Journal of Sound and Vibration (1999) **227**(1), 215–221
Article No. jsvi.1999.2367, available online at http://www.idealibrary.com on IDEAL®

LOCALIZED VIBRATION MODES PROPAGATING ALONG EDGES OF CYLINDRICAL AND CONICAL WEDGE-LIKE STRUCTURES

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(Received 19 March 1999)

1. INTRODUCTION

The existence of antisymmetric localized vibration modes propagating along edges of elastic wedges of linear geometry was first predicted in 1972 by Lagasse [1] and Maradudin *et al*. [2] using numerical calculations. It was shown [1, 2] that such modes, now often called wedge acoustic waves, are characterized by low propagation velocity (generally much lower than that of Rayleigh waves), and their elastic energy is concentrated in the area of about one wavelength near the wedge tips.

Since 1972, wedge acoustic waves have been investigated both theoretically and experimentally in a number of papers with regard to their possible applications to signal processing devices, ultrasonic non-destructive testing, modelling vibration behaviour of special engineering constructions, etc. (see, e.g. references [3–9] and references therein). In particular, in the recently published paper of Hladky-Hennion [7] dealing with finite element calculations of wedge waves, among other results, calculations have been carried out for the velocities of waves propagating along the edge of a cyclindrical wedge-like structure bounded by a circular cylinder and a conical cavity (Figure 1(a)). The results of the calculations demonstrated frequency-dependent behaviour of the velocities of the two lowest order wedge modes as functions of wavenumber.

The aim of this communication is to show that localized wave propagation in this and similar wedge-like structures can be described in a more simple way, using the approximate analytical theory of localized elastic modes in curved solid wedges based on the geometrical-acoustics approach earlier developed by the present author [6]. The results obtained using this theory are in good quantitative agreement with the numerical calculations of reference [7].

2. OUTLINE OF THE THEORY

The approximate analytical theory of localized elastic waves in slender solid wedges which will be used below is based on the geometrical acoustics approach

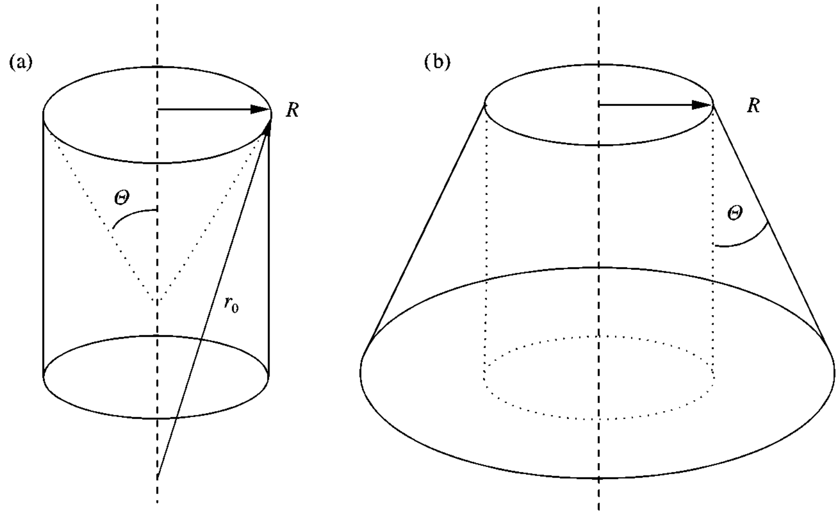

Figure 1. Definition of the in-plane radius of curvature r_0 for the cylindrical wedge-like structure investigated in reference [7] (a), and for the similar structure bounded by a hollow cylinder and an external conical surface (b).

