



AN IMPROVED SUBSPACE ITERATION METHOD WITH SHIFT FOR STRUCTURES WITH MULTIPLE NATURAL FREQUENCIES

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An efficient and numerically stable eigensolution method for structures with multiple natural frequencies is presented. The proposed method is developed by improving the well-known subspace iteration method with shift. A major difficulty of the subspace iteration method with shift is that because of singularity problem, a shift close to an eigenvalue cannot be used, resulting in slower convergence. In this paper, the above singularity problem has been solved by introducing side conditions without sacrifice of convergence. The proposed method is always non-singular even if a shift is on a distinct eigenvalue or multiple ones. This is one of the significant characteristics of the proposed method. The non-singularity is proved analytically. The convergence of the proposed method is at least equal to that of the subspace iteration method with shift, and the operation counts of the above two methods are almost the same when a large number of eigenpairs are required. To show the effectiveness of the proposed method, two numerical examples are considered.

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1. INTRODUCTION

The eigensolution method is very important in a dynamic analysis of structures when the mode superposition method is used. Many eigensolution methods have been developed, and among these methods, the subspace iteration method has hitherto been known to be very efficient, and so has been widely used.

The subspace iteration method was developed and named by Bathe [1, 2]. This method combines the simultaneous inverse iteration method and the Rayleigh–Ritz analysis. The following shortcomings have been identified after extensive use of the method [3]. These include: (1) slow convergence, large computational and storage costs when a relatively large number of eigenpairs are required; (2) significant computational effort required to form and solve the subspace eigenproblem when a large number of eigenpairs are required; and (3) missed eigenvectors caused by a poor choice of starting trial vectors.

To overcome the above shortcomings, many researchers have studied a variety of acceleration procedures of the subspace iteration method. The techniques employed include Chebyshev polynomials [4], over-relaxation method [5, 6],

shifting technique [7], exploitation by partitioning a large structure into a number of substructures [8], improving the selection initial vectors [9], selective repeated inverse iteration and multiple inverse iteration [10, 11], and subspace iteration by omitting some of the Rayleigh–Ritz procedure from certain iteration steps [12, 13].

Among the above accelerated techniques, a shifting technique is well known and effectively used in the commercial FEM programs such as ADINA [14]. However, since the singularity may occur during the use of the shifting technique in the accelerated scheme such as the subspace iteration method with shift, the shift must be carefully chosen to avoid the singularity. It is a significant disadvantage of the subspace iteration method with shift.

Jung *et al.* [15] have developed a method that always guarantees the numerical stability and maintains the convergence rate of the subspace iteration method with shift even if it is an exact eigenvalue itself. However, the method can only be applied to the structures with distinct natural frequencies. If a structure with multiple natural frequencies is analyzed by the method, the singularity problem may still occur.

In this paper, when the eigenvalue analysis for a structure with multiple eigenvalues is performed, an eigensolution technique that always guarantees the numerical stability is developed by improving the method of Jung *et al.* [15]. That is, the proposed method is always numerically stable even if a shift is on a distinct eigenvalue or multiple ones.

The subspace iteration method with shift is briefly reviewed in the next section. Section 3 includes the theory, the proof of the numerical stability, the convergence analysis and the operation counts of the proposed method. The effectiveness of the proposed method is verified by the results of numerical examples in Section 4. Section 5 has concluding remarks.

2. SUBSPACE ITERATION METHOD WITH SHIFT

The eigenproblem of the structural dynamics may be written as follows [16]:

$$KX = MXA, \quad (1)$$

where K and M are the stiffness and mass matrices of the structure of the order n , respectively, the columns of X the eigenvectors, and A a diagonal matrix with eigenvalues on its diagonal.

Applying a shift μ to equation (1) gives

$$(K - \mu M)X = MX\Omega, \quad (2)$$

where

$$\Omega = A - \mu I \quad (3)$$

and I is the unit matrix.

Suppose that the p smallest eigenvalues λ_i ($i = 1, 2, \dots, p$) and corresponding eigenvectors x_i are required. For faster convergence, q trial vectors are normally used with $q = \min\{2p, p + 8\}$. Then, the algorithm of the subspace iteration

method with the shift μ can be described as follows:

Step 1. Find the eigenvector approximations $\bar{X}^{(k+1)}$ by the simultaneous inverse iteration method:

$$(K - \mu M)\bar{X}^{(k+1)} = MX^{(k)}, \quad (4)$$

where $\bar{X}^{(k+1)}$ and $X^{(k)}$ are the $(n \times q)$ matrices.

Step 2. Compute the projections of the matrices $(K - \mu M)$ and M :

$$\bar{K}^{(k+1)} = \bar{X}^{(k+1)\top}(K - \mu M)\bar{X}^{(k+1)}, \quad (5)$$

$$\bar{M}^{(k+1)} = \bar{X}^{(k+1)\top}M\bar{X}^{(k+1)}, \quad (6)$$

where $\bar{K}^{(k+1)}$ and $\bar{M}^{(k+1)}$ are the $(q \times q)$ symmetric matrices.

Step 3. Solve the eigenproblem of the reduced order q :

$$\bar{K}^{(k+1)}Q^{(k+1)} = \bar{M}^{(k+1)}Q^{(k+1)}\Omega^{(k+1)}, \quad (7)$$

where $Q^{(k+1)}$ and $\Omega^{(k+1)}$ are the $(q \times q)$ matrices.

Step 4. Find the improved eigenvectors $X^{(k+1)}$:

$$X^{(k+1)} = \bar{X}^{(k+1)}Q^{(k+1)} \quad (8)$$

and the improved eigenvalues $\Lambda^{(k+1)}$:

$$\Lambda^{(k+1)} = \Omega^{(k+1)} + \mu I. \quad (9)$$

$\Lambda^{(k+1)}$ converges to Λ and $X^{(k+1)}$ converges to X as k approaches infinity. The convergence rate of the subspace iteration method with shift is

$$(\lambda_j - \mu)/(\lambda_{q+1} - \mu). \quad (10)$$

If a shift is an eigenvalue itself or very close to it, the iteration procedure becomes unstable because of the singularity problem occurring during the LDL^T factorization process of the coefficient matrix. To avoid this singularity problem, that is, to guarantee the numerical stability of the subspace iteration method with shift, the following condition was adopted in the subspace iteration method [7];

$$1.01\bar{\lambda}_{s-1} \leq \mu \leq 0.99\bar{\lambda}_s, \quad (11)$$

where $\bar{\lambda}_{s-1}$ is the calculated approximation to the $(s-1)$ th eigenvalue and $\bar{\lambda}_s$ the s th eigenvalue.

It means that a shift must be within a limited region resulting in slow convergence. Moreover, if the calculated approximation to an eigenvalue slightly differs from it, an eigenvalue may be inside the limited region. Then, the singularity may occur although a shift is inside the limited region. These are the significant disadvantages of the subspace iteration method with shift. The purpose of this paper is to remove the limitation in equation (11) for choosing the value of a shift μ .

3. PROPOSED METHOD

3.1. THEORY

Consider the simultaneous inverse iteration step in the subspace iteration method with shift:

$$(K - \mu M)\bar{X}^{(k+1)} = MX^{(k)}. \tag{12}$$

If a shift μ is very close to an eigenvalue, the singularity problem occurs during the LDL^T factorization process of the coefficient matrix $(K - \mu M)$ in equation (12). Then, the $(k + 1)$ th eigenvector approximations $\bar{X}^{(k+1)}$ cannot be acquired, and so the iteration procedure cannot be performed any more. This is a significant disadvantage of the subspace iteration method with shift.

