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LETTERS TO THE EDITOR

LATERAL VIBRATION OF A UNIFORM EULER–BERNOULLI BEAM CARRYING A PARTICLE AT AN INTERMEDIATE POINT

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1. INTRODUCTION

The title problem occurs in several engineering applications and has received attention from several researchers. The theoretical modelling based on Euler–Bernoulli theory of bending is relatively easy but only a limited range of results are found in publications. Some classical boundary conditions were considered in references [1–8] and elastically restrained supports in [9–13]. In references [1–4], approximate fundamental frequencies were presented for clamped–clamped, pinned–pinned and clamped–free cases. In reference [5] a comparison was made of the fundamental frequencies of a pinned–pinned case obtained by Euler–Bernoulli theory and Timoshenko theory. Reference [6] listed 10 frequency equations for combinations of the classical boundary conditions. The frequency equations were first expressed as 8×8 determinants and then generated into transcendental form using *MAPLE* but only the fundamental frequency of the clamped–clamped case was listed. References [7, 8] suggested approximate solutions for large amplitude vibrations. In references [9–13] elastic support resilience was included in the analysis. Reference [9] does not contain any numerical results. In reference [10] results for the pinned–pinned case was included as a special case. In reference [11] fundamental frequencies obtained by approximate methods were compared with experimental results. In reference [12] a theory was developed for elastic supports but only the fundamental frequencies of pinned–pinned and clamped–clamped cases were listed. The only results listed in reference [13] were the fundamental frequencies of beams restrained against angular deflections at the ends.

In the present paper, the frequency equations for all the combinations of the classical boundary conditions are presented as 4×4 determinants equated to zero. These determinants in turn may be expressed as 2×2 determinants. The first three natural frequencies for all the combinations of boundary conditions are tabulated for various magnitudes and positions of the particle mass. Some mode shapes

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(which consist of two portions) are presented and discussed. A similar range of results are not found elsewhere.

2. THEORY

Figure 1a shows a uniform beam O_1O_2 of flexural rigidity. EI, mass per unit length m and length $(R_1 + R_2)L$ carrying a particle of mass M at a distance R_1L from the left end. In the study of lateral vibration of this system. Low [6] used a single co-ordinate system with origin at O_1. In the present note, O_1 an O_2 are the origins of the co-ordinate systems for portions of the beam to the left and to the right of the particle. The use of the two separate co-ordinate systems has some algebraic advantages. In the text subscript $k = 1$ refers to the left portion and subscript $k = 2$ refers to the right portion. For the beam in free vibration at frequency ω, if the amplitude of the deflection is $y_k(x_k)$ at abscissa x_k $(0 \leqslant x_k \leqslant R_kL)$, then the amplitude of the bending moment $M_k(x_k)$, shearing force $Q_k(x_k)$ and the mode shape differential equation for the two portions are

$$M_k(x_k) = EI\,\mathrm{d}^2 y_k(x_k)/\mathrm{d}x_k^2, \qquad Q_k(x_k) = -EI\,\mathrm{d}^3 y_k(x_k)/\mathrm{d}x_k^3,$$

$$EI\,\mathrm{d}^4 y_k(x_k)/\mathrm{d}x_k^4 - m\omega^2 y_k(x_k) = 0. \tag{1}$$

Equations (1) may be expressed in dimensionless form with the choice

$$x_k = X_k L, \qquad y_k(x_k) = Y_k(X_k)L, \quad \delta = M/mL, \qquad \Omega^2 = m\omega^2 L^4/EI. \tag{2}$$

Here δ is the particle mass parameter and Ω is the dimensionless natural frequency. The dimensionless mode shape equations for the left- and right-hand portions are

$$\mathrm{d}^4 Y_k(X_k)/\mathrm{d}X_k^4 - \Omega^2 Y_k(X_k) = 0. \tag{3}$$

The corresponding solutions (the mode shape functions for the left and right portions) are

$$Y_k(X_k) = B_{k1}\sin(\Omega^{1/2}X_k) + B_{k2}\cos(\Omega^{1/2}X_k) + B_{k3}\sinh(\Omega^{1/2}X_k) + B_{k4}\cosh(\Omega^{1/2}X_k). \tag{4}$$

Here B_{k1} through to B_{k4} are the eight constants of integration.

