambda\cos 2\theta}{2}\right\}.$$

The top Lyapunov exponent is thus

$$\lambda^{\varepsilon} = \varepsilon^2\left\{D - \frac{\Lambda}{2}\coth\left(\frac{-\Lambda}{2C}\right)\right\}.$$

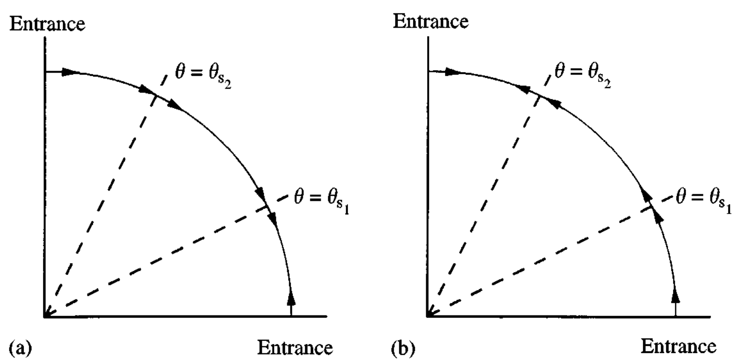

Figure 4. Boundary behaviour for Case 2(a) where (a) $\Lambda > 0$, (b) $\Lambda < 0$.

4.2.3 *Case* 2(*c*): $A \neq 0$, $C = 0$

In this case an additional singular point occurs at $\theta = \pi/4$. As in Case 2(a), the direction of traverse across this point depends on Λ. The corresponding boundary behaviour is illustrated in Figure 5. The associated invariant measures are given below:

$$m(\theta) = \frac{\sin 2\theta}{\alpha \cos^2 2\theta} \exp\left\{ -\frac{\Lambda}{2\alpha \cos 2\theta} \right\}, \quad N_1 = |\Lambda| \exp\left\{ -\frac{\Lambda}{2\alpha} \right\}, \quad N_2 = |\Lambda| \exp\left\{ \frac{\Lambda}{2\alpha} \right\},$$

$$p_0(\theta) = \begin{cases} N_1 m(\theta), & \Lambda > 0, \\ N_2 m(\theta), & \Lambda < 0. \end{cases}$$

Using these the Lyapunov exponent is found to be

$$\lambda^{\varepsilon} = \varepsilon^2 \left\{ D + \frac{|\Lambda|}{2} \right\}.$$

4.3. CASE 3: A $\neq 0$, B $\neq 0$, C $= 0$

In this case the speed density is given by

$$m(\theta) = \frac{\sin 2\theta}{\sigma^2(\theta)} (\alpha^+ + \alpha^- \sec 2\theta)^{-\Lambda/\alpha^-}.$$

Singularities occur at $\theta_f = \pi/4$ and $\theta_s = \frac{1}{2}\cos^{-1}(-\alpha^-/\alpha^+)$. The characteristics of each of these points, and the resulting distribution of probability over the interval $\theta \in (0, \pi/2)$, are influenced by α^- and Λ which are functions of the geometric and operational parameters of the system. The corresponding behaviour of the θ-process is illustrated in Figure 6. The associated stationary densities and maximal Lyapunov exponents are presented below.

$\Lambda > 0$, $\alpha^- > 0$:

$$p_0(\theta) = N_1 \begin{cases} m(\theta), & \theta \in (0, \frac{\pi}{4}), \\ \delta(\theta - \theta_s), & \theta \in (\frac{\pi}{4}, \frac{\pi}{2}), \end{cases}$$

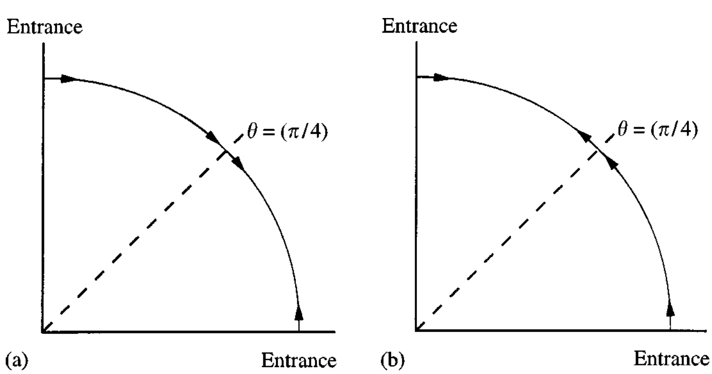

Figure 5. Boundary behaviour for Case 2(c) where (a) $\Lambda < 0$, (b) $\Lambda > 0$.