Jung *et al.* [15] proposed the numerically stable eigensolution method. However, the method can only be applied to the structures with distinct eigenvalues. If structures with multiple eigenvalues are analyzed by the method, the singularity problem may still occur.

In this paper, to solve the above singularity problem that may occur in the case of structures with multiple eigenvalues the following procedures are proposed. First, let us consider a shift close to multiple eigenvalues. To simplify the notation in this discussion, assume that the multiplicity of the lowest eigenvalues is s , that is, $\lambda_1 = \lambda_2 = \dots = \lambda_s$. Then, the inverse iteration step on the multiple eigenvalues can be expressed as follows:

$$(K - \mu M)\bar{X}_s^{(k+1)} = MX_s^{(k)}D_s^{(k+1)}, \tag{13}$$

where the $(n \times s)$ matrices $X_s^{(k)} = [x_1^{(k)}, x_2^{(k)}, \dots, x_s^{(k)}]$, $\bar{X}_s^{(k+1)} = [\bar{x}_1^{(k+1)}, \bar{x}_2^{(k+1)}, \dots, \bar{x}_s^{(k+1)}]$, the $(s \times s)$ matrix $D_s^{(k+1)} = \text{diag}(d_{11}^{(k+1)}, d_{22}^{(k+1)}, \dots, d_{ss}^{(k+1)})$, and the scalar $d_{ii}^{(k+1)}$ controls the length of the vector $\bar{x}_i^{(k+1)}$.

Because there are only $(n \times s)$ equations with $((n + 1) \times s)$ unknowns, $(n \times s)$ components of $\bar{X}_s^{(k+1)}$ and s components of $d_{ii}^{(k+1)}$, in equation (13), s side conditions must be introduced for the solution of equation (13). These conditions are that the current vector set $(X_s^{(k)})$ is orthogonal to the incremental vector set $(\Delta X_s^{(k)})$ with respect to M ; that is,

$$X_s^{(k)T} M \Delta X_s^{(k)} = 0. \tag{14}$$

Adding the mass orthonormality relation, $X_s^{(k)T} M X_s^{(k)} = I_s$, to the side conditions, equation (14) yields

$$X_s^{(k)T} M \bar{X}_s^{(k+1)} = I_s, \tag{15}$$

where

$$\bar{X}_s^{(k+1)} = X_s^{(k)} + \Delta X_s^{(k)}. \tag{16}$$

The inverse iteration step on the other eigenvalues makes use of equation (12); that is,

$$(K - \mu M)\bar{X}_{q-s}^{(k+1)} = MX_{q-s}^{(k)}, \tag{17}$$

where

$$X_{q-s}^{(k)} = [x_{s+1}^{(k)}, x_{s+2}^{(k)}, \dots, x_q^{(k)}]. \quad (18)$$

Writing equations (13), (15) and (17) in matrix form gives

$$\begin{bmatrix} K - \mu M & MX_s^{(k)} \\ X_s^{(k)\top} M & 0 \end{bmatrix} \begin{bmatrix} \bar{X}^{(k+1)} \\ \bar{D}^{(k+1)} \end{bmatrix} = \begin{bmatrix} MX^{(k)} \\ E \end{bmatrix}, \quad (19)$$

where the unknown $(s \times q)$ matrix $\bar{D}^{(k+1)} = [\bar{D}_s^{(k+1)}, 0, \dots, 0]$ and the $(s \times q)$ matrix $E = [I_s, 0, \dots, 0]$.

Note that $\bar{X}^{(k+1)}$ from equation (19) is used for $\bar{X}^{(k+1)}$ in equations (5) and (6) instead of $\bar{X}^{(k+1)}$ in equation (4). Equation (19) is the main linear algebraic equation used in the proposed method.

The coefficient matrix of equation (19) is of the order $(n + s)$, symmetric, and non-singular. The non-singularity is one of the significant advantages of the proposed method and will be shown in the next section.

The proposed method can be applied to practical problems as follows. After assuming that a shift is very close to a distinct eigenvalue, one performs the factorizing process of the coefficient matrix with one side condition. If the shift is not very close to a distinct eigenvalue, but multiple ones (multiplicity = m), the $(n - m)$ th pivot element in the factorizing process of the coefficient matrix usually becomes small compared with its original value and the coefficient matrix becomes singular. To avoid the singularity, the $(m - 1)$ side conditions are added, and then the factorizing process of the coefficient matrix is continued. Since the storage scheme of the proposed method is the skyline algorithm, the extra operation number due to the $(m - 1)$ additional side conditions is small compared with the total operation number of the factorizing process of the coefficient matrix.

3.2. PROOF OF THE NON-SINGULARITY OF THE COEFFICIENT MATRIX [17-19]

The most remarkable characteristic of the proposed method is that the non-singularity is always guaranteed. Let the coefficient matrix of equation (19) be denoted by C , that is,

$$C = \begin{bmatrix} K - \mu M & MX_s^{(k)} \\ X_s^{(k)\top} M & 0 \end{bmatrix}. \quad (20)$$

If C is non-singular when the shift μ becomes multiple eigenvalues, that is, $\mu \cong \lambda_1 = \dots = \lambda_s$, it will also be non-singular for a non-close shift. The resulting C^* will be

$$C^* = \begin{bmatrix} K - \lambda_s M & MX_s \\ X_s^\top M & 0 \end{bmatrix}. \quad (21)$$

The non-singularity of the proposed method is, therefore, proved by introducing the new eigenproblem of the resulting matrix such as

$$C^*Y = M^*YD, \tag{22}$$

where D and Y are the eigenvalue and the associate eigenvector matrices of the new eigenproblem, respectively, and

$$M^* = \begin{bmatrix} M & 0 \\ 0 & I_s \end{bmatrix}, \tag{23}$$

$$Y = [y_1 \ y_2 \ \cdots \ y_{n+s}] \quad \text{and} \quad D = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_{n+s}). \tag{24, 25}$$

The eigenpairs of the eigenproblem equation (22), y_j and γ_j for $j = 1, 2, \dots, n + s$, are as follows:

$$\text{Eigenvector } y_j: \begin{cases} \begin{Bmatrix} x_i \\ e_i \end{Bmatrix}, & \begin{Bmatrix} x_i \\ -e_i \end{Bmatrix}, & \begin{Bmatrix} x_k \\ 0 \end{Bmatrix}, \quad i = 1, 2, \dots, s; k = s + 1, s + 2, \dots, n, \end{cases} \tag{26}$$

$$\text{Eigenvalue } \gamma_j: \begin{cases} 1, \dots, 1 \ (s), \\ -1, \dots, -1 \ (s), & k = s + 1, s + 2, \dots, n, \\ \lambda_k - \lambda_s \ (n - s), \end{cases} \tag{27}$$

where λ_i and x_i are the eigenvalues and eigenvectors of the system $KX = MXA$, and e_i is the $(s \times 1)$ vector of which all elements are zero except for the i th element with unity.

By considering the determinant of equation (22), the relationship can be obtained as follows:

$$\begin{aligned} \det[C^*] &= \det[M^*] \det[D] \\ &= (-1)^s \det[M] \prod_{k=s+1}^n (\lambda_k - \lambda_s). \end{aligned} \tag{28}$$

The determinant of C^* is not zero because of $\det[M] \neq 0$ by definition. The non-singularity of the coefficient matrix in equation (19) is shown. That is, the numerical stability of the proposed method is proved analytically. The proposed method, therefore, has an advantage over the subspace iteration method with shift in that no limited regions are needed in the former.