An advantage of the two separate co-ordinate systems is that two of the constants of integration in each of equations (4) may be eliminated from the boundary conditions at O_1 and O_2 and the two mode shape functions $Y_1(X_1)$ and $Y_2(X_2)$ expressed as

$$Y_k(X_k) = C_{k1}U_k(X_k) + C_{k2}V_k(X_k) \tag{5}$$

and in this problem $U_k(X_k)$, $V_k(X_k)$ are transcendental functions. These functions for the four classical boundary conditions are tabulated in Table 1.

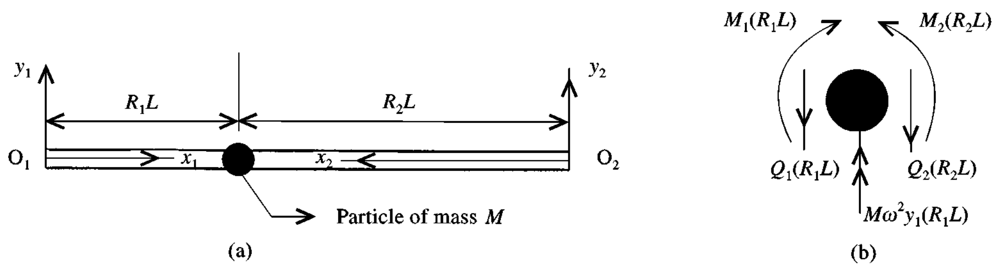

Figure 1. The two separate co-ordinate systems and free body diagram of the particle.

TABLE 1

The functions $U_k(X_k)$ and $V_k(X_k)$ for the classical supports at origin O_k

O_k	$U_k(X_k)$	$V_k(X_k)$
Clamped	$\sin(\Omega^{1/2}X_k) - \sinh(\Omega^{1/2}X_k)$	$\cos(\Omega^{1/2}X_k) - \cos(\Omega^{1/2}X_k)$
Pinned	$\sin(\Omega^{1/2}X_k)$	$\sinh(\Omega^{1/2}X_k)$
Sliding	$\cos(\Omega^{1/2}X_k)$	$\cosh(\Omega^{1/2}X_k)$
Free	$\sin(\Omega^{1/2}X_k) + \sinh(\Omega^{1/2}X_k)$	$\cos(\Omega^{1/2}X_k) + \cosh(\Omega^{1/2}X_k)$

The conditions of continuity of deflection and slope, compatibility of bending moment and shearing force at $x_1 = R_1L$ and $x_2 = R_2L$ as in Figure 1b (which shows the d'Alembert's free body diagram of the particle) are

$$y_1(R_1L) = y_2(R_2L), \qquad \mathrm{d}y_1(R_1L)/\mathrm{d}x_1 = -\,\mathrm{d}y_2(R_2L)/\mathrm{d}x_2,$$

$$M_1(R_1L) = M_2(R_2L), \qquad Q_1(R_1L) + Q_2(R_2L) = M\omega^2 y_1(R_1L). \tag{6}$$

Equations (6) in dimensionless form are

$$Y_1(R_1) - Y_2(R_2) = 0, \qquad \mathrm{d}Y_1(R_1)/\mathrm{d}X_1 + \mathrm{d}Y_2(R_2)/\mathrm{d}X_2 = 0,$$

$$\mathrm{d}^2Y_1(R_1)/\mathrm{d}X_1^2 - \mathrm{d}^2Y_2(R_2)/\mathrm{d}X_2^2 = 0,$$

$$d^3Y_1(R_1)/\mathrm{d}X_1^3 + \mathrm{d}^3Y_2(R_2)/\mathrm{d}X_2^3 + \delta\Omega^2 Y_1(R_1) = 0. \tag{7}$$

