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EIGENFREQUENCY ANALYSIS OF THIN-WALLED GIRDERS CURVED IN PLAN

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An analytical procedure is proposed for the evaluation of the eigenfrequencies of a thin-walled beam curved in plan in response to transverse bending and torsion with various boundary conditions. The analysis is performed through the introduction of four non-dimensionalized geometric parameters which govern the dynamic behaviour of the beam. These parameters lead to an equally non-dimensionalized eigenfrequency parameter through an appropriate treatment of the differential equations on the basis of a computer program specially developed for this purpose. This program is also applied to investigate the influence of the aforementioned parameters on the eigenfrequency behaviour of a thin-walled girder.

1. INTRODUCTION

Thin-walled girders constitute structural elements which, although considered as girders, exhibit a behaviour which cannot be analyzed on the basis of the assumptions used in the technical theory of beams. Although this fact has been recognized as early as the beginning of this century by Timoshenko, it was Vlasov [1], who first developed a consistent theory which allowed the analysis of the thin-walled beam as a linear member, overcoming the fact that the latter is composed of virtually plate elements with a drastic influence on its bending and torsional behaviour.

The theory of the thin-walled beams has not found an extensive use by structural engineers due to the "uncomfortable" deviations from the well-established technical theory. The reason goes back to the more advanced structural perception demanded than the usual practice asked for, although some excellent works like reference [2] have appeared to make the theory more understandable. Also the wide introduction of the finite element method has given perhaps the feeling of independance from theories of such kind. That was an unhappy event as the thin-walled sections are widely used not only in bridge design but also in building

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structures (to speak only of the civil engineering applications), where the need of the designer to have a direct and effectively simple perception of the structural behaviour of his structures without being mainly confronted with the numerical outputs of the finite element programs, is of paramount importance.

The dynamic behaviour of such beams although to some extent investigated in reference [1], has not been given so much attention in research, as was the case for the classical beam members. However, the torsional vibrations of the thin-walled open sections have been treated extensively by Gere [3], and using the matrix-methods techniques, an interesting contribution on the eigenvalue problems of the thin-walled assemblages has been made by Krajcinovic [4]. Very recently a dynamic investigation of the dynamic problem of multicell thin-walled beams with cutouts has been treated by Capuani *et al.* [5], following the needs of high rise building design.

The present paper is a consequence of need to develop a tool to assess the eigenfrequency characteristics of the thin-walled cured beams as they appear mainly in the bridge design, without having to perform the time-consuming finite element analyses which of couse cannot offer parametric study possibilities. For this purpose the static differential equations for curved thin-walled beams already established by Vlasov have been appropriately extended and treated through the introduction of four dimensionless geometric quantities. The practical performance of the proposed solution requires the use of a numerical procedure supported by a computer program specially written for that purpose. This program enables not only the determination of the eigenfrequency spectrum for a given case in any desired extent and for various boundary conditions on the basis of an appropriately formulated and treated frequency equation, but also enables parametric studies for the evaluation of the influence of the aforementioned non-dimensionalized parameters on the eigenfrequency behaviour of the beam.

2. BASIC EQUATIONS

A thin-walled girder of an open or closed cross-section is considered, exhibiting a constant curvature in the horizontal plane. The girder is represented by its centre of gravity axis, which is assumed to coincide with the axis of the shear centers of the girder sections and at the same time lies on the horizontal plane. The girder is loaded transversely to the plane of its curved axis together with a distributed torsional moment acting along the same axis. Each point on the girder is referred to by its co-ordinate s measured from one end along the curved axis. The girder is supported at its ends, its curved length being equal to L (see Figure 1).

It is assumed that although the points of the cross-section as a result of the incoming non-uniform torsion undergo longitudinal displacements so that they do not belong after the deformation to a single plane (warping effect), the profile of the thin-walled girder section remains unchanged under any actions (rigid profile assumption).

The equations governing the statical behaviour of such a beam have been established by Vlasov in reference [1]. In order to examine the dynamic behaviour

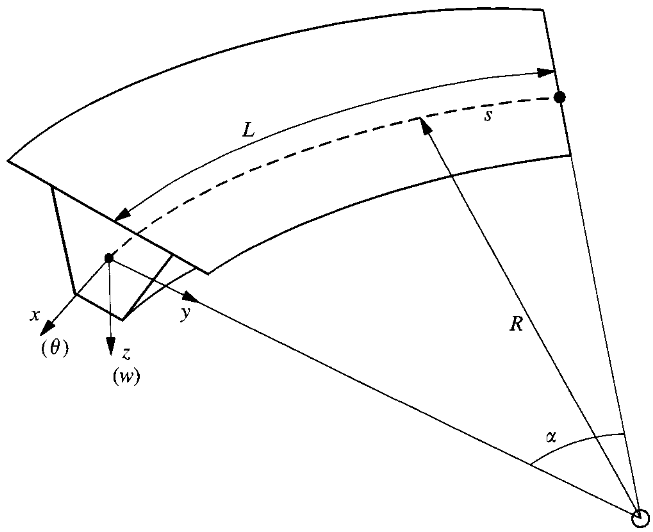

Figure 1. General layout.