3.3. CONVERGENCE ANALYSIS

To analyze the convergence characteristics of the proposed method, we can use the concept of the convergence analysis of the subspace iteration method [20]. The

convergence of the subspace iterations is conveniently studied by first changing the basis from the finite element co-ordinate basis to the basis of the eigenvectors. This change of the basis is achieved using the following relation for the vectors $X^{(k)}$ in equation (13):

$$X^{(k)} = \Phi Z^{(k)}, \tag{29}$$

where Φ is the matrix storing all eigenvectors, $\Phi = [\phi_1, \phi_2, \dots, \phi_n]$. Since Φ is non-singular, there is a unique $Z^{(k)}$ for any $X^{(k)}$, and vice versa.

Introducing the relation of equation (29) to equations (13) and (17), and premultiplying by Φ^T , we obtain the following equations:

$$(A - \mu I_n) \bar{Z}_s^{(k+1)} = Z_s^{(k)} D_s^{(k+1)} \tag{30}$$

and

$$(A - \mu I_n) \bar{Z}_{q-s}^{(k+1)} = Z_{q-s}^{(k)}, \tag{31}$$

where the $(n \times s)$ matrix $\bar{Z}_s^{(k+1)} = [\bar{z}_1^{(k+1)}, \bar{z}_2^{(k+1)}, \dots, \bar{z}_s^{(k+1)}]$ and the $(n \times (q - s))$ matrix $\bar{Z}_{q-s}^{(k+1)} = [\bar{z}_{s+1}^{(k+1)}, \bar{z}_{s+2}^{(k+1)}, \dots, \bar{z}_q^{(k+1)}]$. And then, the equations equivalent to equations (5)–(8), but which express the relations in the new basis, are used to evaluate $Z^{(k+1)}$. The convergence rate of the iteration is established from equations (30) and (31), and using the fact that in the subspace iterations always the optimum approximations to the required eigenvalues and eigenvectors are always calculated.

For the convergence analysis let the iteration matrix $Z^{(k)}$ be denoted as follows:

$$Z^{(k)} = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ 0 & & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & & 0 \\ 0 & & 1 & 0 & & 0 \\ 0 & & 0 & 1 & & 0 \\ 0 & & 0 & 0 & & 0 \\ \vdots & & \vdots & \vdots & & 0 \\ 0 & & 0 & 0 & & 1 \\ z_{q+1,1}^{(k)} & \dots & z_{q+1,s}^{(k)} & z_{q+1,s+1}^{(k)} & \dots & z_{q+1,q}^{(k)} \\ z_{q+2,1}^{(k)} & \dots & z_{q+2,s}^{(k)} & z_{q+2,s+1}^{(k)} & \dots & z_{q+2,q}^{(k)} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{n,1}^{(k)} & \dots & z_{n,s}^{(k)} & z_{n,s+1}^{(k)} & \dots & z_{n,q}^{(k)} \end{bmatrix}, \tag{32}$$

where $Z^{(k)}$ is completely general, because the unit $(q \times q)$ matrix I can always be obtained by linearly combining columns, provided $Z^{(k)}$ is not deficient in the vectors $e_j (j = 1, 2, \dots, q)$, which are vectors of the order n with only zero elements except for the j th location which is unity. Using equations (30) and (31), we then obtain

$$\bar{Z}^{(k+1)} = \left[\begin{array}{cccccc} \frac{d_{11}^{(k+1)}}{\lambda_1 - \mu} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & & 0 \\ 0 & & \frac{d_{ss}^{(k+1)}}{\lambda_s - \mu} & 0 & & 0 \\ 0 & & 0 & \frac{1}{\lambda_{s+1} - \mu} & & 0 \\ 0 & & 0 & 0 & & 0 \\ \vdots & & \vdots & \vdots & & 0 \\ 0 & & 0 & 0 & & \frac{1}{\lambda_q - \mu} \\ z_{q+1,1}^{(k)} \frac{d_{11}^{(k+1)}}{\lambda_{q+1} - \mu} & \cdots & z_{q+1,s}^{(k)} \frac{d_{ss}^{(k+1)}}{\lambda_{q+1} - \mu} & z_{q+1,s+1}^{(k)} \frac{1}{\lambda_{q+1} - \mu} & \cdots & z_{q+1,q}^{(k)} \frac{1}{\lambda_{q+1} - \mu} \\ z_{q+2,1}^{(k)} \frac{d_{11}^{(k+1)}}{\lambda_{q+2} - \mu} & \cdots & z_{q+2,s}^{(k)} \frac{d_{ss}^{(k+1)}}{\lambda_{q+2} - \mu} & z_{q+2,s+1}^{(k)} \frac{1}{\lambda_{q+2} - \mu} & \cdots & z_{q+2,q}^{(k)} \frac{1}{\lambda_{q+2} - \mu} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{n,1}^{(k)} \frac{d_{11}^{(k+1)}}{\lambda_n - \mu} & \cdots & z_{n,s}^{(k)} \frac{d_{ss}^{(k+1)}}{\lambda_n - \mu} & z_{n,s+1}^{(k)} \frac{1}{\lambda_n - \mu} & \cdots & z_{n,q}^{(k)} \frac{1}{\lambda_n - \mu} \end{array} \right] \tag{33}$$

The subspace $E^{(k+1)}$ spanned by $\bar{Z}^{(k+1)}$ is not changed if we multiply the first s columns in $\bar{Z}^{(k+1)}$ by $(\lambda_i - \mu)/d_{ii}^{(k+1)}$ and the other columns by $(\lambda_j - \mu)$, i.e., $E^{(k+1)}$ is also spanned by $\tilde{Z}^{(k+1)}$, where

$$\tilde{Z}^{(k+1)} = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ 0 & & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & & 0 \\ 0 & & 1 & 0 & & 0 \\ 0 & & 0 & 1 & & 0 \\ 0 & & 0 & 0 & & 0 \\ \vdots & & \vdots & \vdots & & 0 \\ 0 & & 0 & 0 & & 1 \\ z_{q+1,1}^{(k)} \frac{\lambda_1 - \mu}{\lambda_{q+1} - \mu} & \dots & z_{q+1,s}^{(k)} \frac{\lambda_s - \mu}{\lambda_{q+1} - \mu} & z_{q+1,s+1}^{(k)} \frac{\lambda_{s+1} - \mu}{\lambda_{q+1} - \mu} & \dots & z_{q+1,q}^{(k)} \frac{\lambda_q - \mu}{\lambda_{q+1} - \mu} \\ z_{q+2,1}^{(k)} \frac{\lambda_1 - \mu}{\lambda_{q+2} - \mu} & \dots & z_{q+2,s}^{(k)} \frac{\lambda_s - \mu}{\lambda_{q+2} - \mu} & z_{q+2,s+1}^{(k)} \frac{\lambda_{s+1} - \mu}{\lambda_{q+2} - \mu} & \dots & z_{q+2,q}^{(k)} \frac{\lambda_q - \mu}{\lambda_{q+2} - \mu} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{n,1}^{(k)} \frac{\lambda_1 - \mu}{\lambda_n - \mu} & \dots & z_{n,s}^{(k)} \frac{\lambda_s - \mu}{\lambda_n - \mu} & z_{n,s+1}^{(k)} \frac{\lambda_{s+1} - \mu}{\lambda_n - \mu} & \dots & z_{n,q}^{(k)} \frac{\lambda_q - \mu}{\lambda_n - \mu} \end{bmatrix}. \tag{34}$$

