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VIBRATIONS OF AN AXIALLY MOVING BEAM WITH TIME-DEPENDENT VELOCITY

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The dynamic response of an axially accelerating, elastic, tensioned beam is investigated. The time-dependent velocity is assumed to vary harmonically about a constant mean velocity. These systems experience a coriolis acceleration component which renders such systems gyroscopic. The equation of motion is solved by using perturbation analysis. Principal parametric resonances and combination resonances are investigated in detail. Stability boundaries are determined analytically. It is found that instabilities occur when the frequency of velocity fluctuations is close to two times the natural frequency of the constant velocity system or when the frequency is close to the sum of any two natural frequencies. When the velocity variation frequency is close to zero or to the difference of two natural frequencies, however, no instabilities are detected up to the first order of perturbation. Numerical results are presented for different flexural stiffness values and for the first two modes.

1. INTRODUCTION

Due to their technological importance, the vibrations of axially moving materials have been investigated by many researchers. Threadlines, high-speed magnetic and paper tapes, strings, power transmission chains and belts, band-saws, fibres, beams, aerial cable tramways and pipes conveying fluid are some of the examples. Ulsoy *et al.* [1] and Wickert and Mote [2] reviewed the literature on axially moving materials. Wickert and Mote [3] investigated the transverse vibrations of travelling strings and beams. They used travelling string eigenfunctions and introduced a convenient orthogonal basis suitable for discretization. The same authors [4] also presented a modal analysis using complex state eigenfunctions and their conjugates. Ulsoy [5] treated a model for the transverse vibration of an axially moving beam which includes elastic coupling between two adjacent spans. The system was then analyzed by using a classical approximate solution method. It was concluded that the presence of a beating phenomenon at low transport velocities was destroyed by higher velocities and/or tension differences in the two spans. Al-Jawi *et al.* [6–8] first investigated the vibration localization phenomenon in dual-span axially moving beams. Wu and Mote [9] studied simple torsion parametric resonances and combination torsion-bending parametric resonances in an axially moving band by an in-plane periodic edge loading that was normal to the longitudinal axis of the band. A theoretical basis for the analysis of band vibration and stability was studied by Ulsoy and Mote [10]. The band natural frequencies were found to decrease with increasing axial velocity at a rate dependent on the wheel support system constant, and to increase with increasing axial tension or "strain".

Pakdemirli and Ulsoy [11] investigated principal parametric resonances and combination resonances for any two modes for an axially accelerating string. They found that for velocity fluctuation frequencies near twice any natural frequency, an instability region occurs whereas for the frequencies close to zero, no instabilities were detected. For combination resonances, instabilities occurred only for those of additive type. No instabilities were detected for difference-type combination resonances in agreement with reference [12]. Oz *et al.* [13] investigated the transition behaviour from strip to beam for axially moving continua. An approximate analytical expression for the natural frequency was given for the problem. For variable velocity profiles, stability borders were determined analytically. The beam effects were studied. Wickert [14] analyzed free non-linear vibrations of a moving beam over the sub- and superharmonic transport speed ranges. Pellicano and Zirilli [15] presented a boundary layer solution for the axially moving beam problem with small flexural stiffnesses. Asokanthan and Ariaratnam [16] investigated flexural instabilities in moving bands under harmonic tension. They discussed the effects due to damping, mean band speed, and the band compliance on the band stability.

In most of the references given, the transport velocity was taken as constant. In reality, the systems are exposed to accelerating and decelerating motions. Miranker [17] took a model for the transverse vibrations of a tape moving between a pair of pulleys and by using a variational procedure derived the equations of motion for time-dependent axial velocity. Mote [18] investigated the problem of an axially accelerating string with harmonic excitation at one end and determined stability by Laplace transform techniques. Pakdemirli *et al.* [19] re-derived the equations of motion for an axially accelerating string using Hamilton's principle and numerically investigated the stability of the response using Floquet theory. A sinusoidal variation of the transport velocity, about a mean velocity of zero, was considered in the analysis. Pakdemirli and Batan [20] considered a different type of velocity variation, namely the periodic, constant acceleration–deceleration type.

In this study, an Euler–Bernoulli beam having different flexural stiffness values and moving with harmonically varying velocities is considered. The beam is simply supported at both ends. The equation of motion is derived by following a method similar to that given in reference [14]. A harmonically varying velocity function is chosen. The equation of motion is solved by directly applying the method of multiple scales to the partial differential system (direct-perturbation method). For higher order expansions, solutions obtained by using this direct-perturbation method better represent the real behaviour of the system than the common method of discretization (discretization–perturbation method) [21–28]. In the present case, which contains only two terms in the expansion, the advantage of using the direct-perturbation method is that the equations need not be cast into a form convenient for orthogonalizing the modes [11]. The natural frequency variation with velocity for various flexural stiffness values are determined for the first two modes. Principal parametric resonances, sum and difference-type combination resonances are investigated. For velocity fluctuation frequencies near twice any natural frequency, an instability region occurs whereas for frequencies close to zero, no instabilities are detected. For combination resonances, instabilities occurred only for those of additive type. No instabilities are detected for difference-type combination resonances up to the first approximation order.

2. APPROXIMATE ANALYSIS

For the axially moving beam illustrated in Figure 1, by following a derivation similar to that given in reference [14], it can be shown that the linear, time-dependent, dimensionless equation of motion is

$$\ddot{w} + 2\dot{w}'v + w'\dot{v} + v_f^2 w^{iv} + (v^2 - 1)w'' = 0, \tag{1}$$

where w is the transverse displacement, v is the axial velocity and v_f is the dimensionless flexural stiffness. The material particle of the travelling beam experiences local $\ddot{w}$, Coriolis $2\dot{w}'v$, and centripetal $v^2 w''$ acceleration components. The boundary conditions are

$$w(0,t) = w(1,t) = 0, \qquad w''(0,t) = w''(1,t) = 0. \tag{2}$$

When $v_f^2 = 0$, equation (1) reduces to that of a travelling string. The dot denotes differentiation with respect to time and the prime denotes differentiation with respect to the spatial variable x.

Assuming that the velocity is harmonically varying about a constant mean velocity v_0, one writes

$$v = v_0 + \varepsilon v_1 \sin \Omega t, \tag{3}$$

where ε is a small parameter and εv_1, which is also small, represents the amplitude of fluctuations, Ω is the fluctuation frequency. Unlike in reference [13], v_f^2 is now an order one term. Substituting equation (3) into equation (1) and keeping terms up to the first order of approximation, one has

$$\ddot{w} + 2v_0\dot{w}' + (v_0^2 - 1)w'' + v_f^2 w^{iv} + \varepsilon(v_1 \Omega \cos \Omega t\, w' + 2v_1 \sin \Omega t\, \dot{w}' + 2v_0 v_1 \sin \Omega t\, w'') = 0. \tag{4}$$

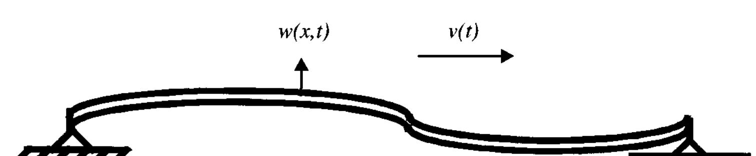

Figure 1. Axially moving beam with time-dependent velocity.