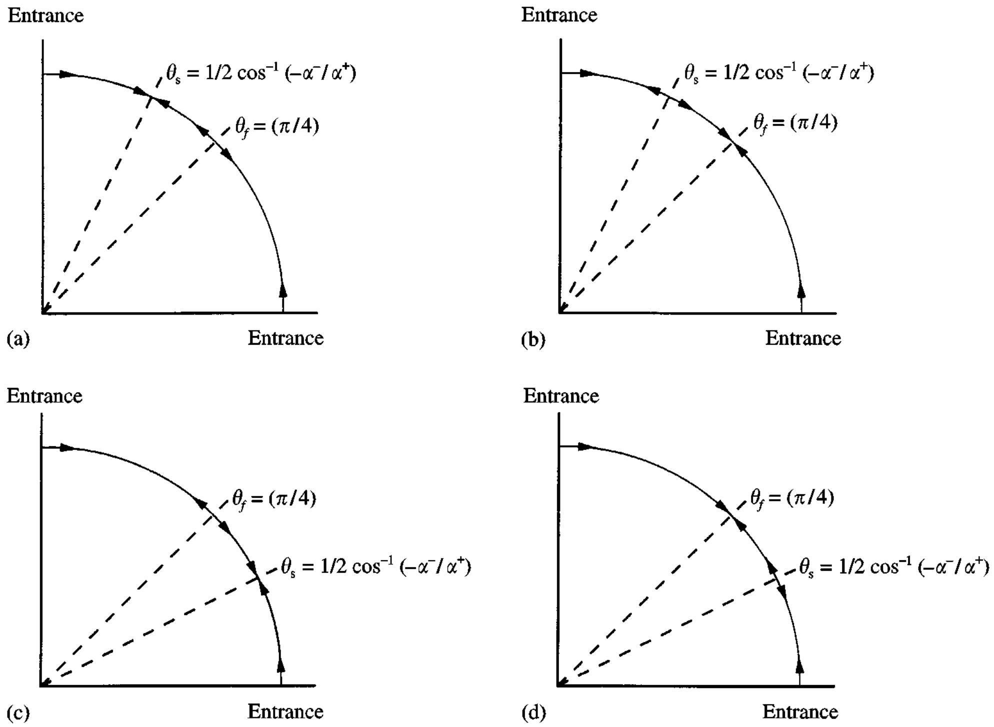

Figure 6. Boundary behaviour for Case 3 where (a) $\Lambda > 0$, $\alpha^- > 0$; (b) $\Lambda < 0$, $\alpha^- > 0$; (c) $\Lambda < 0$, $\alpha^- < 0$; (d) $\Lambda > 0$, $\alpha^- < 0$.

where

$$N_1 = \frac{\Lambda}{\Lambda + (2\alpha_{11})^{-\Lambda/\alpha^-}},$$

$$\lambda^\varepsilon = \varepsilon^2\left\{D + \frac{\Lambda}{2\alpha^+}\left[\frac{\alpha^+(2\alpha_{11})^{-\Lambda/\alpha^-} - \alpha^-\Lambda}{(2\alpha_{11})^{-\Lambda/\alpha^-} + \Lambda}\right]\right\}.$$

$\Lambda < 0$, $\alpha^- > 0$:

$$p_0(\theta) = N_2\begin{cases}\delta(\theta - \frac{\pi}{4}), & \theta \in (0, \theta_s),\\ m(\theta), & \theta \in (\theta_s, \frac{\pi}{2}),\end{cases}$$

where

$$N_2 = \frac{\Lambda}{\Lambda - (2\alpha_{22})^{-\Lambda/\alpha^-}},$$

$$\lambda^\varepsilon = \varepsilon^2\left\{D + \frac{\Lambda}{2}\left[\frac{(2\alpha_{22})^{-\Lambda/\alpha^-}}{\Lambda - (2\alpha_{22})^{-\Lambda/\alpha^-}}\right]\right\}.$$

$\Lambda < 0$, $\alpha^- < 0$:

$$p_0(\theta) = N_2\begin{cases}\delta(\theta - \theta_s), & \theta \in (0, \frac{\pi}{4}),\\ m(\theta), & \theta \in (\frac{\pi}{4}, \frac{\pi}{2}),\end{cases}$$

$$\lambda^\varepsilon = \varepsilon^2\left\{D + \frac{\Lambda}{2\alpha^+}\left[\frac{\alpha^-\Lambda - \alpha^+(2\alpha_{22})^{-\Lambda/\alpha^-}}{(2\alpha_{22})^{-\Lambda/\alpha^-} - \Lambda}\right]\right\}.$$

$\Lambda > 0$, $\alpha^- < 0$:

$$p_0(\theta) = N_1 \begin{cases} m(\theta), & \theta \in (0, \theta_s), \\ \delta(\theta - \frac{\pi}{4}), & \theta \in (\theta_s, \frac{\pi}{2}), \end{cases}$$

$$\lambda^\varepsilon = \varepsilon^2 \left\{ D + \frac{\Lambda}{2} \left[\frac{(2\alpha_{11})^{-\Lambda/\alpha^-}}{\Lambda + (2\alpha_{11})^{-\Lambda/\alpha^-}} \right] \right\}.$$