Equations (7) may now be expressed as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{12} \\ C_{21} \\ C_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{8}$$

where

$$[A_{11}, A_{12}, A_{13}, A_{14}] = [U_1(R_1), V_1(R_1), -U_2(R_2), -V_2(R_2)],$$

$$[A_{21}, A_{22}, A_{23}, A_{24}] = [\mathrm{d}U_1(R_1)/\mathrm{d}X_1, \mathrm{d}V_1(R_1)/\mathrm{d}X_1, \mathrm{d}U_2(R_2)/\mathrm{d}X_2, \mathrm{d}V_2(R_2)/\mathrm{d}X_2],$$

$$[A_{31}, A_{32}, A_{33}, A_{34}] = [\mathrm{d}^2U_1(R_1)/\mathrm{d}X_1^2, \mathrm{d}^2V_1(R_1)/\mathrm{d}X_1^2, -\mathrm{d}^2U_2(R_2)/\mathrm{d}X_2^2, -\mathrm{d}^2V_2(R_2)/\mathrm{d}X_2^2],$$

and

$$[A_{41}, A_{42}, A_{43}, A_{44}] = [\mathrm{d}^3U_1(R_1)/\mathrm{d}X_1^3 + \delta\Omega^2U_1(R_1), \mathrm{d}^3V_1(R_1)/\mathrm{d}X_1^3 + \delta\Omega^2V_1(R_1), \mathrm{d}^3U_2(R_2)/\mathrm{d}X_2^3, \mathrm{d}^3V_2(R_2)/\mathrm{d}X_2^3],$$

The frequency equation is the determinant of the 4×4 matrix equated to zero. There will be a frequency equation for each of the 16 combinations of clamped, pinned, sliding or free boundary conditions at O_1 and O_2. The choice R_1 and $R_2 = 1$, i.e., a beam of length L, will not result in loss of generality and in this case there will be 10 frequency equations. A 4×4 determinant may be expanded manually by inductive development [14]. From the conditions of continuity of deflection and slope at the location of the particle, two more constants of integration may be eliminated and the frequency equation expressed as a 2×2 determinant. The determinant when expanded and simplified will yield the frequency equations listed by Low [6] in which the frequency equations were expressed as 8×8 determinants which needed MAPLE to expand.

2.1. NATURAL FREQUENCY CALCULATIONS

The roots of the frequency equation were determined by a "search" followed by an iterative procedure based on linear interpolation. Corresponding to a selected boundary condition, δ, R_1 and a trial Ω, each element of the determinant was calculated. A "coarse" search was made (starting with $\Omega = 0{\cdot}1$ with a step increase of 0·1) to locate a range at which a sign change occurred in the value of the determinant. A search was now made in this range (with step change of 0·01) to narrow the range of the root. One may go another stage to narrow the range of the root even further. An iterative procedure based on linear interpolation was now invoked to locate the root to a predetermined accuracy. The search was now continued for the next root and so on.

In Table 2, the first three non-zero values of $\Omega^{1/2}$ of a beam of length L are tabulated for 16 combinations of the classical boundary conditions (BC) and for values of $R_1 = 0{\cdot}125$, 0·375, 0·500 and $\delta = 0{\cdot}5$, 1·0, 10·0. The boundary conditions are indicated (i, j) where i or $j = 1$, 2, 3 or 4 denote clamped, pinned, sliding or free support. For example, the boundary condition (2, 4) means pinned at O_1 and free at O_2. If one denotes the dimensionless natural frequency for a certain boundary condition by $\Omega(i, j, \delta, R_1)$, then for the beam of length L (i.e., $R_1 + R_2 = 1$),

$$\Omega(i, j, \delta, R_1) = \Omega(j, i, \delta, 1 - R_1). \tag{9}$$

TABLE 2

The first three non-zero values of $\Omega^{1/2}(i, j, \delta, R_1)$ of a uniform beam with a particle at an intermediate point, i or j = 1, 2, 3 or 4 denote clamped, pinned, sliding or free boundary condition. δ is the particle mass parameter, R_1 is the location of the particle

