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APPLICATION OF CHEBYSHEV SERIES TO SOLUTION OF NON-PRISMATIC BEAM VIBRATION PROBLEMS

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The problem of the vibration of a non-prismatic beam resting on a two-parameter elastic foundation has been solved by applying the approximation by Chebyshev series. As a result, closed analytical formulas defining the coefficients of the sought solutions were obtained. The method was used to solve the eigenproblem for a simply supported beam and a cantilever beam. The obtained results were compared with the results reported by other authors.

1. INTRODUCTION

Variable cross-section bar systems have been gaining popularity as elements of contemporary building structures due to the necessity of rational shaping, an economical design of structures as well as architectonic reasons. Solutions to many structural analysis problems, including stability problems, can be found in a monograph by Krynicki and Mazurkiewicz [1]. The authors of this work considered many complex states of loading. The solutions they found are in an analytical form. They limited themselves, however, to bars for which the moment of inertia of the cross-section can be described by the function $J(\xi) = J_s \cdot (\mu_i(1-\xi) + \mu_k \xi)^n$ $n = 0, 2, 3, 4$. An analytical solution consisting the expansion of the displacement function into a Fourier series, the calculation of the kinetic energy and the elastic strain energy and then solving the Lagrange equation is presented by Heidebrecht in reference [2]. The application of the Fourier series supplemented with power polynomials to the solution of a broad class of problems described by linear variable-coefficient differential equations, stemming from, for example, problems of the vibration of bars with variable cross-section, is presented in a paper by Gonga Rao and Spyrakos [3]. The stiffness matrix and the inertia matrix for a beam with a linearly variable height were determined by Gupta [4] who used them to solve the eigenproblem for a cantilever beam and for a simply supported beam. Eisenberger [5] determined the rigidity-matrix elements for several kinds of non-prismatic beams. Eisenberger and Reich [6] applied the finite element method to the solution of static and dynamic problems, approximating the displacement of the beam by 3rd degree polynomials. The beam's rigidity and

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density in the considered problems was described by power series. Formulas for the determination of the rigidity matrix"s elements for a beam element with variable axial, torsional and flexural rigidity described by power series were presented by Eisenberger in reference [7]. The same polynomial approximation was used by Klasztorny [8] to determine the rigidity matrix and the inertia matrix of Euler- and Timoshenko-type beam finite elements. Many exemplary problems illustrating the use of this method were solved in reference [8]. The generalization of the formulas determined in reference [7], which describe the rigidity and inertia matrices for variable rigidity and density bars resting on a two-parameter foundation, can be found in a paper by Glabisz [9], where the problem of the stability of a non-prismatic rod subjected to non-conservative loads is considered. Similar to references [7, 8], power series were used to approximate the displacement functions.

The present paper deals with the problem of the linear vibration of a beam with variable strength and geometric parameters, resting on a two-parameter, heterogeneous elastic foundation [10]. It is assumed that the variable parameters of the bar, such as flexural and axial rigidity and density, the variable parameters of the foundation and the load can be represented by a series expansion in relation to 1st kind Chebyshev polynomials. Using the theorems and relationships applicable to these polynomials found in the monograph [11], a solution in the form of a Chebyshev series is found. The coefficients of this solution are defined by closed analytical formulas. The longitudinal and transverse vibration of the bar is analyzed. By analogy to the equations, which describe the longitudinal and flexural motion of the bar, the relations derived for the longitudinal vibration can be used to determine the flexural vibration. This method was applied to solve, as an example, the eigenproblem for a simply supported beam and a cantilever beam. The obtained numerical results were compared with those published in references [2, 4].

2. FORMULATION OF PROBLEM

A non-prismatic rectilinear Euler beam 2a in length, resting on a two-parameter elastic foundation, subjected to dynamic normal load $P(X, t)$ and tangent load $R(X, t)$ (Figure 1) is considered.

The linear, transverse vibration of the beam is described by the following partial differential equations:

$$\frac{\partial^2}{\partial X^2}\left(EJ(X)\frac{\partial^2 W}{\partial X^2}\right) - \frac{\partial}{\partial X}\left(N(X)\frac{\partial W}{\partial X}\right) - C(X)\frac{\partial^2 W}{\partial X^2} + K(X)W(X) + \rho(x)\frac{\partial^2 W}{\partial t^2} = P(X, t), \tag{1}$$

$$-\frac{\partial}{\partial X}\left(EA(X)\frac{\partial U}{\partial X}\right) + F(X)U(X) + \rho(X)\frac{\partial^2 U}{\partial t^2} = R(X, t), \tag{2}$$

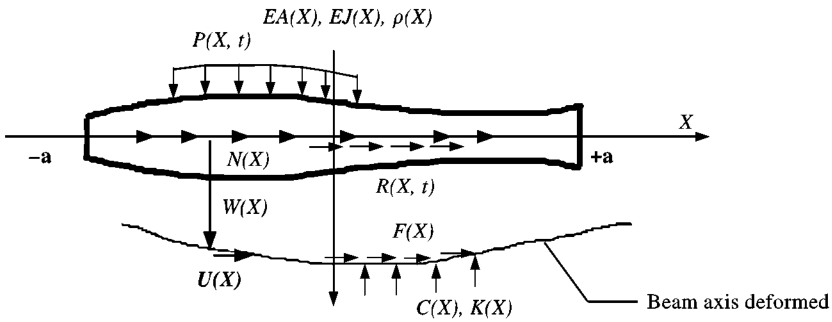

Figure 1. A diagram of a typical non-prismatic beam.

where W and U stand, respectively, for the displacements which are perpendicular and tangent to the beam axis, E is Young's modulus, A and J are the beam's cross-sectional area and moment of inertia, ρ is mass per unit of length, and $F(X)$, $K(X)$, $C(X)$ are functions describing elastic-foundation reactions.