4.4. CASE 4: $A = 0$, $B \neq 0$, $C \neq 0$

In this case, as no additional singularities occur in the interval $\theta \in (0, \pi/2)$, the boundary behaviour is again similar to Case 1, where the speed density is given by

$$m(\theta) = \frac{\sin 2\theta}{\sigma^2(\theta)} (\alpha^- \cos 2\theta + \alpha^+)^{\Lambda/\alpha^-}.$$

Employing equation (18), the top Lyapunov exponent is given by

$$\lambda^\varepsilon = \varepsilon^2 \left\{ D - \frac{\Lambda}{2} \left[\frac{(\alpha_{22})^{\Lambda/\alpha^-} + (\alpha_{11})^{\Lambda/\alpha^-}}{(\alpha_{22})^{\Lambda/\alpha^-} - (\alpha_{11})^{\Lambda/\alpha^-}} \right] \right\}.$$

4.5. CASE 5: $A = 0$, $B \neq 0$, $C = 0$

The conditions that $A = 0$, $B \neq 0$ and $C = 0$ reduce to

$$\alpha_1 + \alpha_2 \neq 0, \ \alpha_1 - \alpha_2 = 0,$$

which cannot be satisfied simultaneously. Therefore, this singular case does not occur in practice and is excluded from subsequent analysis.

4.6. CASE 6: $A = 0$, $B = 0$, $C = 0$

This singular case occurs only under very restrictive circumstances; in addition to restrictions upon the noise spectrum, it requires particular stochastic coupling coefficients. Some suitable conditions are presented below:

(a) $\kappa_1 = \kappa_2, \quad k_{12} = -k_{21} \neq 0, \quad k_{11} = k_{22}, \quad S(0) = S(\omega^-) = S(\omega^+) = 0,$

(b) $\kappa_1 = \kappa_2, \quad k_{12} = k_{21} = 0, \quad k_{11} = k_{22}, \quad S(\omega^+) = 0,$

(c) $\kappa_1 = \kappa_2, \quad k_{12} = k_{21}, \quad k_{11} = -k_{22}, \quad S(2\omega_1) = S(2\omega_2) = S(\omega^-) = 0.$

When any of these combinations is satisfied, the diffusion term is made identically zero for all values of θ. Therefore, the θ-process is governed by

$$\mathrm{d}\theta = \mu(\theta)\,\mathrm{d}t.$$

Clearly, the θ-process is deterministic with fixed points $\theta = 0, \pi/2$. For $\Lambda > 0, \theta = 0$ is stable; for $\Lambda < 0, \theta = \pi/2$ is stable. Thus, there are δ-measures at each fixed point. For $\Lambda > 0$ every initial condition, except that at $\theta = \pi/2$, will reach $\theta = 0$, and the δ-measure at $\theta = 0$ will yield the maximal Lyapunov exponent. Conversely for $\Lambda < 0$, the δ-measure at $\theta = \pi/2$ should be utilized in equation (16). For this singular case the top Lyapunov exponent is given by

$$\lambda^{\varepsilon} = \varepsilon^2 \max(\Lambda_1, \Lambda_2).$$

5. APPLICATION TO ROTATING SHAFT SYSTEM

The results developed in the preceding section are next applied to a typical gyroscopic system; a rotating shaft system studied in a rotating co-ordinate frame. This is subjected to a random axial loading of small intensity as shown in Figure 7.

Stability results are developed for case of symmetric and non-symmetric shafts under white and real noise excitation. The axial loading appears as a multiplicative forcing term in the equations of motion which are presented below:

$$\ddot{q}_1 - 2\Omega\dot{q}_2 + (\omega_{n_1}^2 - \Omega^2)q_1 + \varepsilon f(\xi(t)) \sum_{j=1}^{2} k_{1j}q_j + \varepsilon^2 2\rho\omega_{n_1}\dot{q}_1 = 0,$$

$$\ddot{q}_2 + 2\Omega\dot{q}_1 + (\omega_{n_2}^2 - \Omega^2)q_2 + \varepsilon f(\xi(t)) \sum_{j=1}^{2} k_{2j}q_j + \varepsilon^2 2\rho\omega_{n_2}\dot{q}_2 = 0,$$

where the co-ordinates q_1, q_2 rotate with the shaft at rate Ω. The term ω_{n_i}, $i = 1, 2$, represent the transverse natural frequencies of a beam which arise due to potential forces. Clearly, under the condition that $\Omega = 0$ the equations of motion reduce to those of two stochastically coupled oscillators, studied by Sri Namachchivaya and Van Roessel [7]. As every mechanical system contains some symmetry-breaking imperfection, a perfectly symmetrical shaft is a mathematical idealization. Thus, it is natural to express the natural frequencies of the shaft in terms of an asymmetry imposed on a perfectly symmetrical shaft with natural frequency $\bar{\omega}$, i.e.,