considering a wedge as a plate with a local variable thickness d. For slender wedges of linear geometry $d = x\Theta$, where Θ is the wedge apex angle and x is the distance from the wedge tip measured in the middle plane. The velocities c of the localized wedge modes propagating along the edge of linear wedge (in the y direction) are determined in the geometrical acoustics approximation as solutions of the Bohr–Sommerfeld-type equation [6,9]

$$\int_0^{x_t} [k^2(x) - \beta^2]^{1/2}\,\mathrm{d}x = \pi n, \tag{1}$$

where $\beta = \omega/c$ is a wavenumber of a wedge mode, ω is circular frequency, $k(x)$ is a current local wavenumber of a flexural wave in a plate of variable thickness, $n = 1, 2, 3, \ldots$ is the wedge mode number, and x_t is the so-called ray turning point being determined from the equation $k^2(x) - \beta^2 = 0$.

In the case of a linear wedge in a vacuum $k(x) = 12^{1/4}k_p^{1/2}(\Theta x)^{-1/2}$ and $x_t = 2\sqrt{3}k_p/\Theta\beta^2$, where $k_p = \omega/c_p$ is the wavenumber of a symmetric plate wave, and $c_p = 2c_t(1 - c_t^2/c_l^2)^{1/2}$ is the plate wave velocity, c_l and c_t are propagation velocities of longitudinal and shear acoustic waves in the plate material. Taking the integral in equation (1) and solving the resulting algebraic equation yields the extremely simple analytical expression for wedge wave velocities [6]:

$$c = \frac{c_p n \Theta}{\sqrt{3}}. \tag{2}$$

Expression (2) agrees well with the other theoretical calculations [3–5, 7] and with the experimental results [3]. Although, strictly speaking, the geometrical acoustics approach is not applicable for the lowest order wedge mode ($n = 1$) [6], in practice it provides quite accurate results for wedge wave velocities in this case as well. The analytical expressions for amplitude distributions of wedge modes are rather cumbersome [6] and are not displayed here for brevity.

To calculate the velocities of wedge waves in a curved wedge one has to consider two possible types of curved wedges: wedges curved in their middle plane (in-plane curvature) and wedges curved perpendicular to their middle plane (anti-plane curvature). In both these cases, one assumes that the radius of curvature is large enough in comparison with characteristic wavelengths.

First consider the case of the in-plane curvature and assume for certainty that the radius of curvature is positive (a convex edge) and has a value r_0. Then, using cylindrical coordinates in which the edge of a curved wedge is described as $r = r_0$, one can rewrite the governing equation (1) in the form [6]

$$-\int_{r_0}^{r_t} [k^2(r - r_0) - \beta^2(r_0^2/r^2)]^{1/2}\,\mathrm{d}r = \pi n, \tag{3}$$

where r_t is the co-ordinate of ray turning point. Assuming the radius of curvature r_0 to be large enough ($r_0 \gg |r_0 - r_t|$) and introducing the notation $\xi = r_0 - r$, one can transform from expression (3) to the approximate relation accurate in the first order versus r_0^{-1}:

$$\int_0^{\xi_t} \left(1 + \frac{\xi}{r_0}\right)\left[k^2(\xi)\left(1 - \frac{2\xi}{r_0}\right) - \beta^2\right]^{1/2} \mathrm{d}\xi = \pi n. \tag{4}$$

Here the co-ordinate of the turning point is now $\xi_t = \xi_{t0}(1 - 2\xi_{t0}/r_0)$, where $\xi_{t0} = 2\sqrt{3}k_p/\theta\beta^2$ is the turning point in the absence of curvature.

Evaluating the integral in equation (4), which reduces to a tabulated integral [10], one can find the expression for phase velocities of wedge waves propagating along a convex curved edge:

$$c = \frac{c_p n\Theta}{\sqrt{3}}\left(1 + \frac{\sqrt{3}}{2}\frac{n^2\Theta}{k_p r_0}\right). \tag{5}$$

In the absence of curvature, i.e., for $r_0 \to \infty$, equation (5) reverts to equation (2).