In the subspace iteration using equations (30) and (31), the best eigenvector approximations are extracted from the vectors stored in $\tilde{Z}^{(k+1)}$. But on inspecting the columns of $\tilde{Z}^{(k+1)}$ in equation (34) we find that ultimately the j th column is the best approximation to the vector e_j in the subspace $E^{(k+1)}$. The ultimate rate of convergence to the j th eigenvector is thus obtained by evaluating

$$\frac{\|\tilde{z}_j^{(k+1)} - e_j\|_2}{\|z_j^{(k)} - e_j\|_2} = \frac{\lambda_j - \mu}{\lambda_{q+1} - \mu} \sqrt{\frac{\sum_{i=q+1}^n (z_{i,j}^{(k)})^2 ((\lambda_{q+1} - \mu)/(\lambda_i - \mu))^2}{\sum_{i=q+1}^n (z_{i,j}^{(k)})^2}}, \tag{35}$$

where $z_j^{(k)}$ is the j th column of $Z^{(k)}$, and is similar for $\tilde{z}_j^{(k+1)}$. Hence,

$$\frac{\|\tilde{z}_j^{(k+1)} - e_j\|_2}{\|z_j^{(k)} - e_j\|_2} \leq \frac{\lambda_j - \mu}{\lambda_{q+1} - \mu} \tag{36}$$

and convergence is linear with the rate of convergence equal to $(\lambda_j - \mu)/(\lambda_{q+1} - \mu)$. We, therefore, conclude that provided the starting subspace is not orthogonal to the required least dominant subspace spanned by $\phi_1, \phi_2, \dots, \phi_q$, the j th column in $X^{(k+1)}$ converges linearly with the rate $(\lambda_j - \mu)/(\lambda_{q+1} - \mu)$ to ϕ_j . Since the eigenvalues are calculated using the Rayleigh quotient, the j th eigenvalue in equation (7) converges linearly with the rate $\{(\lambda_j - \mu)/(\lambda_{q+1} - \mu)\}^2$ to λ_j .

TABLE 1
Operation counts for the subspace iteration method with shift

Operation	Calculation	Number of operations
Multiplication	$K - \mu M$	$n(m_M + 1)$
Factorization	$LDL^T = K - \mu M$	$nm_M(m_M + 3)/2$
<i>Iteration</i>		
Multiplication	$MX^{(k)}$	$qn(2m_M + 1)$
Solve for $\bar{X}^{(k+1)}$	$(K - \mu M)\bar{X}^{(k+1)} = MX^{(k)}$	$qn(2m_K + 1)$
Multiplication	$\bar{K}^{(k+1)} = \bar{X}^{(k+1)T}MX^{(k)}$	$qn(q + 1)/2$
Multiplication	$M\bar{X}^{(k+1)}$	$qn(2m_M + 1)$
Multiplication	$\bar{M}^{(k+1)} = \bar{X}^{(k+1)T}M\bar{X}^{(k+1)}$	$qn(q + 1)/2$
Solve for $Z^{(k+1)}$ & $\Omega^{(k+1)}$	$\bar{K}^{(k+1)}Z^{(k+1)}$ $= \bar{M}^{(k+1)}Z^{(k+1)}\Omega^{(k+1)}$	$O(q^3)$ neglected
Multiplication	$X^{(k+1)} = \bar{X}^{(k+1)}Z^{(k+1)}$	nq^2
<i>Sturm sequence check</i>		
Multiplication	$K - \lambda_p M$	$n(m_M + 1)$
Factorization	$LDL^T = K - \lambda_p M$	$nm_M(m_M + 3)/2$
Total	$T_s qn(2m_K + 4m_M + 2q + 4) + n(m_K^2 + 3m_K + 2m_M + 2)$	

3.4. OPERATION COUNTS AND SUMMARY OF ALGORITHM

Let one operation be equal to one multiplication that is nearly always followed by an addition. Assume that the half-bandwidths of K and M are m_K and m_M respectively. The steps for the subspace iteration method with shift with the operations are summarized in Table 1, and for the proposed method in Table 2.

The number of operations of the subspace iteration method with shift is $T_s qn(2m_K + 4m_M + 2q + 4) + n(m_K^2 + 3m_K + 2m_M + 2)$, and that for the proposed method $T_p \{qn(2m_K + 4m_M + 2q + 4 + s) + sn(m_K + (s + 1)/2)\} + n(m_K^2 + 3m_K + 2m_M + 2)$. s is the multiplicity of the multiple eigenvalues which is on or very close to a shift. The proposed method needs more operations per iteration step, $qn + n(m_K + 1)$, than the subspace iteration method with shift. Assume that the ratio is composed of the operation counts per iteration of the proposed method (N_p), that of the subspace iteration method with shift (N_s), and that the difference of the operation counts per iteration for the above two methods ($N_p - N_s$) is as follows:

$$ratio = \frac{N_p - N_s}{N_p} = \frac{qn(1 + s) + sn\{m_K + (s + 1)/2\}}{qn\{(2m_K + 4m_M + 2q + 4 + s) + sn(m_K + (s + 1)/2)\}}. \quad (37)$$

Then, if the half-bandwidth of the stiffness matrix (m_K) is equal to that of the mass matrix (m_M), the above ratio can be approximated as follows:

$$ratio \approx \frac{s}{6q}. \quad (38)$$

TABLE 2
Operation counts for the proposed method

Operation	Calculation	Number of operations
<i>Iteration</i>		
$k = 0$		
Multiplication	$K - \mu M$	$n(m_M + 1)$
Multiplication	$MX^{(0)}$	$qn(2m_M + 1)$
Change the last s columns of $K - \mu M$ to $MX_s^{(0)}$		Neglected
Factorization	$LDL^T = F^{(1)}$	$\{nm_K(m_K + 3 + 2s) + (s + 1)\}/2$
$k = 1, 2, \dots$		
Multiplication	$MX^{(k)}$	$qn(2m_M + 1)$
Change the last s columns of $K - \mu M$ to $MX_s^{(k)}$		Neglected
Factorization	$LDL^T = F^{(k+1)}$	$sn\{m_K + (s + 1)/2\}$
Solve for $\hat{X}^{(k+1)}$	$F^{(k+1)}\hat{X}^{(k+1)} = R$	$qn(2m_K + s + 1)$
Multiplication	$\bar{K}^{(k+1)} = \bar{X}^{(k+1)T}MX^{(k)}$	$qn(q + 1)/2$
Multiplication	$M\bar{X}^{(k+1)}$	$qn(2m_M + 1)$
Multiplication	$\bar{M}^{(k+1)} = \bar{X}^{(k+1)T}M\bar{X}^{(k+1)}$	$qn(q + 1)/2$
Solve for $Z^{(k+1)}$ & $\Omega^{(k+1)}$	$\bar{K}^{(k+1)}Z^{(k+1)}$ $= \bar{M}^{(k+1)}Z^{(k+1)}\Omega^{(k+1)}$	$O(q^3)$ neglected
Multiplication	$X^{(k+1)} = \bar{X}^{(k+1)}Z^{(k+1)}$	nq^2
<i>Sturm sequence check</i>		
Multiplication	$K - \lambda_p M$	$n(m_M + 1)$
Factorization	$LDL^T = \bar{K} - \lambda_p M$	$nm_K(m_K + 3)/2$
Total	$T_p\{qn(2m_K + 4m_M + 2q + 4 + s) + sn(m_K + (s + 1)/2)\}$ $+ n(m_K^2 + 3m_K + 2m_M + 2)$	

where

$$F^{(k+1)} = \begin{bmatrix} K - \mu M & MX_s^{(k)} \\ X_s^{(k)T}M & 0 \end{bmatrix}, \hat{X}^{(k+1)} = \begin{bmatrix} \bar{X}^{(k+1)} \\ \bar{D}_s^{(k+1)} \end{bmatrix}, R = \begin{bmatrix} MX^{(k)} \\ E_s \end{bmatrix} \text{ (see equation (19))}$$

This ratio means that the larger the number of the required eigenpairs, the smaller is the difference of the operation counts between the proposed method and the subspace iteration method with shift. That is, the number of operations for the aforementioned two methods, the subspace iteration method with shift and the proposed method, is almost the same when the required number of eigenpairs is large.