The direct perturbation method will be applied to equation (4) in search of solutions. This method does not require conversion of the equation into other forms as done in the discretization–perturbation method [11]. Using the method of multiple scales [29, 30] and assuming a first order expansion, one writes

$$w(x,t;\varepsilon) = w_0(x,T_0,T_1) + \varepsilon w_1(x,T_0,T_1) + \cdots, \tag{5}$$

where w_0 and w_1 are the displacement functions at orders 1 and ε, $T_0 = t$ and $T_1 = \varepsilon t$ are the usual fast and slow time scales. In terms of the new variables, the time derivatives can be written as

$$\mathrm{d}/\mathrm{d}t = D_0 + \varepsilon D_1 + \cdots, \qquad \mathrm{d}^2/\mathrm{d}t^2 = D_0^2 + 2\varepsilon D_0 D_1 + \cdots, \tag{6}$$

where $D_n = \partial/\partial T_n$. Substituting equations (5) and (6) into equation (4), separating terms at each order of ε, one obtains

$$O(1){:}\ D_0^2 w_0 + 2v_0 D_0 w_0' + (v_0^2 - 1)w_0'' + v_f^2 w_0^{iv} = 0, \tag{7}$$

$$O(\varepsilon){:}\ D_0^2 w_1 + 2v_0 D_0 w_1' + (v_0^2 - 1)w_1'' + v_f^2 w_1^{iv} = -2D_0 D_1 w_0 - 2v_0 D_1 w_0'$$
$$- 2v_1 \sin\Omega T_0 D_0 w_0' - 2v_0 v_1 \sin\Omega T_0 w_0'' - \Omega v_1 \cos\Omega T_0 w_0' \tag{8}$$

The solution at order 1 can be written as follows:

$$w_0(x,T_0,T_1;\varepsilon) = A_n(T_1)\,\mathrm{e}^{i\omega_n T_0} Y_n(x) + \bar{A}_n(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_n T_0} \bar{Y}_n(x), \tag{9}$$

The spatial functions $Y_n(x)$ satisfy the equation

$$v_f^2 Y_n^{iv} + (v_0^2 - 1)Y_n'' + 2\mathrm{i}v_0\omega_n Y_n' - \omega_n^2 Y_n = 0, \tag{10}$$

with the boundary conditions

$$Y_n(0) = 0 \quad Y_n(1) = 0; \quad Y_n''(0) = 0, \quad Y_n''(1) = 0. \tag{11}$$

The solution is

$$Y_n(x) = c_{1n}(\mathrm{e}^{i\beta_{1n}x} + C_{2n}\mathrm{e}^{i\beta_{2n}x} + C_{3n}\mathrm{e}^{i\beta_{3n}x} + C_{4n}\mathrm{e}^{i\beta_{4n}x}). \tag{12}$$

The β_{in} satisfy the dispersive relation

$$v_f^2\beta_{in}^4 + (1 - v_0^2)\beta_{in}^2 - 2\omega_n v_0 \beta_{in} - \omega_n^2 = 0, \quad i = 1,2,3,4\cdots, \quad n = 1,2\cdots. \tag{13}$$

Applying the boundary conditions to the solution, one obtains the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta_{1n}^2 & \beta_{2n}^2 & \beta_{3n}^2 & \beta_{4n}^2 \\ \mathrm{e}^{i\beta_{1n}} & \mathrm{e}^{i\beta_{2n}} & \mathrm{e}^{i\beta_{3n}} & \mathrm{e}^{i\beta_{4n}} \\ \beta_{1n}^2\mathrm{e}^{i\beta_{1n}} & \beta_{2n}^2\mathrm{e}^{i\beta_{2n}} & \beta_{3n}^2\mathrm{e}^{i\beta_{3n}} & \beta_{4n}^2\mathrm{e}^{i\beta_{4n}} \end{bmatrix} \begin{Bmatrix} 1 \\ C_{2n} \\ C_{3n} \\ C_{4n} \end{Bmatrix} c_{1n} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{14}$$

For non-trivial solutions, the determinant of the coefficient matrix must be zero, which yields the support condition

$$
\begin{aligned}
&[\mathrm{e}^{i(\beta_{1n}+\beta_{2n})}+\mathrm{e}^{i(\beta_{3n}+\beta_{4n})}](\beta_{1n}^2-\beta_{2n}^2)(\beta_{3n}^2-\beta_{4n}^2)\\
&\quad+[\mathrm{e}^{i(\beta_{1n}+\beta_{3n})}+\mathrm{e}^{i(\beta_{2n}+\beta_{4n})}](\beta_{2n}^2-\beta_{4n}^2)(\beta_{3n}^2-\beta_{1n}^2)\\
&\quad+[\mathrm{e}^{i(\beta_{2n}+\beta_{3n})}+\mathrm{e}^{i(\beta_{1n}+\beta_{4n})}](\beta_{1n}^2-\beta_{4n}^2)(\beta_{2n}^2-\beta_{3n}^2)=0. \qquad (15)
\end{aligned}
$$

Numerical values of ω_n as well as β_{in} can be calculated by using equations (13) and (15). These results are presented in section 5. Using the set of equations in equation (14), one can find the coefficients C_{2n}, C_{3n} and C_{4n} by elimination:

$$
C_{2n}=-\frac{(\beta_{4n}^2-\beta_{1n}^2)(\mathrm{e}^{i\beta_{3n}}-\mathrm{e}^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(\mathrm{e}^{i\beta_{3n}}-\mathrm{e}^{i\beta_{2n}})}, \qquad C_{3n}=-\frac{(\beta_{4n}^2-\beta_{1n}^2)(\mathrm{e}^{i\beta_{2n}}-\mathrm{e}^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(\mathrm{e}^{i\beta_{2n}}-\mathrm{e}^{i\beta_{3n}})}, \quad (16, 17)
$$

$$
C_{4n}=-1-C_{2n}-C_{3n}. \qquad (18)
$$

Hence, $Y_n(x)$ now reads

$$
\begin{aligned}
Y_n(x)=c_{1n}\Bigg\{&\mathrm{e}^{i\beta_{1n}x}-\frac{(\beta_{4n}^2-\beta_{1n}^2)(\mathrm{e}^{i\beta_{3n}}-\mathrm{e}^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(\mathrm{e}^{i\beta_{3n}}-\mathrm{e}^{i\beta_{2n}})}\mathrm{e}^{i\beta_{2n}x}-\frac{(\beta_{4n}^2-\beta_{1n}^2)(\mathrm{e}^{i\beta_{2n}}-\mathrm{e}^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(\mathrm{e}^{i\beta_{2n}}-\mathrm{e}^{i\beta_{3n}})}\mathrm{e}^{i\beta_{3n}x}\\
&+\left(-1+\frac{(\beta_{4n}^2-\beta_{1n}^2)(\mathrm{e}^{i\beta_{3n}}-\mathrm{e}^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(\mathrm{e}^{i\beta_{3n}}-\mathrm{e}^{i\beta_{2n}})}+\frac{(\beta_{4n}^2-\beta_{1n}^2)(\mathrm{e}^{i\beta_{2n}}-\mathrm{e}^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(\mathrm{e}^{i\beta_{2n}}-\mathrm{e}^{i\beta_{3n}})}\right)\mathrm{e}^{i\beta_{4n}x}\Bigg\}. \qquad (19)
\end{aligned}
$$