For comparison with the numerical calculations of reference [7] it is convenient to rewrite equation (5) in the form

$$c = c_0^{(n)}\left(1 + \frac{3}{2}\frac{n^2}{\beta_0^{(1)} r_0}\right), \tag{6}$$

where $c_0^{(n)} = c_p n\Theta/\sqrt{3}$ is the velocity of a wedge mode characterized by a number n in the absence of curvature, and $\beta_0^{(1)} = \sqrt{3}k_p/\Theta = \sqrt{3}\omega/c_p\Theta$ is a wavenumber of a wedge mode of the first order in the absence of curvature.

According to equations (5) or (6), wedge waves in a wedge with positive in-plane curvature are dispersive and their velocities decrease with the increase of $\beta_0^{(1)}$. For a wedge with a negative edge curvature (a concave edge), equations (5) and (6) remain valid if r_0 is replaced by $-r_0$.

Now consider the case of anti-plane curvature of radius r_{0A}. Obviously, because of the symmetry consideration, wedge wave velocities in such a wedge should not depend on the sign of anti-plane curvature, i.e., they must be invariant when replacing r_{0A} by $-r_{0A}$. This implies that the series expansion of the velocity correction due to the effect of anti-plane curvature must contain only even powers of r_{0A}^{-1}. Therefore, in the first approximation versus r_{0A}^{-1}, anti-plane curvature does not affect wedge wave velocities since the lowest order in the velocity correction is proportional to r_{0A}^{-2}.

Keeping all these in mind, one can easily calculate the velocities of wedge modes propagating along a cylindrical wedge-like structure earlier considered in reference [7] and shown in Figure 1(a). This structure is bounded by an external cylinder of radius R and an internal conical cavity characterized by the rotation angle θ and represents a wedge with the apex angle θ having both in-plane and anti-plane curvatures. According to the discussion above, the only geometrical parameter which matters is the radius of in-plane curvature r_0. For the geometry under consideration the value of r_0 is described as $r_0 = R/\sin(\theta/2)$. Substitution of this value of r_0 into equation (5) or (6) gives the velocities of wedge acoustic modes for the structure under consideration. Obviously, these velocities decrease with the increase of frequency or $\beta_0^{(1)}$.

As an example of another similar structure, one can consider a cylindrical wedge with the apex angle Θ formed by the intersection of a hollow cylinder of radius R and an external conical surface (Figure 1(b)). As one can see, this case differs from the previous one only by the sign of the wedge in-plane curvature which is now negative. Therefore, r_0 in equations (5) and (6) should be replaced by $-r_0$, in agreement with the expression $r_0 = R/\sin(\theta/2)$ for value of apex angle Θ now considered as negative. This implies that the dispersion curves of localized wedge modes in such a structure describe the increase of wedge wave velocities with the increase of frequency or $\beta_0^{(1)}$. They can be obtained by a mirror reflection of the dispersion curves corresponding to Figure 1(a) versus the horizontal lines describing the mode velocities in the absence of curvature.

The developed theory can be applied also to disk-type structures bounded by two conical surfaces characterized by the rotation angles Θ_1 and Θ_2 (Figure 2(a)) or by a conical surface and a plane (Figure 2(b)), the latter configuration being a particular case of the previous structure for $\Theta_2 = \pi/2$. To apply equations (5) or (6) to the structure shown in Figure 2(a) one should determine its apex angle Θ and the radius of in-plane curvature r_0. It is easy to see that the corresponding expressions are $\Theta = \Theta_1 + \Theta_2$ and $r_0 = R/\cos[(\theta_1 - \theta_2)/2]$ respectively. The radius r_0 for such structures is always positive, and the behavior of the velocity dispersion curves of localized modes is similar to that for the cylindrical structure shown in Figure 1(a).