4. NUMERICAL EXAMPLES

The three-dimensional framed structure and the simply supported square plate are analyzed to verify the effectiveness of the proposed method. The solution time spent for the first 10 eigenpairs and the convergence of the proposed method are compared with those of the subspace iteration method with shift which is not used in the limited region (see equation (11)). Each method stops when the error norms are reduced by a factor of 10^{-6} , which yields a stable eigensolution and sufficient

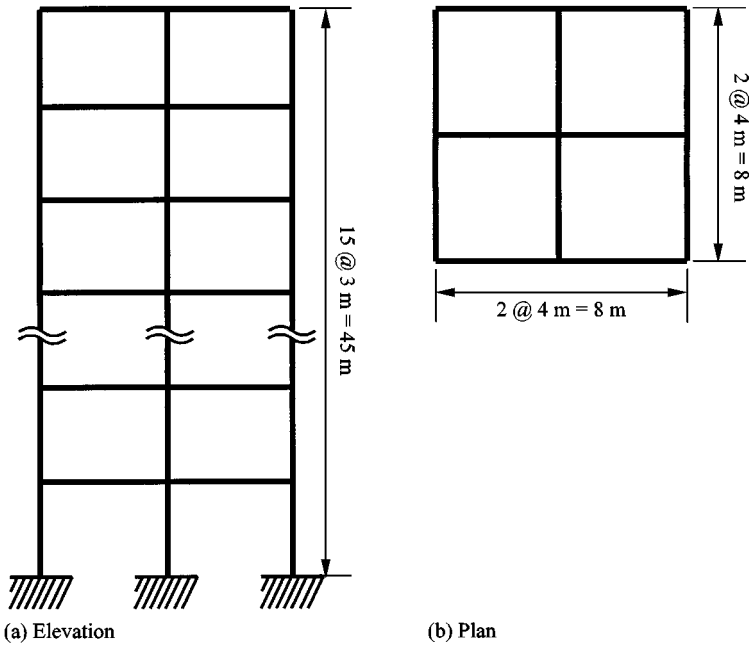


Figure 1. Three-dimensional framed structure. $A = 0.2787 \text{ m}^2$, $I = 8.631 \times 10^{-3} \text{ m}^4$, $E = 2.068 \times 10^{10} \text{ Pa}$, $\rho = 5.154 \times 10^2 \text{ kg/m}^3$.

TABLE 3
The lowest 10 eigenvalues of the three-dimensional framed structure

Mode number	Eigenvalues
1	0.1556E+03
2	0.1556E+03
3	0.3112E+03
4	0.1623E+04
5	0.1623E+04
6	0.2840E+04
7	0.5736E+04
8	0.5736E+04
9	0.8942E+04
10	0.1202E+05

accuracy in the calculated eigenpairs for practical analysis [77]. The error norm [7] is defined as

$$\varepsilon_i^{(k)} = \frac{\|(K - \lambda_i^{(k)}M)x_i^{(k)}\|_2}{\|Kx_i^{(k)}\|_2} \tag{39}$$

All runs are executed in the IRIS4D-20-S17 with 10 Mips and 0.9 Mflops.

TABLE 4

Solution time for the lowest 10 eigenpairs of the three-dimensional framed structure

Analysis methods	Shift = $1.01\lambda_4$	Shift = $1.00001\lambda_4$	Shift = λ_4
Subspace iteration method with shift	409.86 (1.00)	No solution	No solution
Proposed method	426.58 (1.04)	421.69	421.19

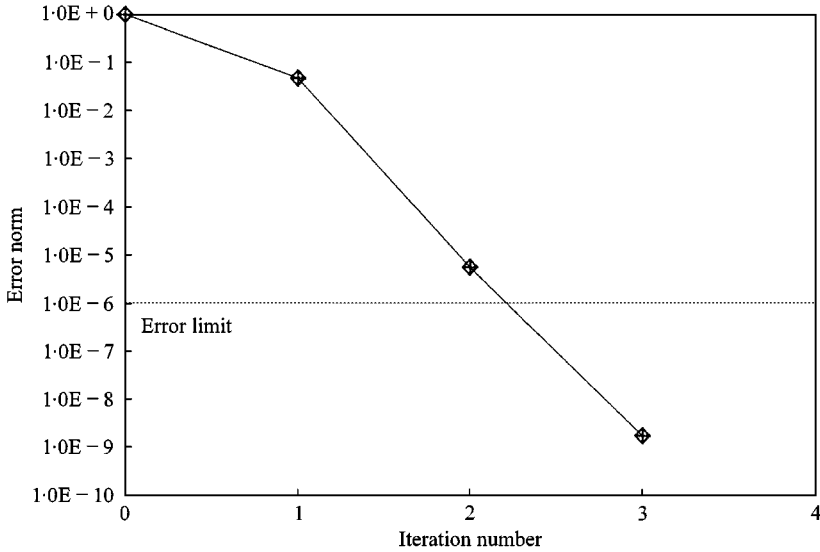


Figure 2. Error norm versus iteration number of the 4th eigenpair in case of shift = $1.01\lambda_4$.
 + Subspace iteration method; \diamond Proposed method.

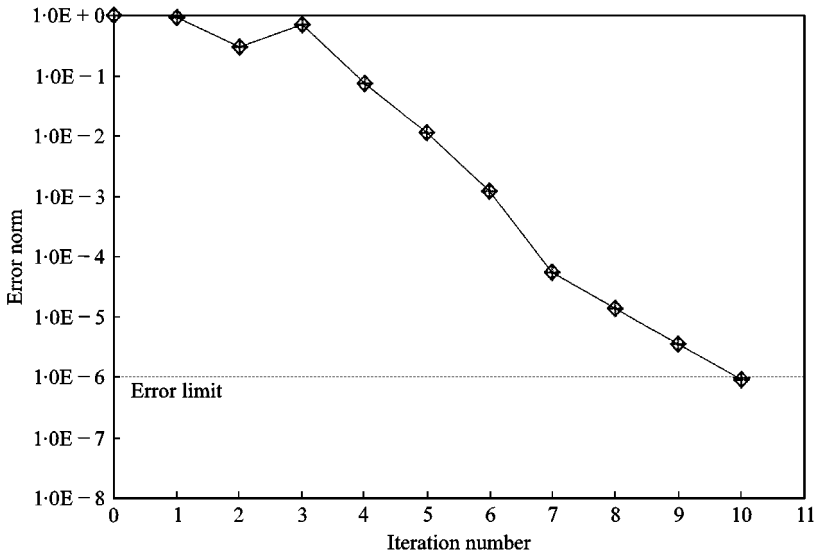


Figure 3. Error norm versus iteration number of the 10th eigenpair in case of shift = $1.01\lambda_4$.
 + Subspace iteration method; \diamond Proposed method.