of the beam, the corresponding inertia terms have to be added and the following equations are then obtained:

$$-E\left(\frac{I_\omega}{R^2}+I_y\right)w^{(4)}+\frac{GI_T}{R^2}w^{(2)}-\frac{EI_\omega}{R}\theta^{(4)}+\frac{EI_y+GI_T}{R}\theta^{(2)}-m\frac{\partial^2 w}{\partial t^2}=-q_z(t), \tag{1}$$

$$-\frac{EI_\omega}{R}w^{(4)}+\frac{EI_y+GI_T}{R}w^{(2)}-EI_\omega\theta^{(4)}+GI_T\theta^{(2)}-\frac{EI_y}{R^2}\theta-I_m\frac{\partial^2\theta}{\partial t^2}=-m_s(t), \tag{2}$$

where, $w(s, t)$ is the vertical deflection of the girder axis, $\theta(s, t)$ the angle of torsional rotation of the girder axis, $w^{(n)}=\partial^n w(s,t)/\partial s^n$, $\theta^{(n)}=\partial^n\theta(s,t)/\partial s^n$,

R the constant radius of curvature of the girder axis, I_y the moment of inertia about the principal y-axis, I_ω the warping resistance of the cross-section, I_T the Saint–Venant torsional inertia of the cross-section, m the mass per unit length of the girder, I_m the rotatory inertia mass of the cross-section per unit length; it is expressed according to the relation $I_m=(m/A)(I_y+I_z)$ where I_z is the moment of inertia about the principal z- axis and A the area of the cross-section, E the modulus of elasticity, G the shear modulus, $q_z(t)$ the distributed load per unit length in direction z, and $m_s(t)$ the distributed torsional moment per unit length about the curved axis.

As already pointed out, the above equations are strictly valid only for girder cross-sections whose centroids coincide with the shear center but, as is also mentioned by Vlasov in reference [1] for the static case, they can also be applied with a practically negligible loss of accuracy for girders of an arbitrary cross-section, whose ratio of the biggest dimension of the cross-section to the radius of curvature does not exceed the value of $\frac{1}{10}$, a condition which is normally satisfied in the case of a bridge girder.

The stress resultants of interest in the examined case are the bending moment M_B about the y-axis, the shearing force Q along the z-axis, the torsional moment M_T and the bimoment M_ω. They are expressed according to reference [1] as follows:

$$M_B = -EI_y(w^{(2)} - \theta/R), \quad M_T = -EI_\omega(\theta^{(3)} + w^{(3)}/R) + GI_T(\theta^{(1)} + w^{(1)}/R), \tag{3,4}$$

$$M_\omega = EI_\omega(\theta^{(2)} + w^{(2)}/R), \tag{5}$$

$$Q = -EI_y(w^{(3)} - \theta^{(1)}/R) - EI_\omega\left(\frac{\theta^{(3)}}{R} + \frac{w^{(3)}}{R^2}\right) + GI_T\left(\frac{\theta^{(1)}}{R} + \frac{w^{(1)}}{R^2}\right) \tag{6}$$

As it is aimed now to investigate the free vibration of the girder, the right-hand sides of the equations (1) and (2) are set equal to zero. By using the classical technique of the separation of variables, the functions $w(s, t)$ and $\theta(s, t)$ are expressed as

$$w(s,t) = w(s)[A_i \sin(\omega_i t) + B_i \cos(\omega_i t)], \quad \theta(s,t) = \theta(s)[A_i \sin(\omega_i t) + B_i \cos(\omega_i t)]. \tag{7,8}$$

Moreover, upon expressing the functions $w(s)$ and $\theta(s)$ in terms of the dimensionless co-ordinate

$$\xi = s/L \tag{9}$$

as $\bar{W}(\xi)$ and $\Theta(\xi)$, respectively, and taking into account that

$$\frac{\mathrm{d}^n w}{\mathrm{d}s^n} = \frac{1}{L^n}\frac{\mathrm{d}^n \bar{W}}{\mathrm{d}\xi^n} \tag{10}$$

as well as

$$\frac{\mathrm{d}^n \theta}{\mathrm{d}s^n} = \frac{1}{L^n}\frac{\mathrm{d}^n \Theta}{\mathrm{d}\xi^n} \tag{11}$$

equations (1) and (2) become

$$E\left(\frac{I_\omega}{R^2} + I_y\right)\frac{1}{L^4}\bar{W}^{(4)} - \frac{GI_T}{R^2L^2}\bar{W}^{(2)} + \frac{EI_\omega}{RL^4}\Theta^{(4)} - \frac{EI_y + GI_T}{RL^2}\Theta^{(2)} - \bar{W}\omega_i^2 m = 0, \tag{12}$$

$$\frac{EI_\omega}{RL^4}\bar{W}^{(4)} - \frac{EI_y + GI_T}{RL^2}\bar{W}^{(2)} + \frac{EI_\omega}{L^4}\Theta^{(4)} - \frac{GI_T}{L^2}\Theta^{(2)} + \left(\frac{EI_y}{R^2} - I_m\omega_i^2\right)\Theta = 0. \tag{13}$$