BC		$R_1 = 0{\cdot}125$			$R_1 = 0{\cdot}250$			$R_1 = 0{\cdot}375$			$R_1 = 0{\cdot}500$		
(i, j)	δ	$\Omega_1^{1/2}$	$\Omega_2^{1/2}$	$\Omega_3^{1/2}$	$\Omega_1^{1/2}$	$\Omega_2^{1/2}$	$\Omega_3^{1/2}$	$\Omega_1^{1/2}$	$\Omega_2^{1/2}$	$\Omega_3^{1/2}$	$\Omega_1^{1/2}$	$\Omega_2^{1/2}$	$\Omega_3^{1/2}$
(1, 1)	0·5	4·6807	7·3885	9·7802	4·3252	6·7751	10·2269	3·9677	7·2682	10·9559	3·8471	7·8532	9·9999
	1·0	4·6271	6·9511	9·3129	4·0152	6·4488	10·1080	3·5710	7·1312	10·9451	3·4378	7·8532	9·7855
	5·0	4·1464	5·7147	8·8427	2·9926	6·0621	9·9884	2·5602	6·9711	10·9312	2·4450	7·8532	9·5378
(1, 2)	0·5	3·9062	6·7666	9·1515	3·7250	6·0651	9·3045	3·4496	6·2821	10·1965	3·2757	6·9236	9·3856
	1·0	3·8847	6·4475	8·6218	3·5457	5·6677	9·1497	3·1563	6·0712	10·1929	2·9492	6·8763	9·2027
	5·0	3·6849	5·1614	8·0001	2·7777	5·1043	8·9926	2·3149	5·8111	10·1883	2·1185	6·8102	8·9921
(1, 3)	0·5	2·3632	5·3998	7·9788	2·3415	4·8805	7·5569	2·2777	4·6303	8·2718	2·1811	4·8166	8·5021
	1·0	2·3613	5·2904	7·4538	2·3180	4·4906	7·2968	2·2006	4·2746	8·1880	2·0487	4·5727	8·4629
	5·0	2·3458	4·4771	6·3988	2·1422	3·5247	7·0142	1·8228	3·6595	8·0904	1·5811	4·2018	8·4120
(1, 4)	0·5	1·8745	4·6474	7·3886	1·8662	4·3206	6·7636	1·8369	4·0313	7·2484	1·7784	4·0327	7·8540
	1·0	1·8738	4·5968	6·9492	1·8573	4·0360	6·4283	1·8009	3·7050	7·1034	1·7004	3·3717	7·8537
	5·0	1·8687	4·1439	5·6943	1·7868	3·1356	6·0256	1·5844	3·0243	6·9325	1·3709	3·3304	7·8533
(2, 1)	0·5	3·7099	6·3133	9·1899	3·3929	6·3501	9·9477	3·2502	6·8942	9·7482	3·2757	6·9236	9·3856
	1·0	3·5270	5·9600	8·9696	3·0853	6·1662	9·8970	2·9194	6·8435	9·6257	2·9492	6·8763	9·2027
	5·0	2·7708	5·4018	8·7411	2·2433	5·9435	9·8408	2·0905	6·7765	9·4755	2·1185	6·8102	8·9921
(2, 2)	0·5	3·0317	5·6623	8·4239	2·8269	5·5194	9·0278	2·6858	5·9154	9·2581	2·6393	6·2832	8·4744
	1·0	2·9328	5·3106	8·1659	2·6174	5·2834	8·9509	2·4381	5·8061	9·2076	2·3832	6·2832	8·2394
	5·0	2·4341	4·6175	7·8836	1·9596	4·9666	8·8658	1·7719	5·6602	9·1405	1·7198	6·2832	7·9491
(2, 3)	0·5	1·5561	4·3853	6·9821	1·5175	4·0716	7·2113	1·4678	4·1000	7·8183	1·4188	4·3726	7·4059
	1·0	1·5420	4·1364	6·6476	1·4715	3·7739	7·0672	1·3908	3·8669	7·8072	1·3197	4·2372	7·2808
	5·0	1·4450	3·3384	6·1986	1·2445	3·1938	6·9013	1·0985	3·4832	7·7918	1·0011	4·0170	7·1217
(2, 4)	0·5	3·7355	6·3060	9·1907	3·4787	6·3333	9·9485	3·4183	6·8783	9·7448	3·5442	6·9483	9·3602
	1·0	3·5753	5·9465	8·9706	3·2330	6·1421	9·8980	3·1990	6·8219	9·6215	3·3896	6·9073	9·1680
	5·0	2·9366	5·3716	8·7426	2·6622	5·9080	9·8420	2·7893	6·7466	9·4706	3·1268	6·8481	8·9446

TABLE 2 (*continued*)