At the second order of approximation, one substitutes equation (9) into equation (8). The result is

$$
\begin{aligned}
&D_0^2w_1+2v_0D_0w_1'+(v_0^2-1)w_1''+v_f^2w_1^{iv}=\\
&\quad-2D_1A_n(\mathrm{i}\omega_nY_n+v_0Y_n')\mathrm{e}^{\mathrm{i}\omega_nT_0}+2D_1\bar{A}_n(\mathrm{i}\omega_n\bar{Y}_n-v_0\bar{Y}_n')\mathrm{e}^{-\mathrm{i}\omega_nT_0}\\
&\quad-\omega_nv_1[A_nY_n'(\mathrm{e}^{i(\omega_n+\Omega)T_0}-\mathrm{e}^{i(\omega_n-\Omega)T_0})-\bar{A}_n\bar{Y}_n'(\mathrm{e}^{i(\Omega-\omega_n)T_0}-\mathrm{e}^{-i(\Omega+\omega_n)T_0})]\\
&\quad+\mathrm{i}v_0v_1[A_nY_n''(\mathrm{e}^{i(\omega_n+\Omega)T_0}-\mathrm{e}^{i(\omega_n-\Omega)T_0})+\bar{A}_n\bar{Y}_n''(\mathrm{e}^{i(\Omega-\omega_n)T_0}-\mathrm{e}^{-i(\Omega+\omega_n)T_0})]\\
&\quad-\frac{\Omega}{2}v_1[A_nY_n'(\mathrm{e}^{i(\omega_n+\Omega)T_0}+\mathrm{e}^{i(\omega_n-\Omega)T_0})+\bar{A}_n\bar{Y}_n'(\mathrm{e}^{i(\Omega-\omega_n)T_0}+\mathrm{e}^{-i(\Omega+\omega_n)T_0})]. \qquad (20)
\end{aligned}
$$

Different cases arise depending on the numerical value of velocity variation frequency. These cases will be treated in the following sections.

3. PRINCIPLE PARAMETRIC RESONANCES

In this section, one assumes that one dominant mode of vibration exists. Depending on the numerical value of frequency, three cases will be investigated separately.

3.1. Ω AWAY FROM $2\omega_n$ AND 0

In this case, equation (20) becomes

$$D_0^2 w_1 + 2v_0 D_0 w_1' + (v_0^2 - 1) w_1'' + v_f^2 w_1^{iv}$$
$$= -2D_1 A_n (\mathrm{i}\omega_n Y_n + v_0 Y_n') \mathrm{e}^{i\omega_n T_0} + cc + NST, \tag{21}$$

where cc and NST denote complex conjugates and non-secular terms respectively. One can take the solution of equation (21) as

$$w_1(x, T_0, T_1) = \phi_n(x, T_1) \mathrm{e}^{i\omega_n T_0} + W(x, T_0, T_1) + cc, \tag{22}$$

The first term is related to secular terms and the second term is related to non-secular terms. If equation (22) is substituted into equation (21), the ϕ_n are found to satisfy the equation

$$v_f^2 \phi_n^{iv} + (v_0^2 - 1)\phi_n'' + 2\mathrm{i}v_0\omega_n\phi_n' - \omega_n^2\phi_n = -2D_1 A_n(\mathrm{i}\omega_n Y_n + v_0 Y_n') \tag{23}$$

and the boundary conditions are

$$\phi_n(0) = 0, \quad \phi_n''(0) = 0, \quad \phi_n(1) = 0, \quad \phi_n''(1) = 0. \tag{24}$$

The solvability condition requires (see reference [29] for details of calculating solvability conditions)

$$D_1 A_n = 0. \tag{25}$$

This means a constant amplitude solution up to the first order of approximation:

$$A_n = A_0. \tag{26}$$

Hence, solutions are bounded for this case up to $O(\varepsilon)$.

3.2. Ω CLOSE TO 0

For this case, the nearness of Ω to zero is expressed as

$$\Omega = \varepsilon\sigma. \tag{27}$$

A similar calculation yields the solvability condition as

$$D_1 A_n + (k_1 \cos\sigma T_1 + k_2 \sin\sigma T_1) A_n = 0, \tag{28}$$

where

$$k_1 = \frac{\Omega v_1 \int_0^1 Y_n' \bar{Y}_n \, \mathrm{d}x}{2\left(\mathrm{i}\omega_n \int_0^1 Y_n \bar{Y}_n \mathrm{d}x + v_0 \int_0^1 Y_n' \bar{Y}_n \, \mathrm{d}x\right)}, \qquad k_2 = \frac{v_1\left(\mathrm{i}\omega_n \int_0^1 Y_n' \bar{Y}_n \, \mathrm{d}x + v_0 \int_0^1 Y_n'' \bar{Y}_n \mathrm{d}x\right)}{\left(\mathrm{i}\omega_n \int_0^1 Y_n \bar{Y}_n \, \mathrm{d}x + v_0 \int_0^1 Y_n' \bar{Y}_n \, \mathrm{d}x\right)}. \tag{29}$$

After solving equation (28), one obtains

$$A_n = A_0 \mathrm{e}^{-(k_1 \sin \sigma T_1/\sigma) + k_2 \cos \sigma T_1/\sigma)}. \tag{30}$$

Since $|\sin \sigma T_1| \leqslant 1$ and $|\cos \sigma T_1| \leqslant 1$, the complex amplitudes are bounded in time. Therefore, there are no instabilities up to this order of approximation.

3.3. Ω CLOSE TO $2\omega_n$

In this case, to represent the nearness of velocity variation frequency to two times one of the natural frequencies, one writes

$$\Omega = 2\omega_n + \varepsilon\sigma, \tag{31}$$

where σ is a detuning parameter. The solvability condition requires

$$D_1 A_n + k_0 \bar{A}_n \mathrm{e}^{i\sigma T_1} = 0, \tag{32}$$

where k_0 is

$$k_0 = \frac{\left\{\frac{1}{2}(\Omega - 2\omega_n)\int_0^1 \bar{Y}_n' \bar{Y}_n \, \mathrm{d}x - \mathrm{i}v_0 \int_0^1 \bar{Y}_n'' \bar{Y}_n \, \mathrm{d}x\right\}}{2\left\{\mathrm{i}\omega_n \int_0^1 \bar{Y}_n Y_n \mathrm{d}x + v_0 \int_0^1 \bar{Y}_n Y_n' \, \mathrm{d}x\right\}} \, v_1, \tag{33}$$

To perform a stability analysis, one introduces the transformation

$$A_n = B_n \mathrm{e}^{i\sigma T_1/2}. \tag{34}$$

B_n can be written in real and imaginary parts in the form

$$B_n = (b_n^R + \mathrm{i}b_n^I)\mathrm{e}^{\lambda T_1}. \tag{35}$$

Substituting equation (34) into equation (32) and equation (35) into the resulting equation, and separating real and imaginary parts, one obtains the matrix equation

$$\begin{bmatrix} (\lambda + k_{0R}) & \left(k_{0I} - \dfrac{\sigma}{2}\right) \\ \left(k_{0I} + \dfrac{\sigma}{2}\right) & (\lambda - k_{0R}) \end{bmatrix} \begin{Bmatrix} b_n^R \\ b_n^I \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{36}$$

where k_{0R} and k_{0I} are the real and imaginary parts of k_0 defined in equation (33). For a non-trivial solution ($b_n^R \neq 0$, $b_n^I \neq 0$), the determinant of the coefficient matrix must be zero:

$$\lambda^2 = -(\sigma^2/4) + k_{0R}^2 + k_{0I}^2. \tag{37}$$

Two roots are obtained from equation (37):

$$\lambda_{1,2} = \mp\sqrt{-(\sigma^2/4) + k_{0R}^2 + k_{0I}^2}. \tag{38}$$

For

$$-\sqrt{k_{0R}^2 + k_{0I}^2} < \sigma/2 < \sqrt{k_{0R}^2 + k_{0I}^2} \tag{39}$$

the response is unstable whereas it is stable outside this region. Hence the stability boundaries are determined by

$$\sigma_{1,2} = \mp 2\sqrt{k_{0R}^2 + k_{0I}^2}. \tag{40}$$

Inserting σ further into equation (31) gives the final result for the frequencies:

$$\Omega = 2\omega_n \mp 2\varepsilon\sqrt{k_{0R}^2 + k_{0I}^2}. \tag{41}$$

The two values of Ω denote the stability boundaries for small ε. For a beam with constant velocity, $v_1 = 0$ and hence $k_0 = 0$ from equation (33) and no instabilities arise for this case. When the amplitude of fluctuations v_1 increases, the stability regions widen.