$$\omega_{n_1} = \bar{\omega} - \nu, \quad \omega_{n_2} = \bar{\omega} + \nu.$$

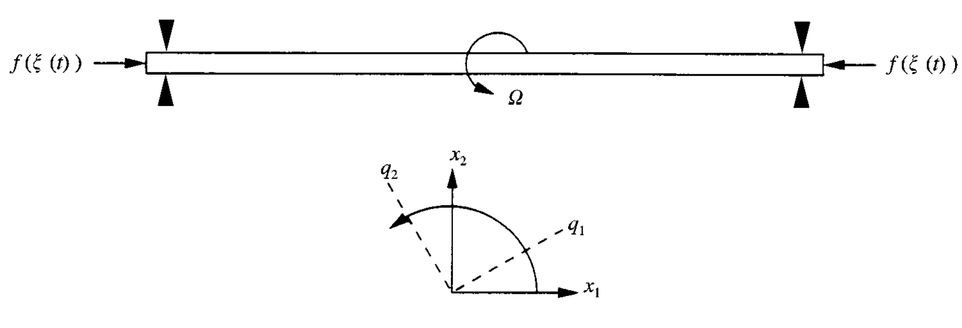

Figure 7. Axially loaded rotating shaft system.

5.1. WHITE NOISE

The first example studied here is that of a symmetric shaft system (i.e., $\nu = 0$) subjected to white noise excitation with constant cosine spectrum, S_0. This shaft system is considered symmetric in the sense that $\omega_{n_1} = \omega_{n_2} \equiv \bar{\omega}$ with a diagonal stochastic coupling matrix, i.e., $k_{11} = k_{22} \equiv k$ and $k_{12} = k_{21} = 0$. For such a system the response frequencies ω_1 and ω_2 are given simply by

$$\omega_1 = \bar{\omega} + \Omega, \quad \omega_2 = \bar{\omega} - \Omega,$$

where $\bar{\omega}$ is the natural frequency of lateral oscillation of the shaft in a non-rotating state. The quantities A, B and C, which characterize the Fokker–Planck equation describing the θ-process, then reduce to

$$A = \frac{S_0 k^2}{8\bar{\omega}^2}, \quad B = 0, \quad C = 0.$$

Clearly, the behaviour of the θ-process corresponds to that of singular case 2(c). The necessary and sufficient condition for almost-sure sample stability is provided by the corresponding Lyapunov exponent,

$$\lambda^{\varepsilon} = \varepsilon^2 \{\rho(\Omega - \bar{\omega}) + A\}.$$

In this case, white noise clearly has a destabilizing influence on the shaft system. The Lyapunov exponent provides the criterion for almost-sure sample stability,

$$\rho \geqslant \frac{S_0 k^2}{8\bar{\omega}^2 \omega_2}. \tag{20}$$

This reproduces the result obtained by Sri Namachchivaya and Talwar [18] using stochastic averaging.

Examined next is the sample stability of a non-symmetric shaft with natural frequencies $\omega_{n_1} = \bar{\omega} - \nu$, $\omega_{n_2} = \bar{\omega} + \nu$ and small internal dissipation given by ρ, where the stochastic coupling terms are as before. Here, the functions A, B and C, which are coefficients of the diffusion term, are given by

$$A = \frac{S_0 k^2}{32\kappa_1\kappa_2(\omega_1^2 - \omega_2^2)^2}\{48\Omega^2\kappa_1\kappa_2 + (\kappa_1 - \kappa_2)^2(\kappa_1 + \kappa_2) + 4\Omega^2(\kappa_1 + \kappa_2)^2\},$$

$$B = \frac{S_0 k^2}{8\kappa_1\kappa_2(\omega_1^2 - \omega_2^2)}(\kappa_1 + \kappa_2), \quad \kappa_i = \omega_{n_i}^2 - \Omega^2,$$

$$C = \frac{3S_0 k^2}{32\kappa_1\kappa_2(\omega_1^2 - \omega_2^2)^2}\{-16\Omega^2\kappa_1\kappa_2 + (\kappa_1 - \kappa_2)^2(\kappa_1 + \kappa_2) + 4\Omega^2(\kappa_1 + \kappa_2)^2\},$$

and each of the above are, in general, non-zero. Stability boundaries are established through a numerical example using the following geometric and operational parameters:

$$\bar{\omega} = \frac{1}{2} + \frac{1}{\sqrt{2}}, \nu = \frac{1}{\sqrt{2}} - \frac{1}{2}, \quad S_0 = 0{\cdot}02, \Gamma_0 = 0, k = 1, \varepsilon = 0{\cdot}1.$$

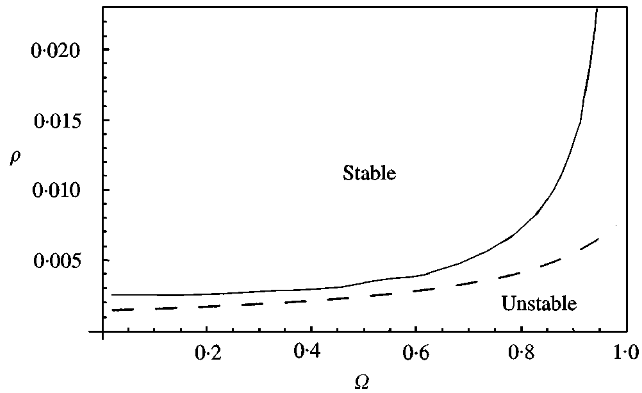

Figure 8. Stability boundaries for symmetric (- - -) and non-symmetric (—) rotating shafts excited by multiplicative white noise.