BC		$R_1 = 0{\cdot}125$			$R_1 = 0{\cdot}250$			$R_1 = 0{\cdot}375$			$R_1 = 0{\cdot}500$		
(i, j)	δ	$\Omega_1^{1/2}$	$\Omega_2^{1/2}$	$\Omega_3^{1/2}$	$\Omega_1^{1/2}$	$\Omega_2^{1/2}$	$\Omega_3^{1/2}$	$\Omega_1^{1/2}$	$\Omega_2^{1/2}$	$\Omega_3^{1/2}$	$\Omega_1^{1/2}$	$\Omega_2^{1/2}$	$\Omega_3^{1/2}$
(3, 1)	0·5	1·9439	5·1809	8·4764	1·9989	5·4734	8·2659	2·0812	5·2946	7·7758	2·1811	4·8166	8·5021
	1·0	1·7431	5·1041	8·4434	1·8090	5·4647	8·1325	1·9122	5·2100	7·5656	2·0487	4·5727	8·4629
	5·0	1·2457	5·0098	8·4058	1·3101	5·4514	7·9415	1·4177	5·0717	7·3142	1·5811	4·2018	8·4120
(3, 2)	0·5	1·3265	4·3665	7·6447	1·3460	4·6213	7·6883	1·3772	4·6837	6·9904	1·4188	4·3726	7·4059
	1·0	1·2000	4·2734	7·6015	1·2242	4·5888	7·6242	1·2640	4·6703	6·7406	1·3197	4·2372	7·2808
	5·0	0·8685	4·1528	7·5518	0·8933	4·5397	7·5253	0·9360	4·6453	6·4079	1·0011	4·0170	7·1217
(3, 3)	0·5	2·8031	5·9901	9·3084	2·9225	6·2832	8·8031	3·0673	5·8433	8·6818	3·1416	5·5109	9·4248
	1·0	2·6865	5·9236	9·2850	2·8369	6·2832	8·6048	3·0321	5·6808	8·5243	3·1416	5·2554	9·4248
	5·0	2·5091	5·8442	9·2582	2·6966	6·2832	8·3394	2·9653	5·4367	8·3472	3·1416	4·8767	9·4248
(3, 4)	0·5	2·1578	5·1648	8·4771	2·2157	5·4661	8·2628	2·2940	5·3286	7·7568	2·3572	4·9459	8·4846
	1·0	2·0815	5·0824	8·4443	2·1562	5·4545	8·1282	2·2619	5·2562	7·5384	2·3532	4·7514	8·4394
	5·0	1·9609	4·9803	8·4068	2·0572	5·4367	7·9356	2·2034	5·1348	7·2743	2·3451	4·4553	8·3800
(4, 1)	0·5	1·5102	4·5446	7·8509	1·6041	4·6701	7·2222	1·6967	4·3155	1·3517	1·7784	4·0327	7·8540
	1·0	1·3469	4·5091	7·8500	1·4580	4·6607	7·0355	1·5789	4·1694	7·2263	1·7004	3·7717	7·8537
	5·0	0·9561	4·4657	7·8489	1·0628	4·6458	6·7884	1·1980	3·9385	7·0734	1·3709	3·3304	7·8533
(4, 2)	0·5	3·7178	7·0433	10·1279	3·9240	6·6065	9·4573	3·7923	6·4264	10·2026	3·5442	6·9483	9·3602
	1·0	3·6642	7·0374	10·1058	3·9229	6·4535	9·3125	3·7342	6·2472	10·2005	3·3896	6·9073	9·1680
	5·0	3·5961	7·0303	10·0773	3·9213	6·2380	9·1551	3·6335	6·0137	10·1977	3·1268	6·8481	8·9446
(4, 3)	0·5	2·0792	5·3906	8·6372	2·2409	5·3881	7·8869	2·3473	4·9608	8·3183	2·3572	4·9459	8·4846
	1·0	1·9858	5·3657	8·6367	2·1893	5·3459	7·6926	2·3384	4·7732	8·2404	2·3532	4·7514	8·4394
	5·0	1·8470	5·3354	8·6360	2·1012	5·2796	7·4530	2·3207	4·4909	8·1466	2·3451	4·4553	8·3800
(4, 4)	0·5	4·5633	7·8496	10·8236	4·7167	7·2111	10·3312	4·4333	7·3301	10·9428	4·2724	7·8532	9·9639
	1·0	4·5226	7·8487	10·7765	4·7714	7·0201	10·2168	4·3189	7·1973	10·9279	4·1079	7·8532	9·7372
	5·0	4·4720	7·8477	10·7162	4·7027	6·7659	10·0963	4·1373	7·0338	10·9084	3·8525	7·8532	9·4721