Now, the following non-dimensionalized parameters are introduced:

$$C_1 = \frac{I_\omega}{R^2 I_y}, \qquad C_2 = \frac{GI_T}{EI_y}, \qquad \alpha = \frac{L}{R}, \qquad T = \frac{I_m}{mR^2} \tag{14–17}$$

together with the non-dimensionalized eigenfrequency parameter

$$\Omega_i = (L^4 m/EI_y)\,\omega_i^2. \tag{18}$$

The above parameters are referred as the wraping parameter (C_1), the torsional parameter (C_2), the central angle (α) and the rotatory mass parameter (T) respectively.

Upon expressing the non-dimensionalized deflection as

$$W(\xi) = \overline{W(\xi)}/L, \tag{19}$$

equations (12) and (13) take the non-dimensionalized forms

$$\alpha(1 + C_1)W^{(4)} - \alpha^3 C_2 W^{(2)} + C_1\Theta^{(4)} - \alpha^2(1 + C_2)\Theta^{(2)} - \alpha\Omega_i W = 0, \tag{20}$$

$$\alpha C_1 W^{(4)} - \alpha^3(1 + C_2)W^{(2)} + C_1\Theta^{(4)} - \alpha^2 C_2\Theta^{(2)} + \alpha^4\Theta - T\Omega_i\Theta = 0. \tag{21}$$

3. BOUNDARY CONDITIONS

Four boundary conditions have to be satisfied at each end. According to the type of support at each end of the girder, the following conditions have to be satisfied in each case.

3.1. SIMPLE SUPPORT

The girder end is allowed to rotate freely about its y-axis and moreover the section is allowed to warp also freely, and at the same time it is restrained in rotation about the tangent of the curved axis: i.e.,

$$w = 0, \qquad \theta = 0, \qquad M_B = 0, \qquad M_\omega = 0. \tag{22a–d}$$

These relations, upon taking into account the above expressions as well as the non-dimensionalized parameters, take the following form:

$$W = 0, \qquad \Theta = 0, \qquad W^{(2)} - \alpha\Theta = 0, \quad \Theta^{(2)} + \alpha W^{(2)} = 0. \tag{23a–d}$$

3.2. FIXED SUPPORT

The girder end is totally restrained also against warping: i.e.,

$$w = 0, \qquad \theta = 0, \qquad w^{(1)} = 0, \qquad \theta^{(1)} = 0. \tag{24a–d}$$

Expressing the above relations in terms of the non-dimensionalized deformations yields

$$W = 0, \qquad \Theta = 0, \qquad W^{(1)} = 0, \qquad \Theta^{(1)} = 0. \tag{25a–d}$$

3.3. FREE END

The girder end is totally free from any stress; i.e.,

$$M_B = 0, \qquad M_\omega = 0, \qquad M_T = 0, \qquad Q = 0. \tag{26a–d}$$

In terms of the introduced non-dimensionalized parameters the following form is equivalently obtained:

$$W^{(2)} - \alpha\Theta = 0, \qquad \Theta^{(2)} + \alpha W^{(2)} = 0, \tag{27a, b}$$

$$-C_1(\Theta^{(3)} + \alpha W^{(3)}) + \alpha^2 C_2(\Theta^{(1)} + \alpha W^{(1)}) = 0, \qquad W^{(3)} - \alpha\Theta^{(1)} = 0. \tag{27c, d}$$

4. TREATMENT OF THE DIFFERENTIAL SYSTEM

The differential system of equations (20) and (21) is written more conveniently as

$$A_1 W^{(4)} + A_2 W^{(2)} + A_3\Theta^{(4)} + A_4\Theta^{(2)} + A_5 W = 0, \tag{28}$$

$$B_1 W^{(4)} + B_2 W^{(2)} + B_3\Theta^{(4)} + B_4\Theta^{(2)} + B_5\Theta = 0 \tag{29}$$

and treated in the following way.

Considering the system as a linear one with unknowns $W^{(4)}$ and $W^{(2)}$ and solving it, one obtains

$$W^{(4)} = \frac{(A_2B_3 - B_2A_3)\Theta^{(4)} + (A_2B_4 - A_4B_2)\Theta^{(2)} + A_2B_5\Theta - A_5B_2W}{(A_1B_2 - B_1A_2)}. \tag{30}$$

$$W^{(2)} = \frac{(B_1A_3 - A_1B_3)\Theta^{(4)} + (B_1A_4 - A_1B_4)\Theta^{(2)} + B_1A_5W - A_1B_5\Theta}{(A_1B_2 - B_1A_2)}. \tag{31}$$

After double differentiation of equation (31), the following relation is also obtained:

$$W^{(4)} = \frac{(B_1A_3 - A_1B_3)\Theta^{(6)} + (B_1A_4 - A_1B_4)\Theta^{(4)} + B_1A_5W^{(2)} - A_1B_5\Theta^{(2)}}{(A_1B_2 - B_1A_2)}. \tag{32}$$