4. COMBINATION RESONANCES

In this section, it is assumed that there are two dominant modes. Two cases are significant. The velocity variation frequency may either be nearly equal to the sum of any two modes or to the difference of any two modes.

4.1. COMBINATION RESONANCES OF SUM TYPE

Upon taking two dominant modes (i.e. the nth and mth modes)

$$\Omega = \omega_m + \omega_n + \varepsilon\sigma, \tag{42}$$

the solution can be written at O(1) as

$$w_0(x, T_0, T_1) = A_n(T_1)\mathrm{e}^{i\omega_n T_0} Y_n(x) + A_m(T_1)\mathrm{e}^{i\omega_m T_0} Y_m(x) + cc, \tag{43}$$

where Y_n is defined in equation (19). Substituting equation (43) into equation (8) and using relation (42), one obtains

$$
\begin{aligned}
&D_0^2 w_1 + 2v_0 D_0 w_1' + (v_0^2 - 1) w_1'' + v_f^2 w_1^{iv} \\
&\quad = \{ -2D_1 A_n (\mathrm{i}\omega_n Y_n + v_0 Y_n') \\
&\qquad + ((\omega_m - \tfrac{1}{2}\Omega)\bar{Y}_m' + (\mathrm{i}v_0 \bar{Y}_m'')\bar{A}_m v_1 \mathrm{e}^{i\sigma T_1} \} \mathrm{e}^{i\omega_n T_0} \\
&\qquad + \{ -2D_1 A_m (\mathrm{i}\omega_m Y_m + v_0 Y_m') \\
&\qquad + ((\omega_n - \tfrac{1}{2}\Omega)\bar{Y}_n' + \mathrm{i}v_0 \bar{Y}_n'')\bar{A}_n v_1 \mathrm{e}^{i\sigma T_1} \} \mathrm{e}^{i\omega_m T_0} + cc + NST.
\end{aligned} \tag{44}
$$

Taking the solution for O(ε) as

$$
w_1(x, T_0, T_1) = \phi_n(x, T_1)\mathrm{e}^{i\omega_n T_0} + \phi_m(x, T_1)\mathrm{e}^{i\omega_m T_0} + W(x, T_0, T_1) + cc, \tag{45}
$$

where the first two terms are related to secular terms and the third term is for non-secular terms. Substituting equation (45) into equation (44) and separating the solutions for nth and mth modes, one obtains

$$
\begin{aligned}
&v_f^2 \phi_n^{iv} + (v_0^2 - 1)\phi_n'' + 2\mathrm{i}v_0\omega_n\phi_n' - \omega_n^2\phi_n \\
&\quad = -2D_1 A_n(\mathrm{i}\omega_n Y_n + v_0 Y_n') + \{(\omega_m - \tfrac{1}{2}\Omega)\bar{Y}_m' + \mathrm{i}v_0 \bar{Y}_m''\}\bar{A}_m v_1 \mathrm{e}^{i\sigma T_1},
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
&v_f^2 \phi_m^{iv} + (v_0^2 - 1)\phi_m'' + 2\mathrm{i}v_0\omega_m\phi_m' - \omega_m^2\phi_m \\
&\quad = -2D_1 A_m(\mathrm{i}\omega_m Y_m + v_0 Y_m') + \{(\omega_n - \tfrac{1}{2}\Omega)\bar{Y}_n' + \mathrm{i}v_0 \bar{Y}_n''\}\bar{A}_n v_1 \mathrm{e}^{i\sigma T_1}.
\end{aligned} \tag{47}
$$

These two equations can be solved in a way similar to that for the principal parametric resonance case and the complex amplitude modulation equations are obtained as

$$
D_1 A_n + k_3 \bar{A}_m \mathrm{e}^{i\sigma T_1} = 0, \qquad D_1 A_m + k_4 \bar{A}_n \mathrm{e}^{i\sigma T_1} = 0, \tag{48, 49}
$$

where

$$
k_3 = -\frac{\left\{(\omega_m - \tfrac{1}{2}\Omega)\displaystyle\int_0^1 \bar{Y}_m' \bar{Y}_n \,\mathrm{d}x + \mathrm{i}v_0 \int_0^1 \bar{Y}_m'' \bar{Y}_n \,\mathrm{d}x\right\} v_1}{2\left(\mathrm{i}\omega_n \displaystyle\int_0^1 Y_n \bar{Y}_n \,\mathrm{d}x + v_0 \int_0^1 Y_n' \bar{Y}_n \,\mathrm{d}x\right)}, \tag{50}
$$

$$
k_4 = -\frac{\left\{(\omega_n - \tfrac{1}{2}\Omega)\displaystyle\int_0^1 \bar{Y}_n' \bar{Y}_m \,\mathrm{d}x + \mathrm{i}v_0 \int_0^1 \bar{Y}_n'' \bar{Y}_m \,\mathrm{d}x\right\} v_1}{2\left(\mathrm{i}\omega_m \displaystyle\int_0^1 Y_m \bar{Y}_m \,\mathrm{d}x + v_0 \int_0^1 Y_m' \bar{Y}_m \,\mathrm{d}x\right)}. \tag{51}
$$

For equations (48) and (49), one introduces the transformations

$$A_n = B_n e^{i\sigma T_1/2}, \qquad A_m = B_m e^{i\sigma T_1/2} \tag{52}$$

and obtains

$$D_1 B_n + \mathrm{i}\frac{\sigma}{2} B_n + k_3 \bar{B}_m = 0, \qquad D_1 B_m + \mathrm{i}\frac{\sigma}{2} B_m + k_4 \bar{B}_n = 0. \tag{53, 54}$$

The stability may be directly determined from the above equations. One assumes that equations (53) and (54) possess solutions of the form

$$B_n = b_n e^{\lambda T_1}, \qquad B_m = b_m e^{\bar{\lambda} T_1}, \tag{55}$$

where b_n and b_m are real. Substituting (55) into the equations (53) and (54), taking the complex conjugate of the second equation, one obtains for non-trivial solutions

$$\lambda = \mp\sqrt{-(\sigma^2/4) + k_3 \bar{k}_4}. \tag{56}$$

Using definitions (50) and (51), one may show that $k_3\bar{k}_4$ is always real. If $k_3\bar{k}_4 < 0$, then the system is stable everywhere and if $k_3\bar{k}_4 \geqslant 0$ then the stability boundaries are determined by the lines

$$\sigma = \mp 2\sqrt{k_3 \bar{k}_4}, \tag{57}$$

or

$$\Omega = \omega_m + \omega_n \mp 2\varepsilon\sqrt{k_3 \bar{k}_4}. \tag{58}$$

Numerical treatment of the above equation is given in the next section.