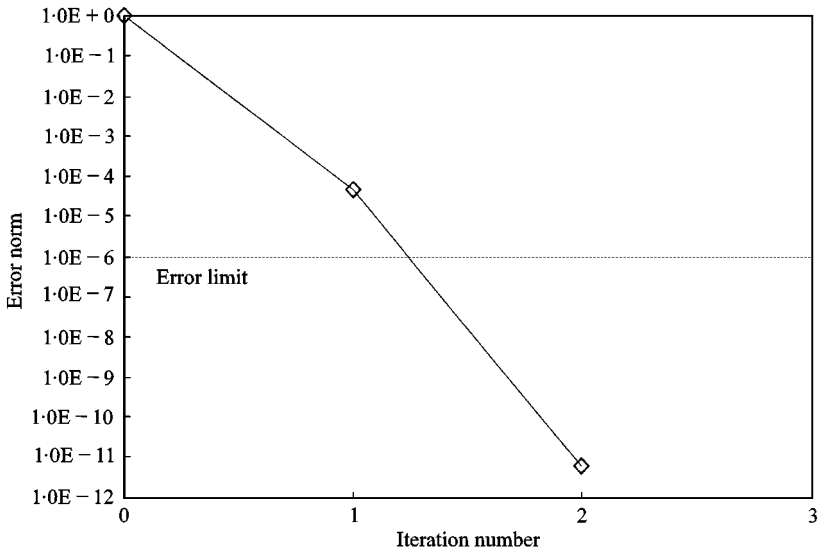


Figure 4. Error norm versus iteration number of the 4th eigenpair in case of shift = $1.00001 \lambda_4$.
 ◇ Proposed method.

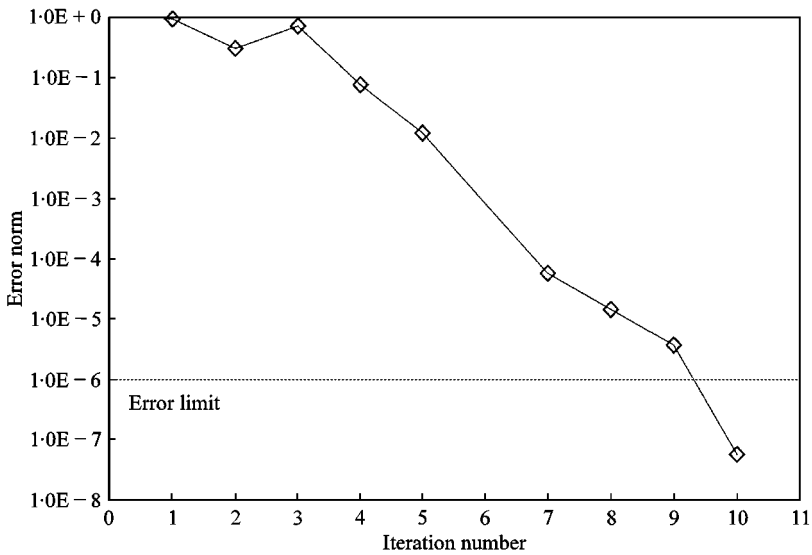


Figure 5. Error norm versus iteration number of the 10th eigenpair in case of shift = $1.00001 \lambda_4$.
 ◇ Proposed method.

4.1. THREE-DIMENSIONAL FRAMED STRUCTURE

The first example is a three-dimensional framed structure. The geometric configuration and the material properties are shown in Figure 1. The structure is discretized by using 315 beam elements resulting in a system of dynamic equations with a total of 810 degrees of freedom. The consistent mass matrix is used for M .

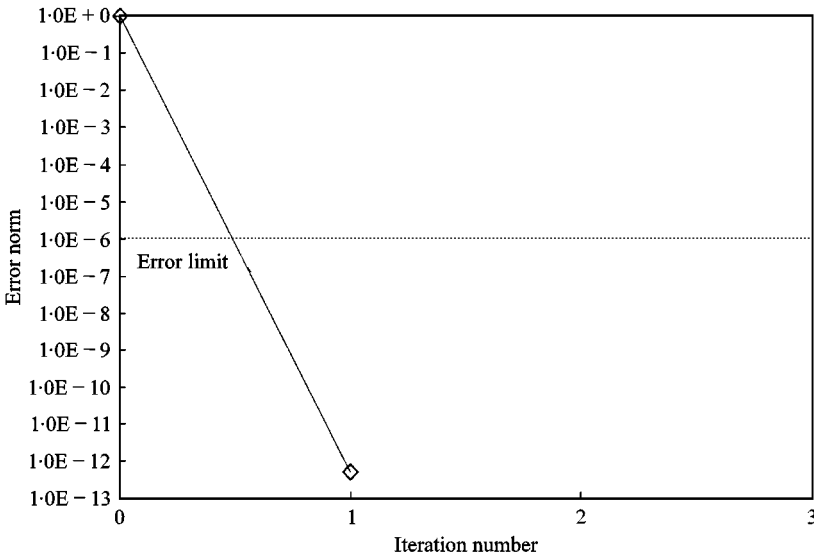


Figure 6. Error norm versus iteration number of the 4th eigenpair in case of shift = λ_4 .
 ◇ Proposed method.

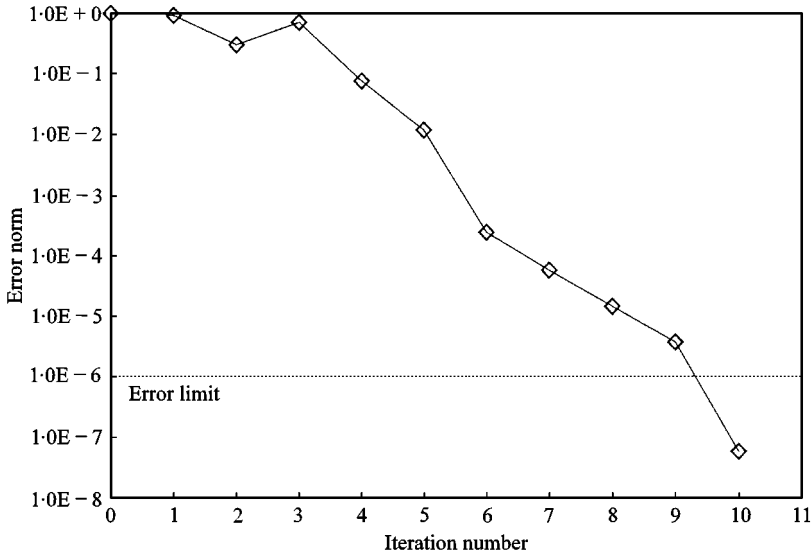


Figure 7. Error norm versus iteration number of the 10th eigenpair in case of shift = λ_4 .
 ◇ Proposed method.

The lowest 10 eigenvalues of the model are shown in Table 3. The eigenvalues of the model are distinct root or multiple ones.