A stability boundary is obtained by solving numerically for those values of the damping parameter, ρ, at which the top Lyapunov exponent becomes identically zero. This is presented in Figure 8 as a function of the rotational speed, Ω where parameter space to the left of the curves represents stable behaviour.

Using the condition of equation (20), a stability boundary is obtained for the corresponding symmetric shaft with natural frequency $\bar{\omega}$. It is clear that the stable region for the non-symmetric shaft is smaller than that for the symmetric shaft.

5.2. REAL NOISE

Next, conditions for sample stability are investigated where the excitation to the system is a real noise process generated by passing a white noise, $W(t)$ through a second order filter, i.e.,

$$\ddot{\xi} + 2\alpha\dot{\xi} + \beta^2 \xi = W(t).$$

The height of the power spectral density of $W(t)$ is denoted by S_0. The parameter β represents the carrying frequency of the ξ-process, the spectral profile of which is given by

$$S_{\xi}(\omega) = \frac{S_0}{(\beta^2 - \omega^2)^2 + 4\alpha^2\omega^2}.$$

It would be expected that changes to the carrying frequency of the noise process should affect system stability; their effect would be particularly pronounced at values $\beta = 2\omega_i$, $i = 1, 2$, and $\beta = |\omega_1 \pm \omega_2|$. The system is initially considered to be symmetric where $\omega_{n_1} = \omega_{n_2} \equiv \bar{\omega}$ and $k_{11} = k_{22} \equiv k$ and $k_{12} = k_{21} = 0$. In this case the functions A, B and C reduce respectively to

$$A = \frac{S(\omega^+)k^2}{8(\bar{\omega}^2 + \Omega^2)^2}, \quad B = 0, \quad C = 0, \quad \omega^{\pm} = \omega_1 \pm \omega_2.$$

Thus, the behaviour of the θ-process, whose probability density function is defined by equation (15), corresponds to that treated in singular case 2(c). The associated top Lyapunov exponent is then given by

$$\lambda^{\varepsilon} = \varepsilon^2 \left\{ D + \frac{|A|}{2} \right\} = \varepsilon^2 \left\{ \frac{k^2 S(\omega^+)}{8\bar{\omega}^2} - \rho(\bar{\omega} - \Omega) \right\}. \tag{21}$$

From this expression a number of comments can be made. As positive values of λ^{ε} indicate sample instability w.p. 1, it is clear that, for all values of α and β, the noise process has a destabilizing influence on the symmetric rotating shaft system. It is interesting to note that, in this example, centering the carrying frequency around $2\omega_1$, $2\omega_2$ or $|\omega_1 - \omega_2|$ does not affect sample stability.

Similar to the preceding example, consider the non-symmetric gyroscopic system with the following geometric and operational parameters:

$$\bar{\omega} = \frac{1}{2} + \frac{1}{\sqrt{2}},\ v = \frac{1}{\sqrt{2}} - \frac{1}{2},\ \Omega = 0{\cdot}4,\ \Gamma_0 = 0,\ \rho = 0{\cdot}005,\ \varepsilon = 0{\cdot}1,\ k = 1.$$

For values of $\Omega < \omega_{n_1}$, Figure 9 illustrates the height of the spectrum of the white noise process driving the filter equation required to overcome the stabilizing effects of internal damping. In this case, the response frequencies ω_1, ω_2 are $\omega_1 = 1{\cdot}6613$, $\omega_2 = 0{\cdot}7483$. The influence of centering the noise process around $2\omega_1$, $2\omega_2$ or $\omega_1 + \omega_2$ greatly reduces the spectral height of the noise process required to cause instability. For comparison, the stability boundary for the symmetric shaft is also illustrated. In the symmetric case the response frequencies ω_1, ω_2 are $\omega_1 = 1{\cdot}6071$, $\omega_2 = 0{\cdot}8071$.