4.2. COMBINATION RESONANCES OF DIFFERENCE TYPE

Upon assuming $m > n$ without loss of generality, the nearness of frequency to the difference of mth and nth modes is expressed as

$$\Omega = \omega_m - \omega_n + \varepsilon\sigma. \tag{59}$$

A calculation similar to that for the sum-type resonances yields the complex amplitude equations

$$D_1 A_n + k_5 A_m e^{-i\sigma T_1} = 0, \qquad D_1 A_m + k_6 A_n e^{i\sigma T_1} = 0, \tag{60, 61}$$

where

$$k_5 = -\frac{\left\{(\omega_m - \tfrac{1}{2}\Omega)\int_0^1 Y_m' \bar{Y}_n \,\mathrm{d}x - \mathrm{i}v_0 \int_0^1 Y_m'' \bar{Y}_n \,\mathrm{d}x\right\} v_1}{2\left(\mathrm{i}\omega_n \int_0^1 Y_n \bar{Y}_n \,\mathrm{d}x + v_0 \int_0^1 Y_n' \bar{Y}_n \,\mathrm{d}x\right)}, \tag{62}$$

$$k_6 = -\frac{\left\{-(\omega_n + \tfrac{1}{2}\Omega)\int_0^1 Y_n' \bar{Y}_m \,\mathrm{d}x + \mathrm{i}v_0 \int_0^1 Y_n'' \bar{Y}_m \,\mathrm{d}x\right\} v_1}{2\left(\mathrm{i}\omega_m \int_0^1 Y_m \bar{Y}_m \,\mathrm{d}x + v_0 \int_0^1 Y_m' \bar{Y}_m \,\mathrm{d}x\right)}, \tag{63}$$

A suitable transformation for this case is

$$A_n = B_n \mathrm{e}^{-i\sigma T_1/2}, \qquad A_m = B_m \mathrm{e}^{i\sigma T_1/2}. \tag{64}$$

Substituting into the equations yields

$$D_1 B_n - \mathrm{i}\frac{\sigma}{2} B_n + k_5 B_m = 0, \qquad D_1 B_m + \mathrm{i}\frac{\sigma}{2} B_m + k_6 B_n = 0. \tag{65, 66}$$

One assumes solutions of the form

$$B_n = b_n \mathrm{e}^{\lambda T_1}, \qquad B_m = b_m \mathrm{e}^{\lambda T_1}, \tag{67}$$

For non-trivial solutions,

$$\lambda = \mp\sqrt{-(\sigma^2/4) + k_5 k_6}. \tag{68}$$

From the definitions given in equations (62) and (63) it can be shown that $k_5 k_6$ is always a negative real number,

$$k_5 k_6 < 0. \tag{69}$$

In equation (68), the region inside of the root is always negative. λ is then a pure imaginary number denoting that the solutions are always bounded. In conclusion, for difference-type combination resonances, no instabilities arise up to O(ε).

5. NUMERICAL EXAMPLES

In this section, numerical plots for the natural frequencies and stability borders will be presented.

Natural frequencies are found by solving equations (13) and (15) simultaneously and plotted in Figures 2 and 3 for the first and second modes respectively for six different flexural stiffness values. As seen, with increasing mean velocity, natural frequencies decrease. Increasing flexural stiffness (introducing beam effects) on the other hand causes an increase in natural frequencies.

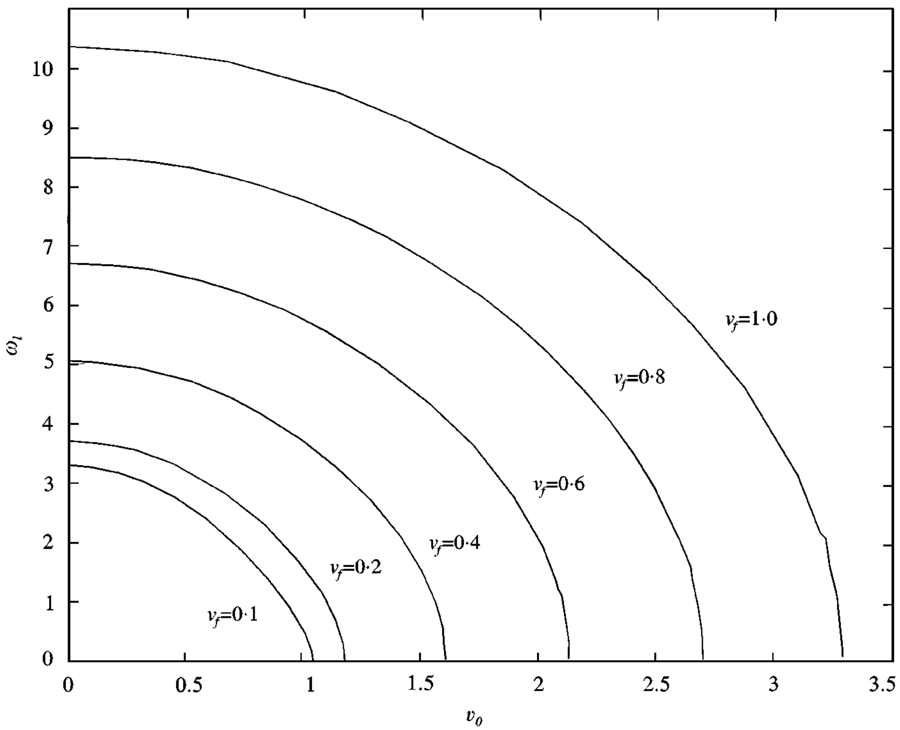

Figure 2. Comparisons of first natural frequency values for different flexural stiffnesses.

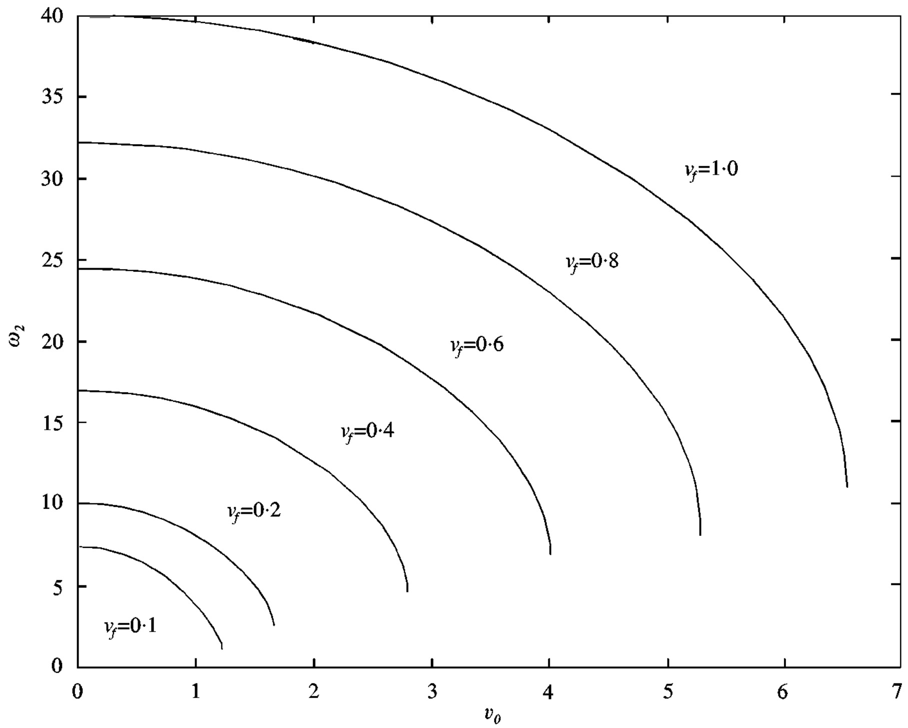

Figure 3. Comparisons of second natural frequency values for different flexural stiffnesses.