The next example of wedge-like structure can be obtained by intersection of two conical surfaces as shown in Figure 3(a) (note that such a structure generalizes both case shown in Figure 1). The expressions for wedge apex angle and in-plane radius of curvature in this case are $\Theta = \Theta_1 + \Theta_2$ and $r_0 = R/\sin[(\Theta_1 - \Theta_2)/2]$. It is interesting to see that for $\Theta_1 = \Theta_2$ (Figure 3(b)) the in-plane radius of curvature r_0

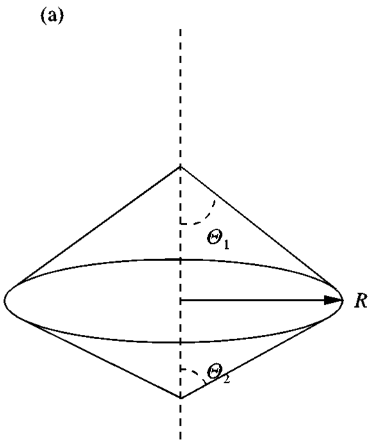

Figure 2. Examples of disk-type structures bounded by two conical surfaces (a), and by a conical surface and a plane (b).

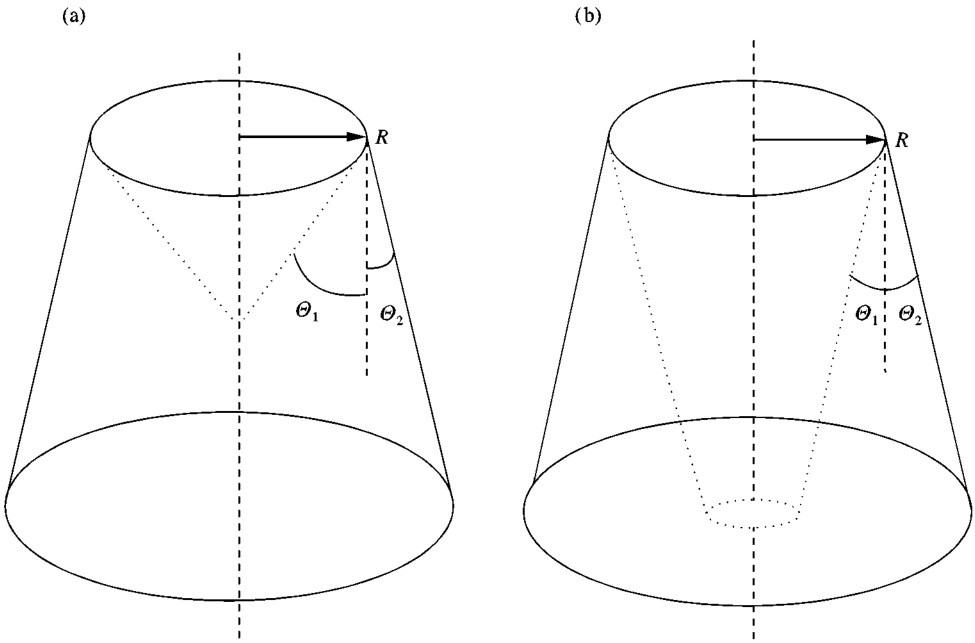

Figure 3. Examples of ridge-type structures bounded by two conical surfaces for $\Theta_1 \neq \Theta_2$ (a), and $\Theta_1 = \Theta_2$ ("dispersion-free" structure) (b).

tends to infinity, $r_0 = \infty$. Therefore, in the first approximation versus r_0^{-1} wedge waves in such a conical structure with $\Theta_1 = \Theta_2$ propagate without dispersion.

One can easily invent other examples of curved wedge-like structures that can be described by the present theory. Such structures may be formed by different combinations of cylindrical, conical and plane surfaces. In particular, these may include asymmetrical structures with variable wedge apex angles (local wedge wave velocities in such structures are functions of a position along the edge). However, the discussion of all such cases is beyond the scope of this communication.