The stability boundaries in Figure 9 are for shaft systems featuring diagonal stochastic coupling matrices. Results obtained by altering these coupling matrices are presented next. Figures 10 and 11 show stability boundaries for symmetric and non-symmetric shafts obtained where, other than altering k_{ij}, the geometric and

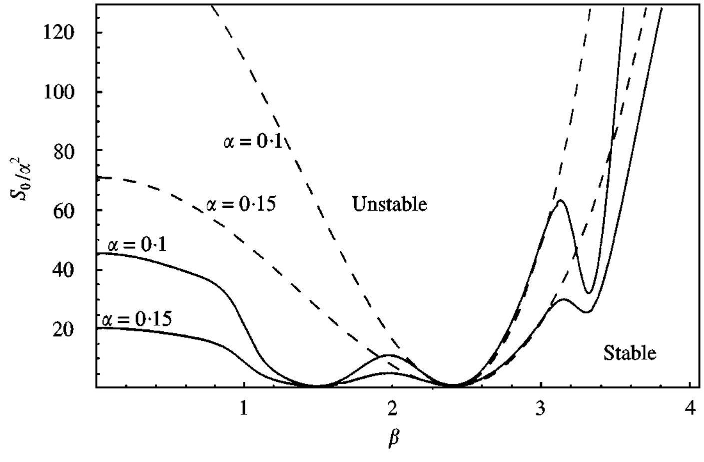

Figure 9. Stability boundaries for rotating shafts of symmetric (– – –) and non-symmetric (—) cross-section. $k_{11} = 1$, $k_{12} = 0$, $k_{21} = 0$, $k_{22} = 1$, $\Omega = 0{\cdot}4$.

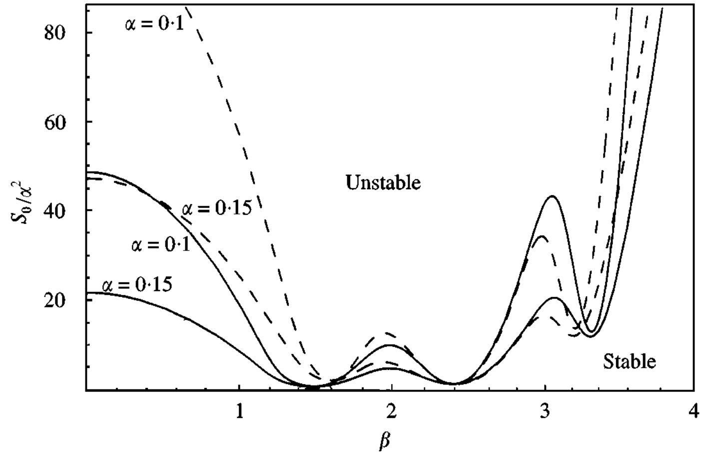

Figure 10. Stability boundaries for rotating shafts of symmetric (– – –) and non-symmetric (—) cross-section. $k_{11} = 1$, $k_{12} = 0{\cdot}5$, $k_{21} = 0{\cdot}5$, $k_{22} = 1$, $\Omega = 0{\cdot}4$.

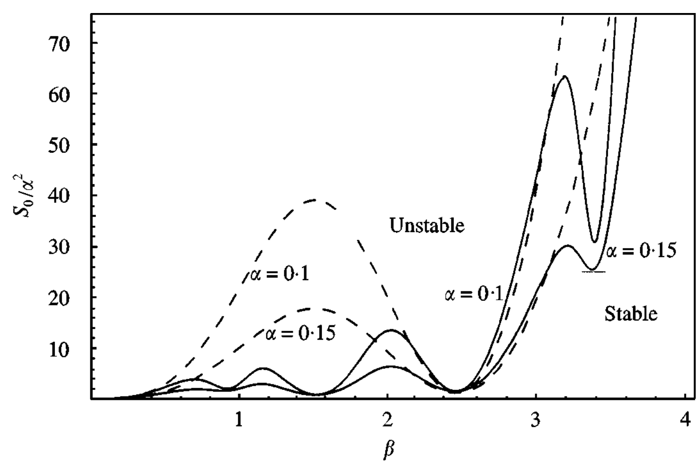

Figure 11. Stability boundaries for rotating shafts of symmetric (– – –) and non-symmetric (—) cross-section. $k_{11} = 1$, $k_{12} = 0{\cdot}5$, $k_{21} = -0{\cdot}5$, $k_{22} = 1$, $\Omega = 0{\cdot}4$.

operational parameters used are identical to those of the case studied above. Figure 10 contains the stability boundaries for shaft systems with stochastic coupling terms given by $k_{11} = 1$, $k_{12} = 0{\cdot}5$, $k_{21} = 0{\cdot}5$, $k_{22} = 1$. Unlike the previous case, the stability boundary of the symmetric shaft system is qualitatively similar to that of the non-symmetric shaft system.

For completeness, the case of a stochastic coupling matrix which is skew-symmetric will be admitted. Using the parameters $k_{11} = 1$, $k_{12} = 0{\cdot}5$, $k_{21} = -0{\cdot}5$, $k_{22} = 1$ yields the stability boundaries shown in Figure 11. In this case, the sample stability of the non-symmetric shaft system is also influenced by centering the excitation process around the difference $|\omega_1 - \omega_2|$.