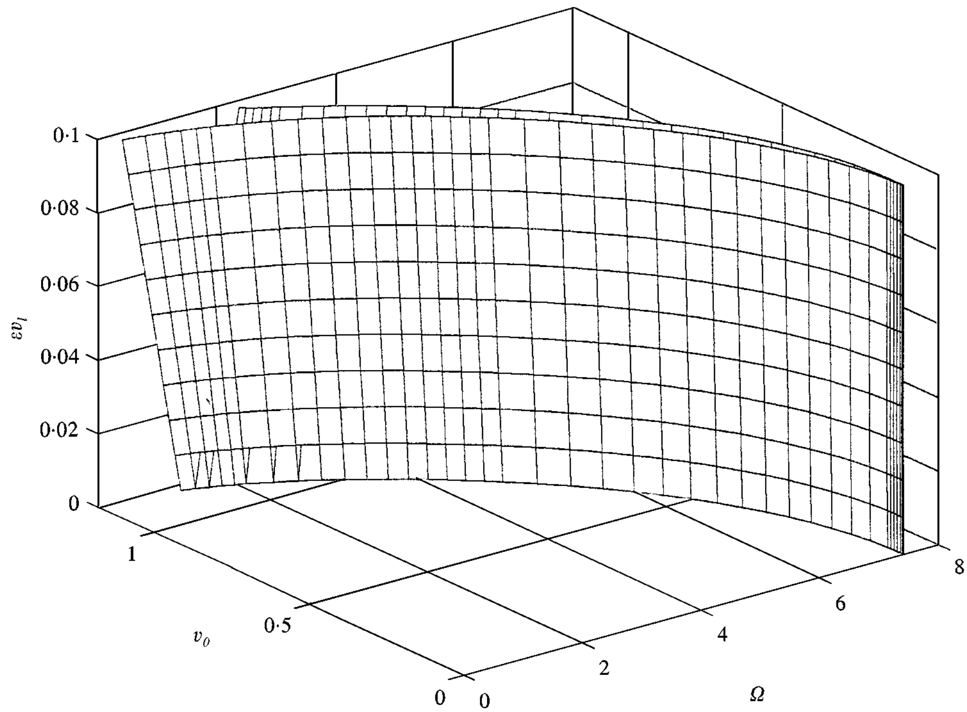

Figure 4. Stable and unstable regions for principle parametric resonances for the first mode ($n = 1$, $v_f = 0{\cdot}2$).

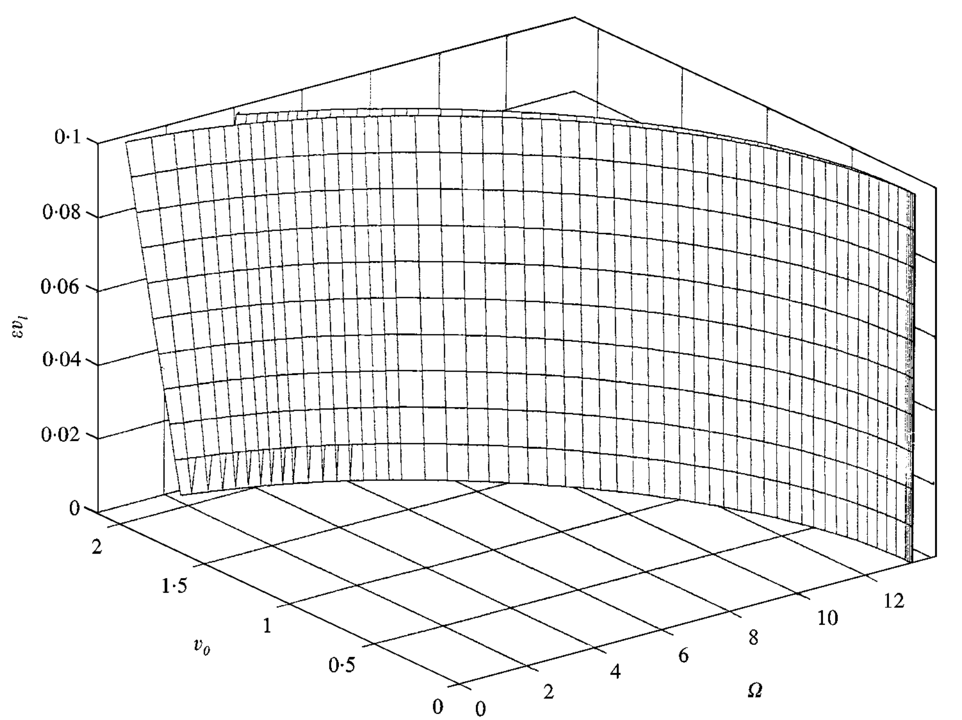

Figure 5. Stable and unstable regions for principle parametric resonances for the first mode ($n = 1$, $v_f = 0{\cdot}6$).

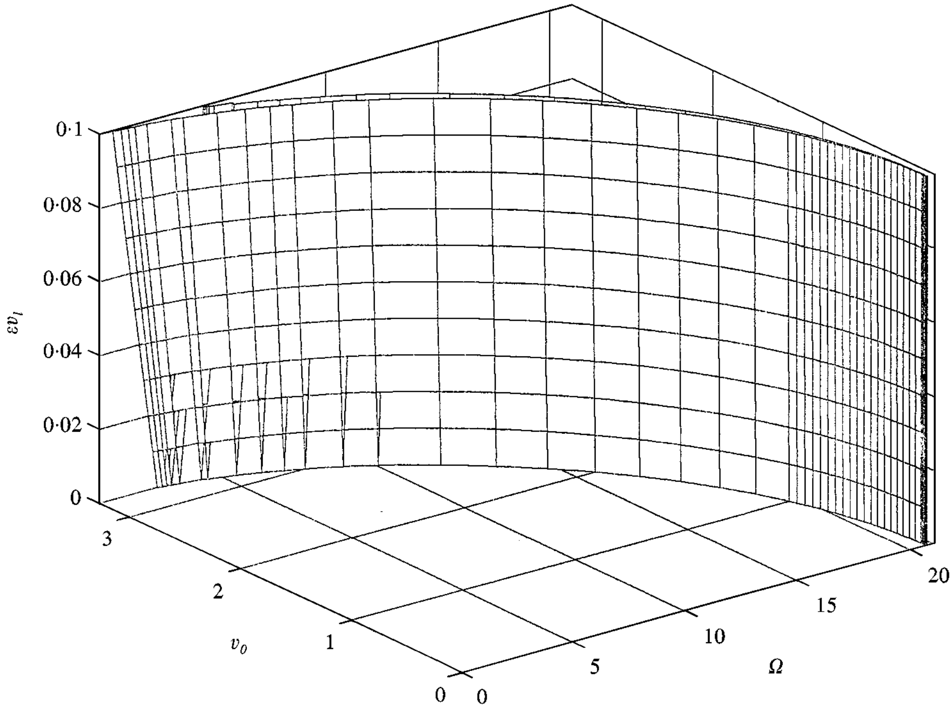

Figure 6. Stable and unstable regions for principle parametric resonances for the first mode ($n = 1$, $v_f = 1{\cdot}0$).