If now equations (30)–(32) are considered as a linear system with unknowns $W^{(4)}$, $W^{(2)}$ and W and solved in terms of $\Theta^{(6)}$, $\Theta^{(4)}$, $\Theta^{(2)}$ and Θ, then by substituting the resulting expression for $W^{(2)}$ in equation (29), the following differential equation of eighth degree for Θ is obtained:

$$D_8\Theta^{(8)} + D_6\Theta^{(6)} + D_4\Theta^{(4)} + D_2\Theta^{(2)} + D_0\Theta = 0. \tag{33}$$

Moreover, the function W is expressed through the function Θ according to the relation:

$$W = F_6\Theta^{(6)} + F_4\Theta^{(4)} + F_2\Theta^{(2)} + F_0\Theta. \tag{34}$$

The coefficients D and F of the above equations, respectively, can be evaluated in terms of the non-dimensionalized parameters α, C_1, C_2, T as well as the non-dimensionalized eigenfrequency Ω_i, according to the expressions given in the appendix.

Now the frequency equation can be formulated by appropriate satisfaction of the eight relevant boundary conditions (four for each end) of the respective beam.

More concisely, the solution of the differential equation (33) is built in terms of the eight conjugate roots of its characteristic algebraic equation:

$$D_8 r^8 + D_6 r^6 + D_4 r^4 + D_2 r^2 + D_0 = 0. \tag{35}$$

The expression for Θ has then the form

$$\Theta = \sum_i K_i e^{a_i \xi}(\sin(b_i \xi) + \cos(b_i \xi)) + \sum_j K_j e^{r_j \xi}, \tag{36}$$

where $(a_i \pm i \cdot b_i)$ and $\pm r_j$, are the conjugate imaginary and real roots of equation (35), respectively, and K_i and K_j, $(i + j = 8)$, are appropriate coefficients. Consequently the function $W(\xi)$ can also be determined by direct substitution into the expression (34).

The expressions (36) and (34) can be substituted into the respective eight boundary conditions of the specific girder considered, so that a homogeneous linear system of eight equations with the eight unknown K is obtained. In order that a solution exists the determinant of the corresponding coefficients has to be equal to zero for appropriate values of the quantities Ω_i, a condition known as the frequency equation of the problem.

5. NUMERICAL PROCEDURE

Since it is practically impossible to provide analytic expressions for the coefficients of the unknown K's, and the problem is even worse for their determinant, the frequency equation of the problem cannot be explicitly obtained.

Therefore, a simple iterative technique is developed in order to detect the relevant non-dimensionalized eigenfrequencies Ω_i, through a computer program specially written for this purpose.

Beginning from a safe lower limit set by the value $\Omega_0 = (0{\cdot}5)^2$ which is definitely less than the one corresponding to the cantilever beam, an increase is used by a constant step set equal to one tenth of that value used in subsequent cycles. The program adheres to the following scheme:

1. Assignment of an initial value of $\Omega_0 = (0{\cdot}5)^2$ for the non-dimensionalized eigenfrequency Ω_i.
2. Determination of the coefficients D and F in equations (33) and (34) respectively.
3. Determination of the roots of the algebraic equation (35).
4. Determination of the coefficients of K's on the basis of the established eight boundary conditions of the beam, according to the equations (36) and (34).
5. Evaluation of the determinant of the above coefficients.
6. Steps 1–5 are repeated with an incremental step equal to $\Omega_0/10$ so many times continuously, until the last evaluated determinant shows a change of sign, respectively, to the previous one.

7. Steps 1–6 are repeated with a halved value of the incremental step i.e. $\Omega_0/20$-times, until a satisfactory approximation of the relevant non-dimensionalized eigenfrequency Ω_i is reached.
8. Steps 6–7 are repeated until the desired number of eigenfrequencies is obtained.

6. THE CASE $I_\omega \sim 0$

6.1. THE BASIC EQUATIONS

The vanishing of I_ω, which expresses the absence of warping and the subsequent consideration of a uniform torsion, deserves special attention because it corresponds to a practically adopted assumption for closed celled girders as they are used in bridge design.