One may deduce frequencies for $R_1 = 0{\cdot}625$, $0{\cdot}75$, $0{\cdot}875$ from the results in Table 2. For $R_1 = 0{\cdot}5$, the second non-zero frequencies for boundary conditions (1, 1), (2, 2) and (4, 4) and the first non-zero frequencies of (3, 3) are independent of δ. In these cases because of symmetry, there is a node at $R_1 = 0{\cdot}5$.

2.2. MODE SHAPE CALCULATIONS

The mode shapes, position of nodes, etc. are useful tools in vibration analysis. To establish the modes shape for a particular condition (i, j), δ and R_1, the natural frequency Ω was calculated. In this note the mode shape (two separate curves) was normalized with the factor R_0 (without loss of generality) so that

$$Y_1(R_0) = C_{11}U_1(R_0) + C_{12}V_1(R_0) = Z_0, \tag{10}$$

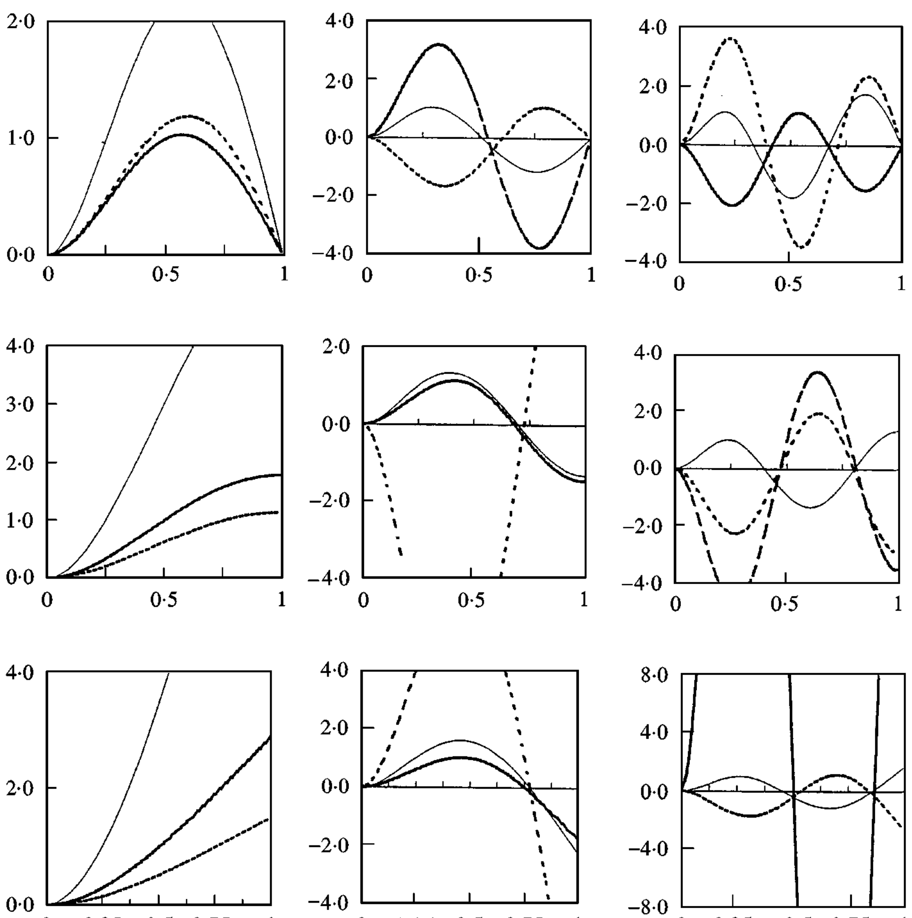

Figure 2. The first three mode shapes (normalized with $R_0 = R_1$ and $Z_0 = 1{\cdot}0$, first mode in first column, etc.) of clamped–pinned (first row), clamped–sliding (second row) and clamped–free (third row) beams. Thin line $R_1 = 0{\cdot}25$, thick line $R_1 = 0{\cdot}5$ and discontinuous line $R_1 = 0{\cdot}75$. For all cases $\delta = 0{\cdot}2$.