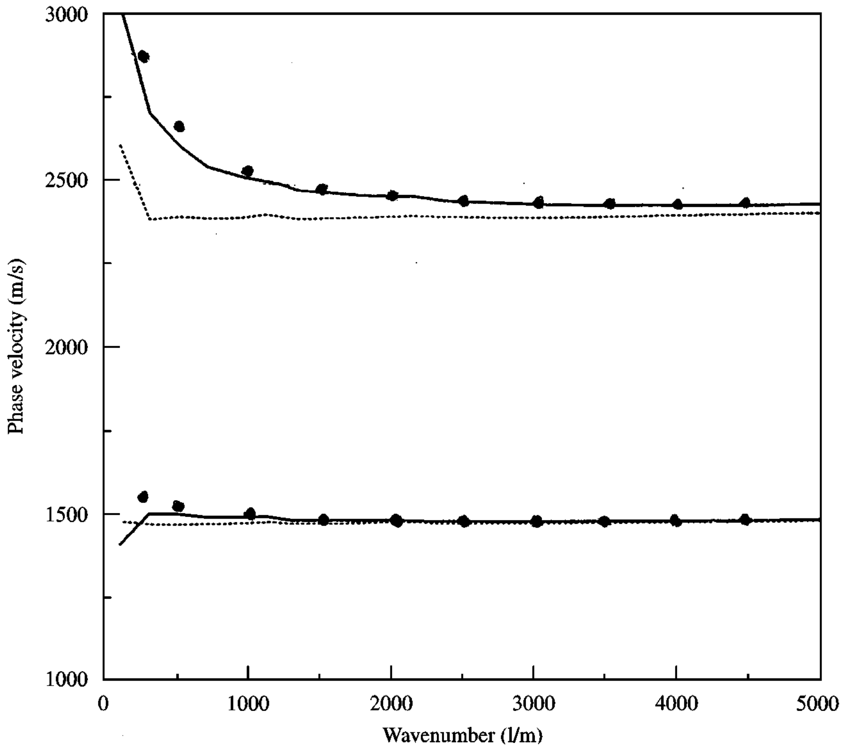

Figure 4. Calculated velocities c of the first and second modes propagating in the duraluminium cylindrical wedge structure shown in Figure 1(a) as functions of a wavenumber $\beta_0^{(1)}$ of the lowest order mode of a linear wedge: point indicate the results calculated using formula (6); solid curves show the results of finite element calculations [7]; broken lines indicate the velocities of the same modes in the absence of wedge curvature, according to reference [7].

3. COMPARISON WITH FINITE ELEMENT CALCULATIONS

To compare the results of the present theory with the existing finite element calculations, a duraluminium cylindrical wedge structure studied in the paper [7] has been considered (see Figure 1(a)). The velocities c of two lowest order wedge modes have been determined according to equatiom (6) as functions of a wavenumber $\beta_0^{(1)}$ of the first mode in the absence of curvature. Resonance effects due to circumpropagation of localized wedge modes, which introduce the discreteness in their wavenumbers, have been neglected. The parameters of structure were the following: Young's modulus $E = 6{\cdot}972 \times 10^{10}$ pa, mass density $\rho = 2700\,\mathrm{kg/m^3}$, the Poisson's ratio $\sigma = 0{\cdot}344$, wedge apex angle $\theta = 30°$, the external radius $R = 19{\cdot}5$ mm, and the height of the structure $h = 47$ mm. The results of the current calculations (points) are shown in Figure 4 together with the results of finite element calculations of reference [7] (curves). One can see that the agreement between the above described analytical theory and the results of finite element calculations is very good. This proves that the approximate analytical theory works well for curved wedge-like structure under consideration.

4. CONCLUSIONS

The approximate analytical theory of localized vibration modes in cylindrical and conical wedge-like structures based on the geometrical-acoustics approch gives the results which agree well with the results of more laborious finite element calculations. Because of its simplicity, this theory provides clear understanding of the phenomena involved and allows one to make quick and easy calculations of mode velocities for any cylindrical and conical wedge structures if their radii of curvature are much larger than the wavelengths of propagating waves.

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