In this case the above formulation cannot be applied firstly because the basic equations (1) and (2) change their structure and secondly because three boundary conditions instead of four for each end have to be taken into account. The basic equations (1) and (2) are rewritten as

$$-EI_y w^{(4)} + \frac{GI_T}{R^2} w^{(2)} + \frac{EI_y + GI_T}{R}\theta^{(2)} - m\frac{\partial^2 w}{\partial t^2} = -q_z, \tag{37}$$

$$\frac{EI_y + GI_T}{R} w^{(2)} + GI_T\theta^{(2)} - \frac{EI_y}{R^2}\theta - I_m\frac{\partial^2 \theta}{\partial t^2} = -m_s. \tag{38}$$

In an analogous way as in section 2, the following non-dimensionalized equations are correspondingly obtained:

$$W^{(4)} - \alpha^2 C_2 W^{(2)} - \alpha(1 + C_2)\Theta^{(2)} - \Omega_i W = 0, \tag{39}$$

$$-\alpha^3(1 + C_2)W^{(2)} - \alpha^2 C_2 \Theta^{(2)} + \alpha^4\Theta - T\Omega_i\Theta = 0. \tag{40}$$

6.2. THE BOUNDARY CONDITIONS

In the case of fixed support the equations (24a–c) are valid. Expressed in terms of non-dimensionalized deformations, they take the form

$$W = 0, \qquad \Theta = 0, \qquad W^{(1)} = 0. \tag{41a–c}$$

In the case of a free end equations (26a), (26c) and (26d) are valid. Expressed in terms of non-dimensionalized parameters they take the form

$$W^{(2)} - \alpha\Theta = 0, \qquad (\Theta^{(1)} + \alpha W^{(1)}) = 0, \qquad W^{(3)} - \alpha\Theta^{(1)} = 0. \tag{42a–c}$$

6.3. TREATMENT OF THE DIFFERENTIAL SYSTEM

The differential system of equations (39) and (40) is written more conveniently in the form

$$A_1 W^{(4)} + A_2 W^{(2)} + A_4\Theta^{(2)} + A_5 W = 0, \tag{43}$$

$$B_2 W^{(2)} + B_4\Theta^{(2)} + B_5\Theta = 0. \tag{44}$$

By eliminating from the above equations $\Theta^{(2)}$ the following form is obtained:

$$\Theta = (1/A_4 B_5)[A_1 B_4 W^{(4)} + (A_2 B_4 - B_2 A_4) W^{(2)} + A_5 B_4 W] \tag{45}$$

and substituting in equation (44) one obtains

$$(A_1 B_4) W^{(6)} + (A_2 B_4 + A_1 B_5 - B_2 A_4) W^{(4)} + (A_2 B_5 + A_5 B_4) W^{(2)} + (A_5 B_5) W = 0, \tag{46}$$

where

$$A_1 = 1, \qquad A_2 = -\alpha^2 C_2, \qquad A_4 = -\alpha(1 + C_2), \qquad A_5 = -M_i, \tag{47a–d}$$

$$B_2 = -\alpha^3(1 + C_2), \qquad B_4 = -\alpha^2 C_2, \qquad B_5 = (\alpha^4 - TM_i). \tag{48a–c}$$

The procedure for assembling the six homogeneous linear equations coresponding to the respective boundary conditions at both ends of the girder and subsequently the numerical evaluation of the frequency equation, is to the one previously described in sections 4 and 5.

7. PARAMETRIC STUDIES

In order to assess the influence of the warping parameter C_1, the torsional parameter C_2, the rotatory mass parameter T and the central angle α on the fundamental non-dimensionalized frequency Ω_1, a series of parametric studies is

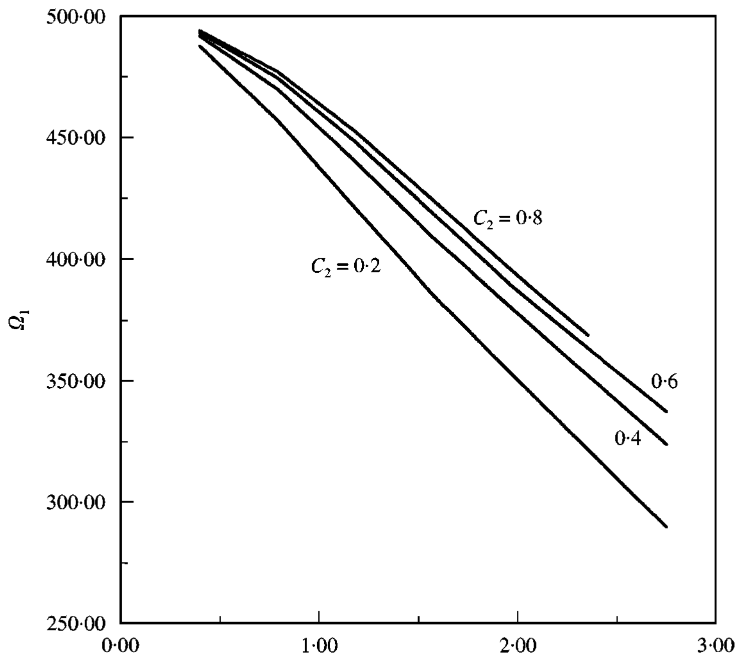

Figure 2. Influence of the variation of the central angle and torsional parameter on the fundamental eigenfrequency (Case 1). $C_1 = 0{\cdot}0$, $T = 0{\cdot}0$.

carried through, for the case of a girder fixed at its both ends, by applying the aforementioned computer program. The results are presented in the following figures and tables.