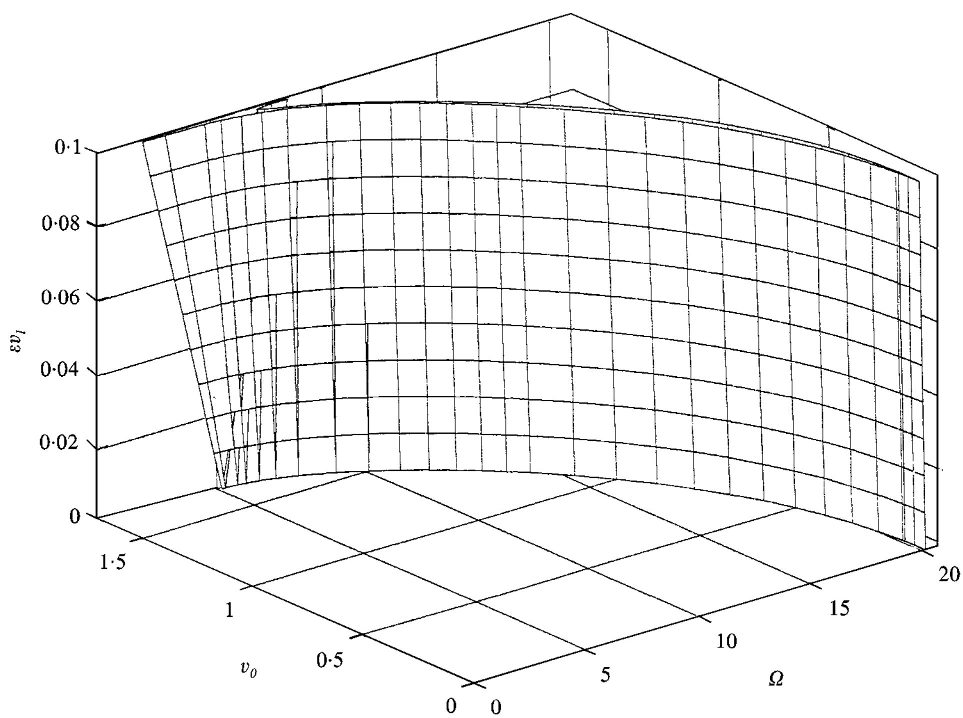

Figure 7. Stable and unstable regions for principle parametric resonances for the second mode ($n = 2$, $v_f = 0{\cdot}2$).

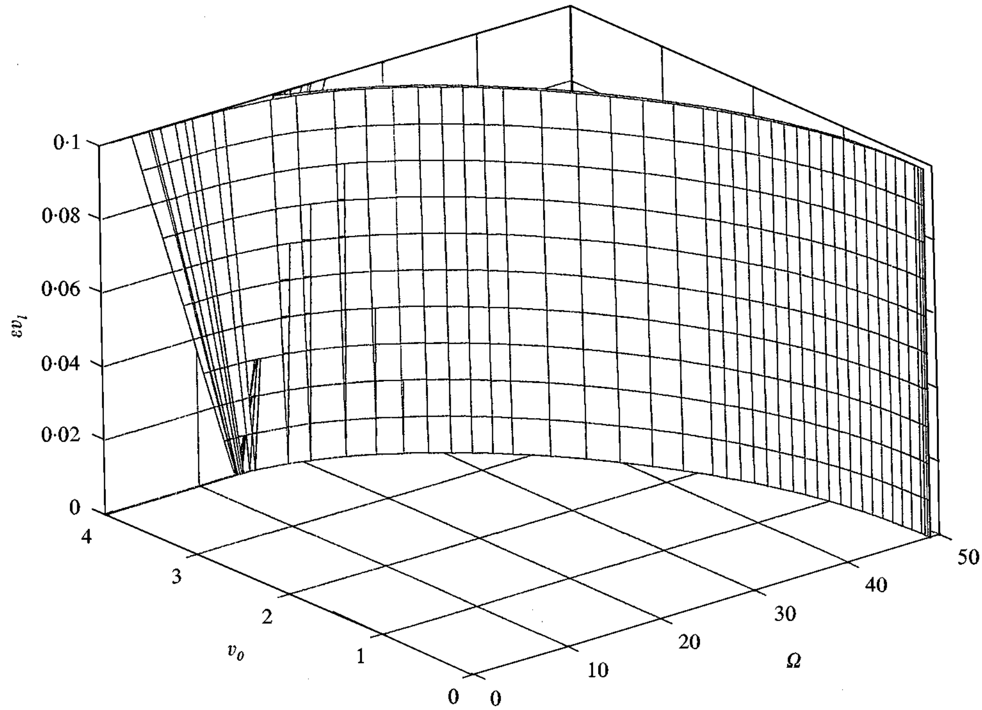

Figure 8. Stable and unstable regions for principle parametric resonances for the second mode ($n = 2$, $v_f = 0{\cdot}6$).

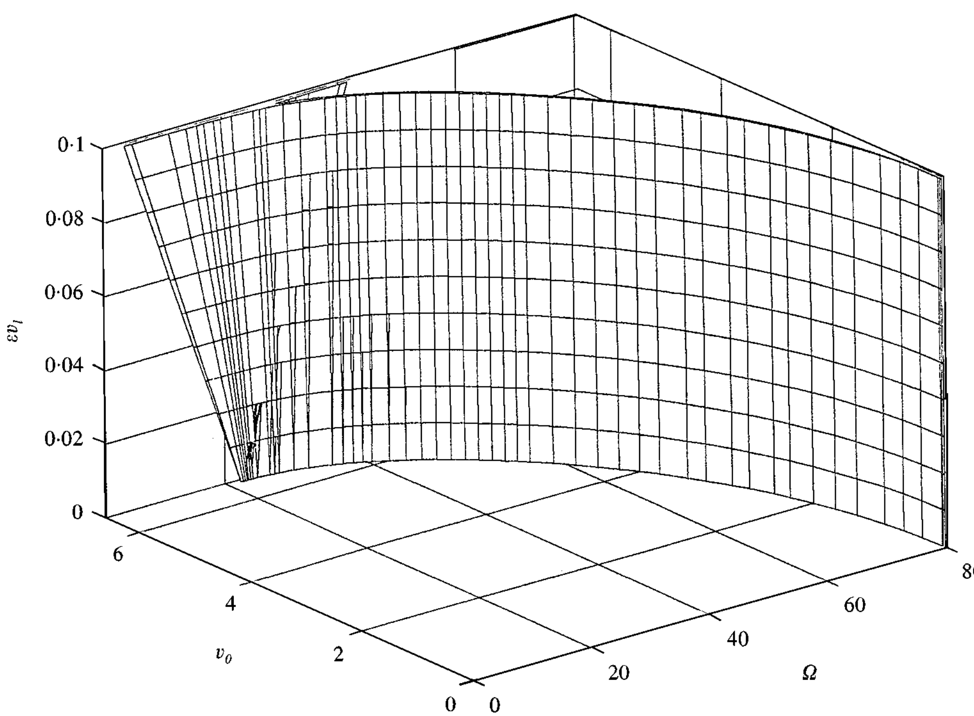

Figure 9. Stable and unstable regions for principle parametric resonances for the second mode ($n = 2$, $v_f = 1{\cdot}0$).

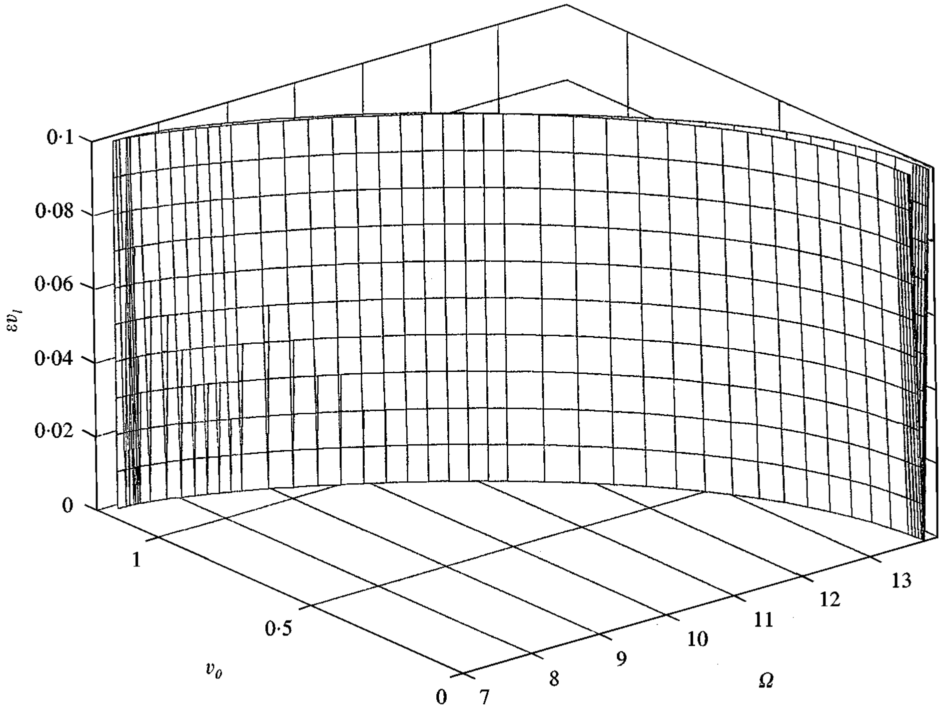

Figure 10. Stable and unstable regions for combination resonances of sum type for the first two modes ($n = 1$, $m = 2$, $v_f = 0{\cdot}2$).