Some results are shown in Table 4 and in Figures 2–7. The solution time for the two methods are summarized in Table 4. When a shift is on $1.01 \lambda_4$, the subspace iteration method with shift and the proposed method obtain the required 10

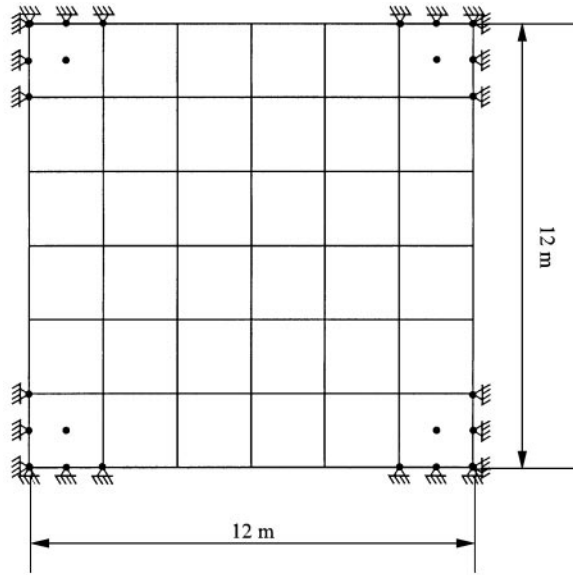


Figure 8. Simply supported square plate. $E = 2.0 \times 10^{11}$ Pa, $\rho = 7.85 \times 10^3$ kg/m³, Poisson ratio = 0.3, Shell thickness = 0.01 m.

TABLE 5
The lowest 10 eigenvalues of the simply supported square plate

Mode number	Eigenvalues
1	0.4435E+01
2	0.2914E+02
3	0.2914E+02
4	0.7367E+02
5	0.1305E+03
6	0.1305E+03
7	0.2087E+03
8	0.2087E+03
9	0.4010E+03
10	0.4418E+03

TABLE 6
Solution time for the lowest 10 eigenpairs of the simply supported square plate

Analysis methods	Shift = 1.01λ ₂	Shift = 1.00001λ ₂	Shift = λ ₂
Subspace iteration method with shift	723.34 (1.00)	No solution	No solution
Proposed method	749.38 (1.03)	747.09	750.35

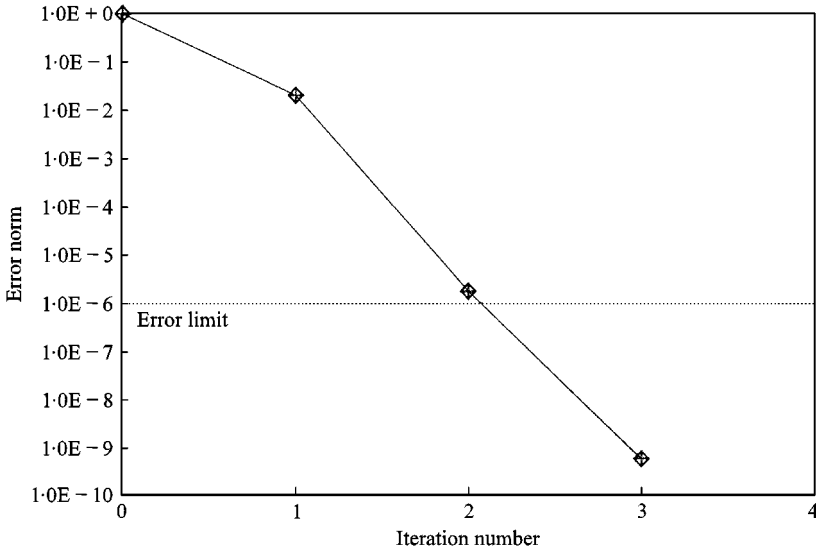


Figure 9. Error norm versus iteration number of the 2nd eigenpair in case of shift = $1.01\lambda_2$.
 + Subspace iteration method; \diamond Proposed method.

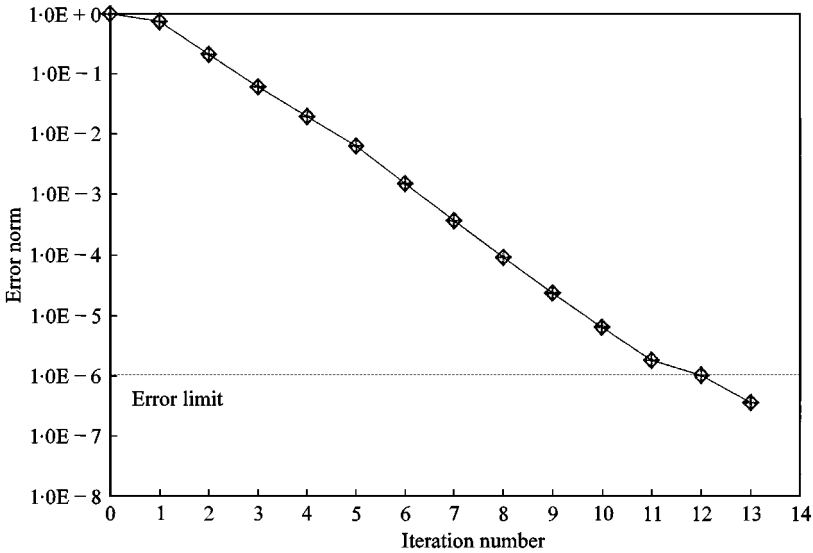


Figure 10. Error norm versus iteration number of the 10th eigenpair in case of shift = $1.01\lambda_2$.
 + Subspace iteration method; \diamond Proposed method.

eigenpairs. However, when the shift is on $1.00001\lambda_4$ or on λ_4 , the subspace iteration method with shift does not calculate the solutions while the proposed method finds the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is exactly the same as the multiple eigenvalues, as analytically proved in the Section 3.2.

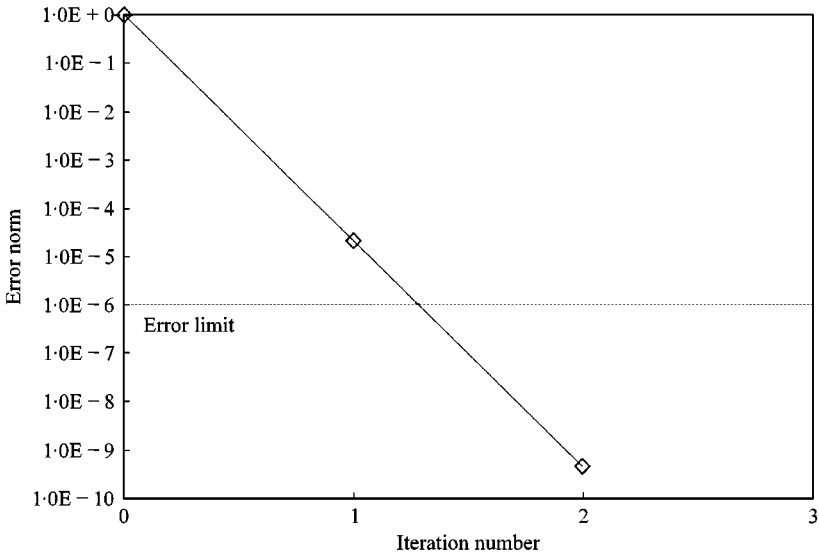


Figure 11. Error norm versus iteration number of the 2nd eigenpair in case of shift = $1.00001 \lambda_2$.
 ◇ Proposed method.