In Figures 2 and 3 two extreme cases are treated regarding the parameters C_1 and T, while the parameters C_2 and α are allowed to vary in a region of practical interest. Table 1 shows also some characteristic numerical results taken from these figures. It is seen that with increasing value of the central angle (L/R) the eigenfrequency is reduced almost linearly, having a direct inflence on the results. The increase of the torsional parameter C_2 leads to a slight increase of the eigenfrequency Ω_1, while the influence of the warping parameter C_1 is also small.

In Figures 4 and 5 the parameters C_1 and C_2 are varied. Two fixed values for L/R are, respectively, used with a constant value for T as shown. Table 2 shows also

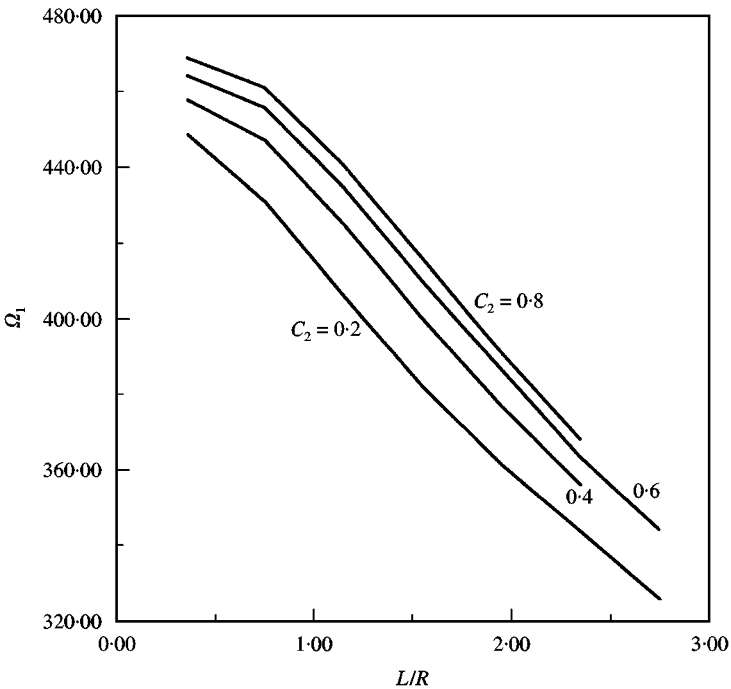

Figure 3. Influence of the variation of the central angle and torsional parameter on the fundamental eigenfrequency (Case 2). $C_1 = 10^{-2}$, $T = 10^{-2}$.

TABLE 1

Influence of L/R and C_2 on the range of values of Ω_1

L/R	C_2	$C_1 = T = 0$	$C_1 = T = 10^{-2}$	Δ%
0·50	0·2	481	445	− 7·5
	0·8	491	469	− 4·5
2·5	0·2	309	335	+ 8·4
	0·8	360	359	0

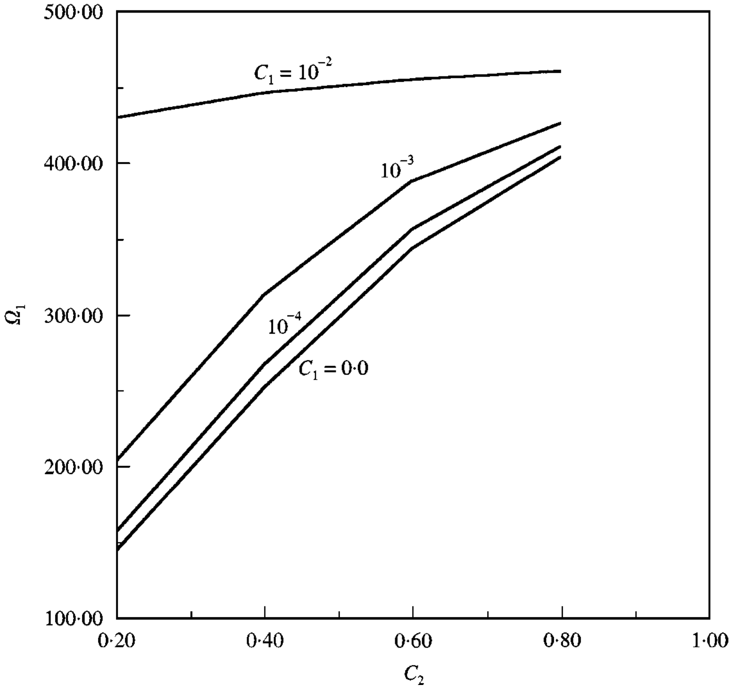

Figure 4. Influence of the variation of the warping and torsional parameter on the fundamental eigenfrequency (Case 1). $L/R = \pi/4$, $T = 10^{-2}$.