6. CONCLUSIONS

In this work, the procedure established by Sri Namachchivaya *et al.* [1] for the computation of moment and maximal Lyapunov exponents for coupled oscillators, has been employed to create an asymptotic expansion for the maximal Lyapunov exponent of a four-dimensional gyroscopic system subjected to a small-intensity real noise.

The results derived were then applied to a rotating shaft system subjected to real and white noise excitation where the shaft itself was treated individually as being alternately symmetric and non-symmetric. Necessary and sufficient conditions for sample stability and corresponding stability boundaries were established for various operating conditions.

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APPENDIX A: LOGARITHMIC-POLAR EQUATIONS OF MOTION OF A STOCHASTICALLY PERTURBED GYROSCOPIC SYSTEM

From section 2, the equations of motion of a four-dimensional gyroscopic system, transformed into logarithmic-polar co-ordinates, are written,

$$\dot{\rho} = \sum_{j=0}^{2} \varepsilon^j q^j(\phi_1, \phi_2, \theta, \xi), \quad \dot{\theta} = \sum_{j=0}^{2} \varepsilon^j s^j(\phi_1, \phi_2, \theta, \xi),$$

$$\dot{\phi}_i = \omega_i + \sum_{j=0}^{2} \varepsilon^j h_i^j(\phi_1, \phi_2, \theta, \xi), \quad i = 1, 2,$$

$$\mathrm{d}\xi = \mu_\xi(\xi)\mathrm{d}t + \sigma_\xi(\xi) \circ \mathrm{d}W_t, \quad \xi \in M.$$

The coefficients of each of these equations are presented explicitly here, where the following rotation has been used:

$$a_{ij}^{\pm} = \Delta_1 \tilde{\alpha}_i \tilde{\alpha}_j k_{12} \pm \Delta_2 k_{21}, \; b_{ij}^{k\pm} = \omega_k(\Delta_1 \zeta_1 \tilde{\alpha}_i \pm \Delta_2 \zeta_2 \tilde{\alpha}_j),$$

$$c_{ij}^{\pm} = k_{11}\Delta_1 \tilde{\alpha}_i \pm k_{22}\Delta_2 \tilde{\alpha}_j, \; \Delta = \Delta_1 \Delta_2$$

$$\Delta_1 = \tilde{\alpha}_1 \omega_1 - \tilde{\alpha}_2 \omega_2, \quad \Delta_2 = \tilde{\alpha}_2 \omega_1 - \tilde{\alpha}_1 \omega_2, \quad \phi^{\pm} = \phi_1 \pm \phi_2.$$

$$q^0(\phi_1, \phi_2, \theta, \xi) = 0, \quad s^0(\phi_1, \phi_2, \theta, \xi) = 0, \quad h_i^0(\phi_1, \phi_2, \theta, \xi) = 0,$$

$$q^1(\phi_1, \phi_2, \theta, \xi) = \tfrac{1}{4\Delta} f(\xi)(q_0^1(\phi_1, \phi_2) + q_c^1(\phi_1, \phi_2)\cos 2\theta + q_s^1(\phi_1, \phi_2)\sin 2\theta),$$

$$q^2(\phi_1, \phi_2, \theta, \xi) = \tfrac{1}{4\Delta}(q_0^2(\phi_1, \phi_2) + q_c^2(\phi_1, \phi_2)\cos 2\theta + q_s^2(\phi_1, \phi_2)\sin 2\theta),$$

$$q_0^1(\phi_1, \phi_2) = c_{21}^- \sin 2\phi_1 - c_{12}^- \sin 2\phi_2 + a_{21}^+ \cos 2\phi_1 - a_{12}^+ \cos 2\phi_2,$$

$$q_c^1(\phi_1, \phi_2) = -2a_{12}^- + c_{21}^- \sin 2\phi_1 + c_{12}^- \sin 2\phi_2 + a_{21}^+ \cos 2\phi_1 + a_{12}^+ \cos 2\phi_2,$$

$$q_s^1(\phi_1, \phi_2) = [c_{22}^- - c_{11}^-]\sin \phi^+ + [c_{12}^+ + c_{21}^+]\sin \phi^- - [c_{11}^+ - c_{22}^+]\cos \phi^+ + [c_{11}^+ - c_{22}^+]\cos \phi^-,$$

$$q_0^2(\phi_1, \phi_2) = -\, b_{21}^{1+} + b_{12}^{2+} + b_{21}^{1-} \cos 2\phi_1 - b_{12}^{2-} \cos 2\phi_2,$$

$$q_c^2(\phi_1, \phi_2) = -\, b_{21}^{1+} - b_{12}^{2+} + b_{21}^{1-} \cos 2\phi_1 + b_{12}^{2-} \cos 2\phi_2,$$