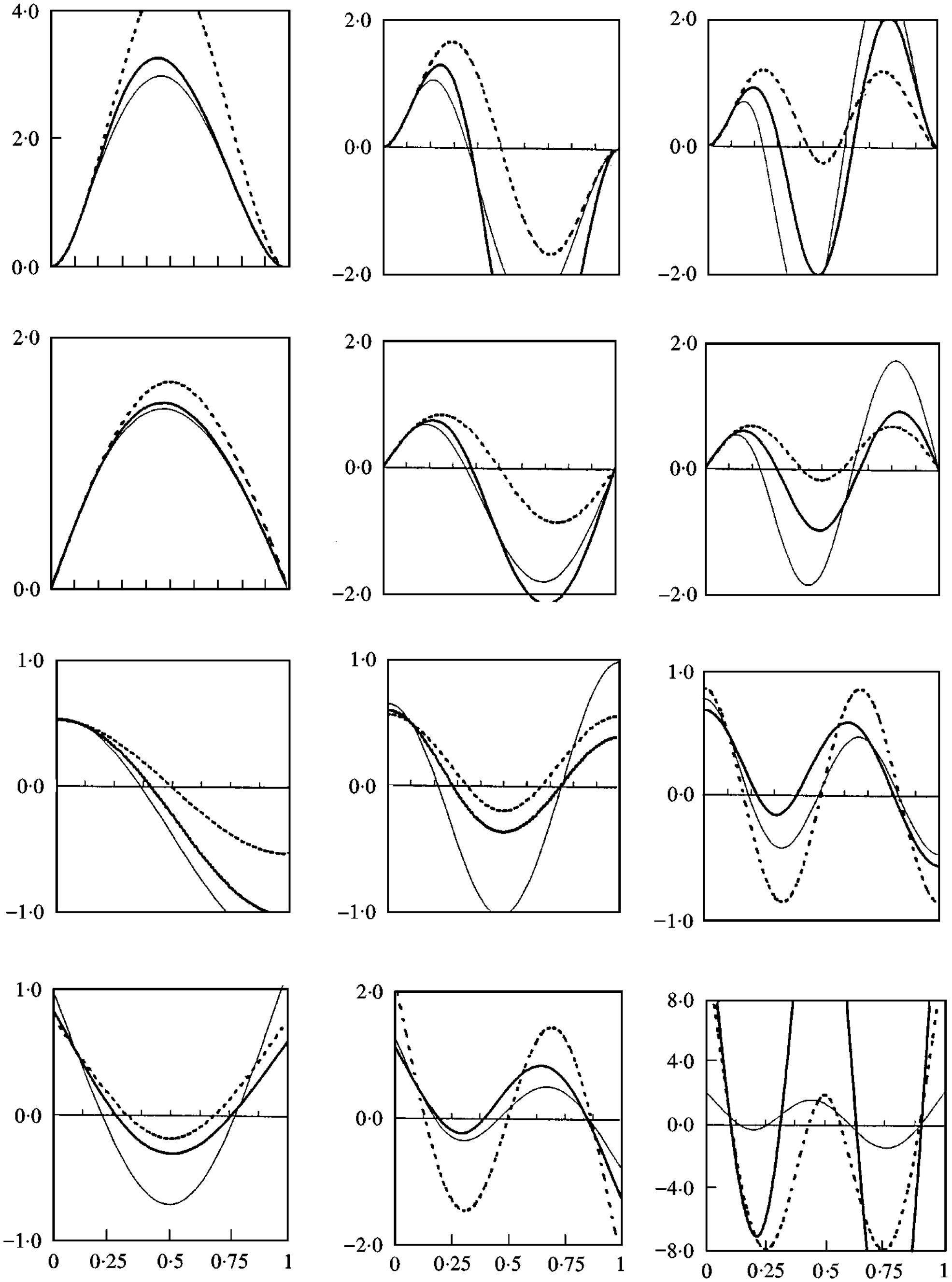

Figure 3. The first three mode shapes (normalized with $R_0 = 0{\cdot}1$ and $Z_0 = 0{\cdot}5$, first mode in first column, etc.) of clamped–clamped (first row), pinned–pinned (second row), sliding–sliding (third row) and free–free (fourth row) beams. Thin line for $R_1 = 0{\cdot}2$, thick line $R_1 = 0{\cdot}3$ and discontinuous line $R_1 = 0{\cdot}5$. For all cases $\delta = 1{\cdot}0$.

where Z_0 is arbitrarily chosen and the normalizing factor $R_0 \leqslant R_1$. The four constants of integration in equation (5) were obtained from

$$\begin{bmatrix} A_{01} & A_{02} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{12} \\ C_{21} \\ C_{22} \end{bmatrix} = \begin{bmatrix} Z_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (11)$$

where $A_{01} = U_1(R_0)$ and $A_{02} = V_1(R_0)$.

Figure 2 illustrates the change in the first three modes shapes of clamped–pinned, clamped–sliding and clamped–free beams when a particle of $\delta = 0{\cdot}2$ is placed at $R_1 = 0{\cdot}25$ or $0{\cdot}5$ or $0{\cdot}75$ and the mode shapes were normalized with $Z_0 = 1{\cdot}0$ and $R_0 = R_1$. By substituting C_{11} and C_{12} from equation (11) into equation (8), the normalized left portion of the mode shape $Y_1(X_1)$ was established by increasing X_1 in small steps from 0 to R_1. The right portion was similarly established. A clamped–free uniform beam has a node at $0{\cdot}7834L$ in its second mode and nodes at $0{\cdot}5035L$ and at $0{\cdot}8677L$ at its third mode; see for example [15]. This explains the "bulgy" second normalized mode shape of the clamped–free case in Figure 2 when the particle is located at $R_1 = 0{\cdot}75$ (close to a node) and the excessively "bulgy" normalized third mode shape with the particle at $R_1 = 0{\cdot}5$ (too close to a node). The "bulgy" normalized second mode shape of the clamped–sliding beam is because of a node in the vicinity of $0{\cdot}75L$. "Bulgy" normalized mode shapes are due to the accidental choice of the normalizing factor R_0 and does not imply large deflections. Figure 3 shows the normalized mode shapes of clamped–clamped, pinned–pinned, sliding–sliding and free–free beams for particle of $\delta = 1{\cdot}0$ and $R_1 = 0{\cdot}2$, $0{\cdot}3$ and $0{\cdot}5$ and the choice made for equation (11) are $R_0 = 0{\cdot}1$ and $Z_0 = 0{\cdot}5$. Note the symmetrical mode shapes for $R_1 = 0{\cdot}5$ and in this case one avoids the choice of $R_0 = 0{\cdot}5$.

3. CONCLUSIONS

Low (6) presented the frequency equations of a uniform beam with a particle at an intermediate point as 8×8 determinants, used MAPLE software to expand them but presented only the fundamental frequencies of the clamped–clamped case. In the present note, the choice of two separate co-ordinate system enabled the frequency equations to be expressed as 4×4 determinants (which if needed may be expanded manually) equated to zero. Two more constants of integration may be eliminated from the conditions of continuity of deflection and slope at the position of the particle and the frequency equations may now be expressed as 2×2 determinants, but it was found that this additional manual operation did not offer much overt advantage.

For 16 combinations of classical boundary conditions, the first three frequencies are presented in Table 2 for $R_1 = 0{\cdot}125$, $0{\cdot}250$, $0{\cdot}375$ and $0{\cdot}500$ and $\delta = 0{\cdot}5$, $1{\cdot}0$ and $5{\cdot}0$. Equation (9) enables the frequencies for $R_1 = 0{\cdot}625$, $0{\cdot}750$, $0{\cdot}875$ to be deduced from the table.

Typical normalized mode shapes are presented. If accidentally or otherwise, the normalizing factor R_0 is near a node, large values will result for the normalized mode shapes which means that there is a node in the vicinity of R_0.

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