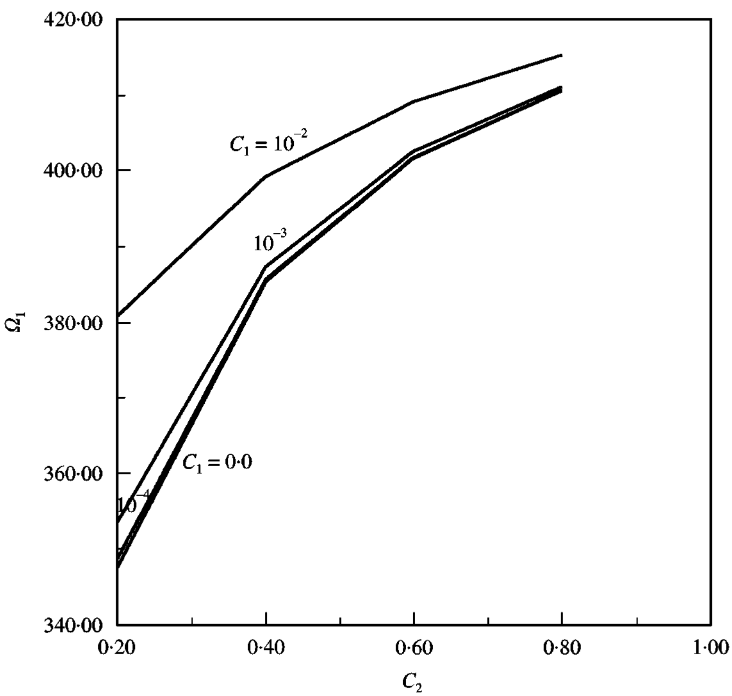

Figure 5. Influence of the variation of the warping and torsional parameter on the fundamental eigenfrequency (Case 2). $L/R = \pi/2$, $T = 10^{-2}$.

TABLE 2

Influence of C_1, C_2 and L/R on the range of values of Ω_1

C_2	C_2	$L/R = \pi/4$	$L/R = \pi/2$	$\Delta\%$
0·20	0·0	144	348	+ 141·7
	10^{-3}	204	354	+ 73·5
0·40	0·0	252	385	+ 52·8
	10^{-3}	313	387	+ 23.6
0·60	0·0	343	402	+ 17·2
	10^{-3}	388	402	+ 3.6
0·80	0·0	403	410	+ 1·7
	10^{-3}	426	411	− 3.5

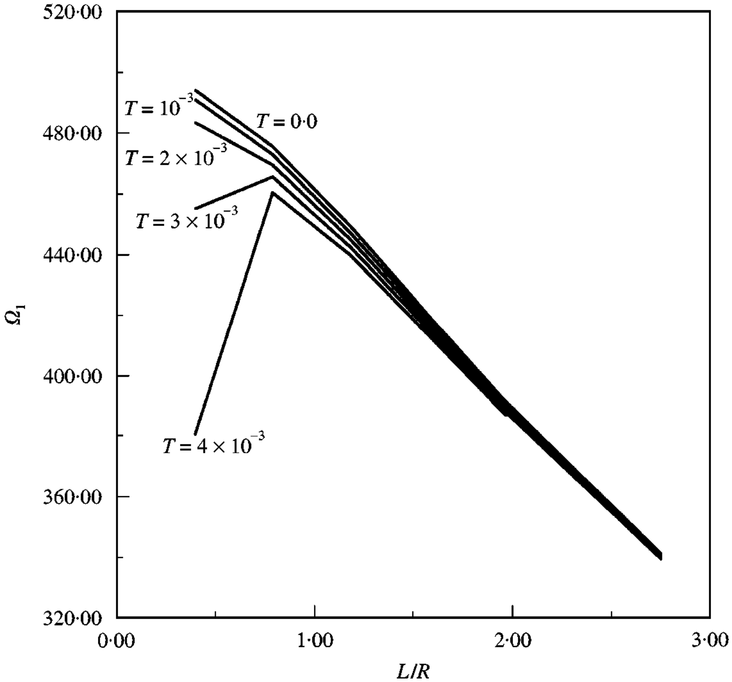

Figure 6. Influence of the variation of the rotatory mass parameter and the central angle on the fundamental eigenfrequency (Case 1). $C_1 = 10^{-3}$, $C_2 = 0{\cdot}6$.

some characteristic numerical results taken from these figures. The increase of the eigenfrequency Ω_1 together with the torsional parameter C_2 is reconfirmed while, as also previously mentioned, the warping parameter C_1 does not have a strong influence on the results especially for girders used in bridge design with big values of radii of curvature as imposed by the roadway alignment. The influence of the central angle (L/R) is more pronounced for the smaller values of the torsional parameter C_2.