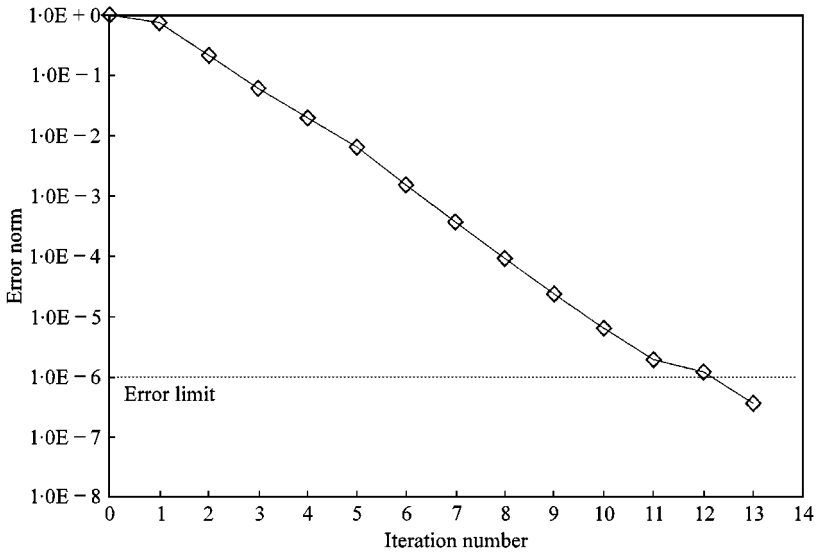


Figure 12. Error norm versus iteration number of the 10th eigenpair in case of shift = $1.00001 \lambda_2$.
 ◇ Proposed method.

For each solution method, the convergence of each eigenpair is depicted in Figures 2–7. Figures 2 and 3 show that when the shift is on $1.01 \lambda_4$ the convergence of the proposed method is nearly equal to that of the subspace iteration method with shift. Figures 4 and 5 show that when the shift is on $1.00001 \lambda_4$, the proposed method converges well without any singularity while the subspace iteration method

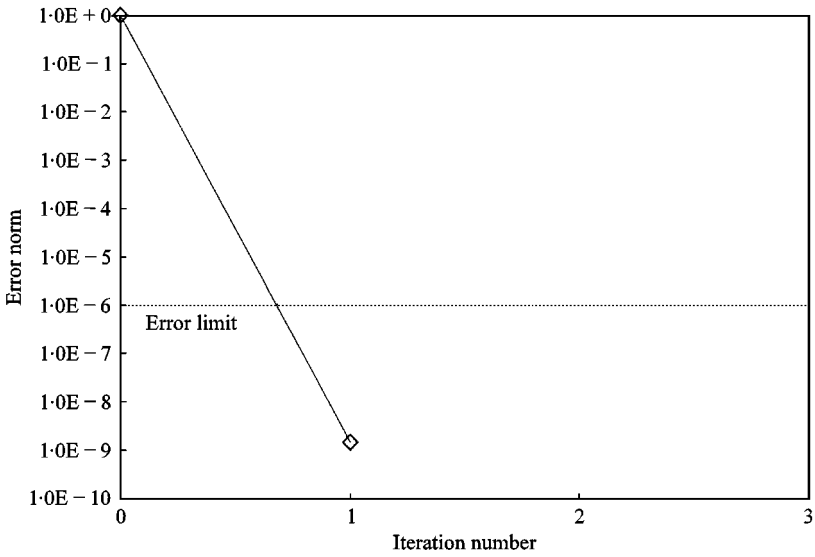


Figure 13. Error norm versus iteration number of the 2nd eigenpair in case of shift = λ_2 .
 ◇ Proposed method.

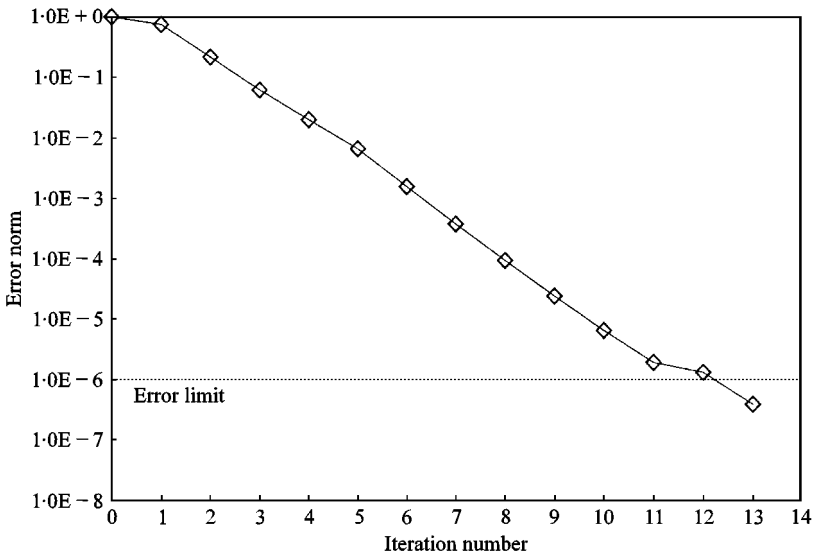


Figure 14. Error norm versus iteration number of the 10th eigenpair in case of shift = λ_2 .
 ◇ Proposed method.

with shift cannot converge due to the singularity. Figures 6 and 7 show that when the shift is exactly the same as the fourth eigenvalue, the proposed method only converges well without any singularity. These results are the same as a shift is on $1.00001\lambda_4$. From the above results, it can be seen that the proposed method can choose a more exact shift than the subspace iteration method with shift, and thus the proposed method can be more computationally efficient.

4.2. SIMPLY SUPPORTED SQUARE PLATE

The second example is the simply supported square plate. Figure 8 shows the geometric configuration and material properties. The structure is discretized by using 36 shell elements (nine node/element) resulting in a system of dynamic equations with a total of 701 degrees of freedom. The consistent mass matrix is used for M .

The lowest 10 eigenvalues of the model are shown in Table 5. The eigenvalues of the model are distinct root or multiple ones.

Some results are shown in Table 6 and in Figures 9–14. The solution time for the two methods are summarized in Table 6. When a shift is on $1.01 \lambda_2$, the subspace iteration method with shift and the proposed method obtain the required 10 eigenpairs. However, when the shift is on $1.0001 \lambda_2$ or on λ_2 , the subspace iteration method with shift does not calculate the solutions while the proposed method finds the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is exactly the same as the multiple eigenvalues.

For each solution method, the convergence of each eigenpair is depicted in Figures 9–14. Figures 9 and 10 show that when the shift is on $1.01 \lambda_2$ the convergence of the proposed method is nearly equal to that of the subspace iteration method with shift. Figures 11 and 12 show that when the shift is on $1.00001 \lambda_2$ the proposed method converges well without any singularity while the subspace iteration method with shift cannot converge due to the singularity. Figures 13 and 14 show that when the shift is exactly the same as the second eigenvalue, the proposed method only converges well without any singularity.

5. CONCLUSIONS

A numerically stable technique using side conditions for improving the subspace iteration method with shift has been presented. The characteristics of the proposed method identified by the analytical and the numerical results from numerical examples are summarized as follows:

- (1) The non-singularity of the proposed method is always guaranteed, which is proved analytically; even if the shift is on or very close to multiple eigenvalues, the proposed method can obtain the solutions without any singularity.
- (2) The convergence rate of the proposed method is at least equal to that of the subspace iteration method with shift, and the operation counts of the proposed method and the subspace iteration method with shift are almost the same when the required number of eigenpairs is large.

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