$$q_s^2(\phi_1, \phi_2) = [b_{11}^{1-} - b_{22}^{2-}] \cos \phi^+ + [b_{11}^{1+} - b_{22}^{2+}] \cos \phi^-,$$

$$s^1(\phi_1, \phi_2, \theta, \xi) = \tfrac{1}{4\Delta} f(\xi)(s_0^1(\phi_1, \phi_2) + s_c^1(\phi_1, \phi_2) \cos 2\theta + s_s^1(\phi_1, \phi_2) \sin 2\theta),$$

$$s^2(\phi_1, \phi_2, \theta, \xi) = \tfrac{1}{4\Delta}(s_0^2(\phi_1, \phi_2) + s_c^2(\phi_1, \phi_2) \cos 2\theta + s_s^2(\phi_1, \phi_2) \sin 2\theta),$$

$$s_0^1(\phi_1, \phi_2) = [-c_{11}^- - c_{22}^-] \sin \phi^+ + [c_{11}^+ - c_{22}^+] \sin \phi^- - [a_{11}^+ + a_{22}^+] \cos \phi^+ - [a_{11}^- + a_{22}^-] \cos \phi^-,$$

$$s_c^1(\phi_1, \phi_2) = [-c_{11}^- + c_{22}^-] \sin \phi^+ + [c_{11}^+ + c_{22}^+] \sin \phi^- - [c_{11}^+ - c_{22}^+] \cos \phi^+ + [c_{11}^+ - c_{22}^+] \cos \phi^-,$$

$$s_s^1(\phi_1, \phi_2) = 2a_{12}^- - c_{21}^- \sin 2\phi_1 - c_{12}^- \sin 2\phi_2 - a_{21}^+ \cos 2\phi_1 - a_{12}^+ \cos 2\phi_2,$$

$$s_0^2(\phi_1, \phi_2) = [-\, b_{11}^{1-} - b_{22}^{2-}] \cos \phi^+ + [b_{11}^{1+} + b_{22}^{2+}] \cos \phi^-,$$

$$s_c^2(\phi_1, \phi_2) = [-\, b_{11}^{1-} + b_{22}^{2-}] \cos \phi^+ + [b_{11}^{1+} - b_{22}^{2+}] \cos \phi^-,$$

$$s_s^2(\phi_1, \phi_2) = b_{21}^{1+} + b_{12}^{2+} - b_{21}^{1-} \cos 2\phi_1 - b_{12}^{2-} \cos 2\phi_2,$$

$$h_1^1(\phi_1, \phi_2, \theta, \xi) = \tfrac{1}{2\Delta} f(\xi)(h_{1(0)}^1(\phi_1) + h_{1(\theta)}^1(\phi_1, \phi_2) \tan \theta),$$

$$h_1^2(\phi_1, \phi_2, \theta, \xi) = \tfrac{1}{2\Delta}(h_{1(0)}^2(\phi_1) + h_{1(\theta)}^2(\phi_1, \phi_2) \tan \theta),$$

$$h_{1(0)}^1(\phi_1) = c_{21}^+ - a_{12}^+ \sin 2\phi_1 + c_{21}^- \cos 2\phi_1,$$

$$h_{1(\theta)}^1(\phi_1, \phi_2) = -\, a_{22}^+ \sin \phi^+ + a_{22}^- \sin \phi^- + c_{22}^- \cos \phi^+ + c_{22}^+ \cos \phi^-,$$

$$h_{1(0)}^2(\phi_1, \phi_2) = -\, b_{21}^{1-} \sin 2\phi_1, \;\; h_{1(\theta)}^2(\phi_1, \phi_2) = -\, b_{22}^{2-} \sin \phi^+ + b_{22}^{2+} \sin \phi^-,$$

$$h_2^1(\phi_1, \phi_2, \theta, \xi) = \tfrac{1}{2\Delta} f(\xi)(h_{2(0)}^1(\phi_1) + h_{2(\theta)}^1(\phi_1, \phi_2)\cot\theta),$$

$$h_2^2(\phi_1, \phi_2, \theta, \xi) = \tfrac{1}{2\Delta}(h_{2(0)}^2(\phi_1) + h_{2(\theta)}^2(\phi_1, \phi_2)\cot\theta),$$

$$h_{2(0)}^1(\phi_2) = -c_{12}^+ + a_{12}^+ \sin 2\phi_2 + c_{12}^- \cos 2\phi_2,$$

$$h_{2(\theta)}^1(\phi_1, \phi_2) = a_{11}^+ \sin\phi^+ + a_{11}^- \sin\phi^- - c_{11}^- \cos\phi^+ - c_{11}^+ \cos\phi^-,$$

$$h_{2(0)}^2(\phi_1, \phi_2) = b_{12}^{2-} \sin 2\phi_2, h_{2(\theta)}^2(\phi_1, \phi_2) = b_{11}^{1-} \sin\phi^+ + b_{11}^{1+} \sin\phi^-.$$