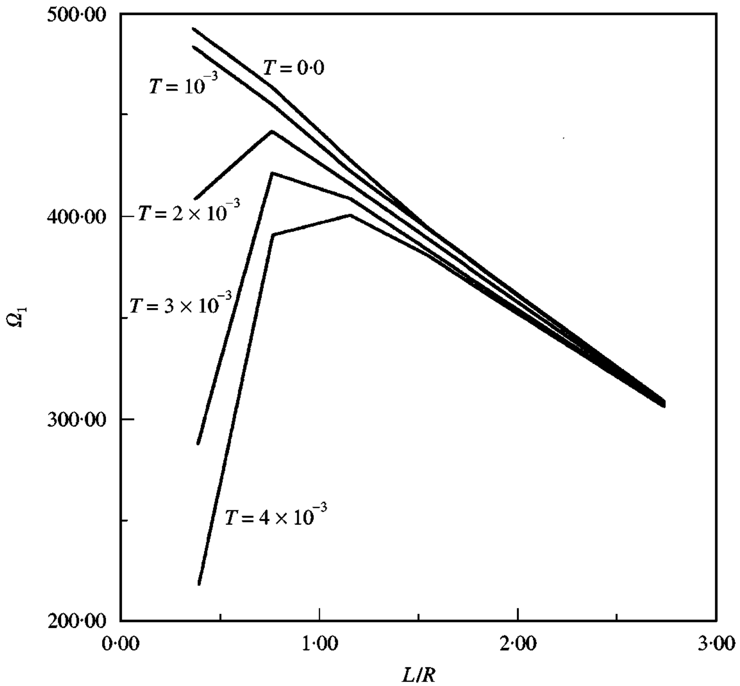

Figure 7. Influence of the variation of the rotatory mass parameter and the central angle on the fundamental eigenfrequency (Case 2). $C_1 = 10^{-3}$, $C_2 = 0{\cdot}2$.

From Figures 6 and 7 the effect of the rotatory mass parameter T in two cases with fixed values of C_1 and C_2 can be seen. Although this effect is in general of no special importance as shown, intense changes take place for low values of the central angle.

8. CONCLUSIONS

According to the described procedure, the evaluation of the eigenfrequency analysis of the transverse response of the thin-walled beams curved in plan is made possible by considering four non-dimensionalized parameters regarding the central angle of curvature, the torsional rigidity, the warping resistance and the rotatory mass of the cross-section, on the basis of a computer program, specially written for this purpose. The influence of all these factors has been investigated by using this program and shown in respective curves and tables, whereby it is found that the last two parameters at least in bridge construction do not play such an important role. It has to be emphasized that the above procedure, due to its dimensionless character, is much more effective than the appropriate use of a finite element program. Moreover, as the range of the values of the non-dimensionalized eigenfrequencies Ω_1 is rather restricted, a direct comparison with the "critical" frequency region between 2 and 5 Hz in roadway bridges could be easily made in order to draw relevant conclusions during also the preliminary design calculations also.

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APPENDIX

The coefficients in equations (22 and 23) are as follows:

$$A_1 = \alpha(1 + C_1), \quad A_2 = \alpha^3 C_2, \quad A_3 = C_1, \quad A_4 = -\alpha^2(1 + C_2), \quad A_5 = -\alpha M_i, \tag{A1–A5}$$

$$B_1 = \alpha C_1, \quad B_2 = -\alpha^3(1 + C_2), \quad B_3 = C_1, \quad B_4 = -\alpha^2 C_2, \quad B_5 = \alpha^4 - TM_i, \tag{B1–B5}$$

Moreover the coefficients "D" and "F" of equations (27 and 28) are expressed as follows:

$$D_0 = B_2(p_4k_3 - k_4p_3) - B_5(k_4^2 + p_4), \tag{C1}$$

$$D_2 = B_1(p_4k_3 - k_4p_3) - B_2(k_4\colon p_2 + k_2p_4 + k_4k_3), - B_4(k_4^2 + p_4) \tag{C2}$$

$$D_4 = -B_1(k_4p_2 + p_4k_2 + k_4k_3) - B_2(k_4p_1 + p_4k_1 - k_4k_2) - B_3(k_4^2 + p_4) \tag{C3}$$

$$D_6 = B_1(k_4k_2 - p_4k_1 - k_4p_1) + B_2k_1k_4, \quad D_8 = B_1k_4k_1 \tag{C4, C5}$$

and

$$F_0 = \frac{p_3 + k_3k_4}{k_4^2 + p_4}, \qquad F_2 = -\frac{k_2k_4 - p_2 - k_3}{k_4^2 + p_4}, \qquad F_6 = -\frac{k_1k_4 - p_1 + k_2}{k_4^2 + p_4},$$

$$F_6 = -\frac{k_1}{k_4^2 + p_4}, \tag{D1–D4}$$

where

$$p_1 = \frac{A_2B_3 - B_2A_3}{A_1B_2 - B_1A_2}, \qquad p_2 = \frac{A_2B_4 - A_4B_2}{A_1B_2 - B_1A_2}, \qquad p_3 = \frac{A_2B_5}{A_1B_2 - B_1A_2},$$

$$p_4 = \frac{B_2A_5}{A_1B_2 - B_1A_2}, \tag{E1–E4}$$

$$k_1 = \frac{B_1A_3 - A_1B_3}{A_1B_2 - B_1A_2}, \qquad k_2 = \frac{B_1A_4 - A_1B_4}{A_1B_2 - B_1A_2}, \qquad k_3 = \frac{A_1B_5}{A_1B_2 - B_1A_2},$$

$$k_4 = \frac{B_1A_5}{A_1B_2 - B_1A_2}. \tag{F1–F4}$$