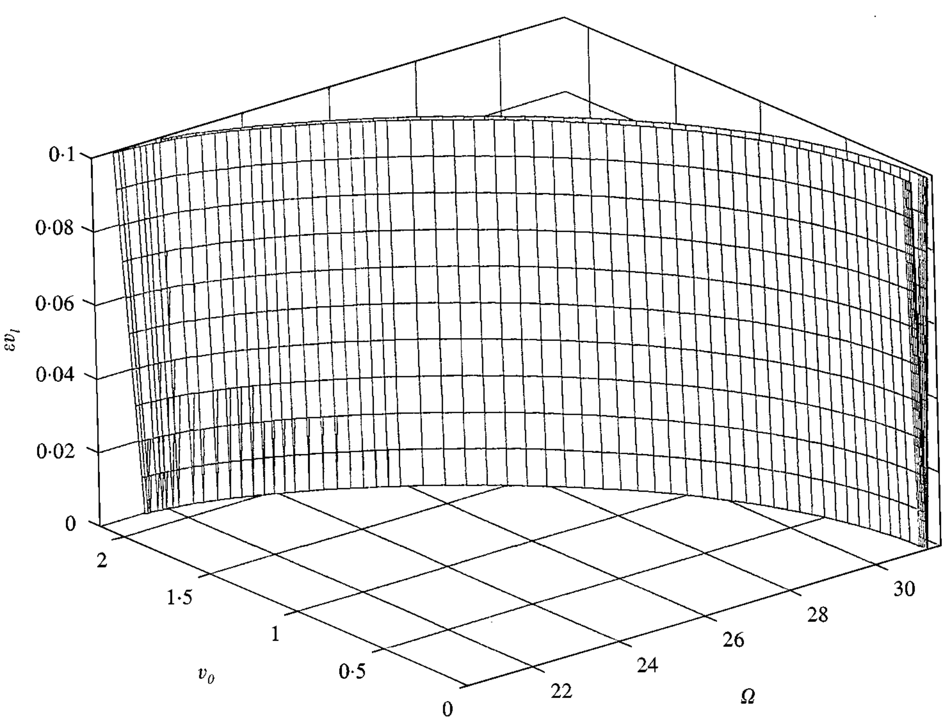

Figure 11. Stable and unstable regions for combination resonances of sum type for the first two modes ($n = 1$, $m = 2$, $v_f = 0{\cdot}6$).

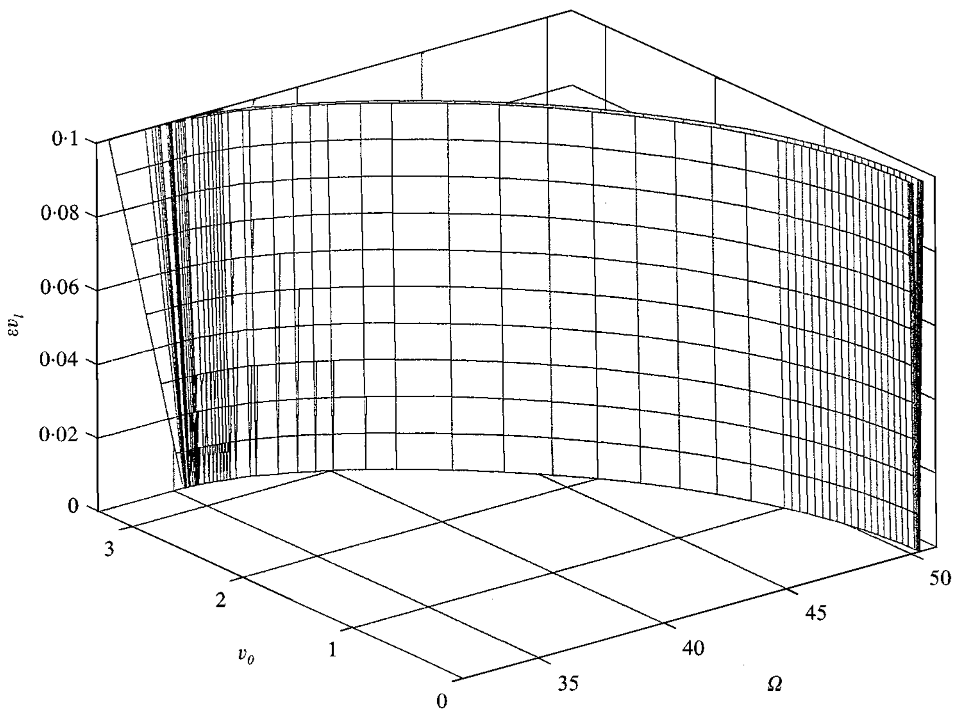

Figure 12. Stable and unstable regions for combination resonances of sum type for the first two modes ($n = 1$, $m = 2$, $v_f = 1{\cdot}0$).

In Figures 4–6, stable and unstable regions are plotted for principle parametric resonances case for the first mode for three different flexural stiffness values ($v_f = 0{\cdot}2$, $0{\cdot}6$ and 1). In all figures (Figures 4–12), the regions in between the planar surfaces are unstable whereas the remaining regions are stable. With increasing flexural stiffness value, the stability regions shift to higher Ω values. Increasing the velocity variation amplitude enlarges the stability regions. In Figures 7–9, stable and unstable regions are given for the second mode. Similar conclusions can be drawn, as written for Figures 4–6.

In Figures 10–12, stable and unstable regions are plotted for combination resonances of sum-type case for the first two modes for three flexural stiffness values. As the flexural stiffness value increases, the stability regions shift to higher Ω values. Increasing the velocity variation amplitude causes the stability regions to be wider.

6. CONCLUDING REMARKS

In this study, the vibrations of an axially moving Euler–Bernoulli beam has been investigated. The velocity is assumed to be harmonically changing about a mean value. The method of multiple scales is applied to the equation of motion. Velocity-dependent natural frequencies are found by using a standard root-finding algorithm for different flexural stiffnesses for the first two modes.

The influence of small fluctuations of velocity on the stability of the system is investigated. The boundaries separating stable and unstable regions are calculated. Principal parametric resonances and combination resonances of sum and difference type for any two modes are considered in the analysis. It is found that for velocity fluctuation frequencies near twice any natural frequency, an instability region occurs whereas for the frequencies close to zero, no instabilities are detected up to the first order of approximation. For sum-type combination resonances, instabilities do occur. On the contrary, no instabilities are detected for difference-type resonances up to the first order of approximation. Boundaries separating stable and unstable regions are plotted for principle parametric and combination resonances of sum type. It is shown that beam effects cause the stability boundaries to shift to higher frequency values and increasing the velocity variation amplitude causes the stability regions to be wider.

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