To define boundary conditions, which follow from the type of support at the points $\pm a$, the relations defining the beam cross-section angular displacement, the bending moments, the shearing forces and the axial forces will be used:

$$\Phi(X, t) = \frac{\partial W}{\partial X},$$

$$M(X, t) = -EJ\frac{\partial^2 W}{\partial X^2},$$

$$T(X, t) = -\frac{\partial}{\partial X}\left(EJ\frac{\partial^2 W}{\partial X^2}\right) - N\frac{\partial W}{\partial X},$$

$$S(X, t) = EA\frac{\partial U}{\partial X}, \tag{3}$$

As dimensionless quantities $x = X/a$, $w = W/a$, $u = U/a$ are introduced, equations (1) and (2) assume the following form:

$$\overline{EJ}(x)\frac{\partial^4 w}{\partial x^4} + \left(2\frac{\partial\overline{EJ}(x)}{\partial x}\right)\frac{\partial^3 w}{\partial x^3} + \left(\frac{\partial^2\overline{EJ}(x)}{\partial x^2} - n(\bar{N}(x) + \bar{C}(x))\right)\frac{\partial^2 w}{\partial x^2} - n\frac{\partial\bar{N}(x)}{\partial x}\frac{\partial w}{\partial x}$$

$$+ n\bar{K}(x)w + g\bar{\rho}(x)\frac{\partial^2 w}{\partial t^2} = n\bar{p}(x, t), \tag{4}$$

$$-d\left(\overline{EA}(x)\frac{\partial^2 u}{\partial x^2} + \frac{\partial\overline{EA}(x)}{\partial x}\frac{\partial u}{\partial x}\right) + n\bar{F}(x)u + g\bar{\rho}(x)\frac{\partial^2 u}{\partial t^2} = n\bar{r}(x, t), \tag{5}$$

and relations (3) are expressed by these formulas

$$\phi(x,t) = \Phi(ax,t) = \frac{\partial w}{\partial x},$$

$$m(x,t) = \frac{M(ax,t)a}{EJ_0} = -\overline{EJ}\frac{\partial^2 w}{\partial x^2},$$

$$t(x,t) = \frac{T(ax,t)a^2}{EJ_0} = -\left(\frac{\partial}{\partial x}\overline{EJ}\right)\frac{\partial^2 w}{\partial x^2} - \overline{EJ}\frac{\partial^3 w}{\partial x^3} - n\bar{N}\frac{\partial w}{\partial x},$$

$$s(x,t) = \frac{S(ax,t)}{EA_0} = \overline{EA}\frac{\partial u}{\partial x}, \tag{6}$$

where

$$EJ = EJ_0\overline{EJ}, \quad N = P_0\bar{N}, \quad C = P_0\bar{C}, \quad K = \frac{P_0}{a^2}\bar{K}, \quad \rho = \rho_0\bar{\rho}, \quad P = \frac{P_0}{a}\bar{p},$$

$$EA = EA_0\overline{EA}, \quad F = \frac{P_0}{a^2}\bar{F}, \quad R = \frac{P_0}{a}\bar{r}, \quad n = \frac{a^2P_0}{EJ_0}, \quad g = \frac{a^4\rho_0}{EJ_0}, \quad d = \frac{a^2EA_0}{EJ_0}, \tag{7}$$

and EJ_0, EA_0, ρ_0, P_0 are reference quantities.

To simplify the notation, we shall assume consistently EJ, EA, N, C, K, ρ, F instead of $\overline{EJ}$, $\overline{EA}$, $\bar{N}$, $\bar{C}$, $\bar{K}$, $\bar{\rho}$, $\bar{F}$.

The boundary conditions for the basic types of support of the beam at points $\pm a$ are specified in Table 1.

Equations (4) and (5) supplemented with the boundary conditions will constitute a basis for the solution of the formulated problem.

3. SOLUTION OF THE PROBLEM

To solve equations (4) and (5), the following theorem concerning ordinary differential equations [11] will be used:

Theorem. *If a function f satisfies the following $n > 0$-order linear differential equation*

$$\sum_{m=0}^{m} \hat{P}_m(x) f^{(n-m)}(x) = \hat{P}(x), \tag{8}$$

and

$$Q_m(x) = \sum_{j=0}^{m} (-1)^{m+j}\binom{n-j}{m-j}\hat{P}_j^{(m-j)}(x), \quad m = 0, 1, \ldots, n, \tag{9}$$

and the Chebyshev series coefficients of functions $(Q_0 f)^{(n)}$, $(Q_1 f)^{(n-1)}, \ldots, Q_n f$, $\hat{P}$ are defined for each integer k, the following identity is true:

$$\sum_{m=0}^{n} 2^{n-m}\sum_{j=0}^{m} b_{nmj}(k)a_{k-m+2j}[Q_m(x)f(x)] = \sum_{j=0}^{m} b_{nnj}(k)a_{k-n+2j}[\hat{P}(x)], \tag{10}$$

TABLE 1

Boundary conditions for basic beam support types

Type of support	w	φ	t	m	u	s
			0	0		0
	0			0		0
	0			0	0	
	0	0			0	
		0	0		0	

where $b_{nmj}(k)$ are polynomials of integer variable k

$$b_{nmj}(k) = (-1)^j \binom{m}{j} (k-n)_{n-m+j}(k-m+2j)(k+j+1)_{n-j}(k^2-n^2)^{-1},$$

$$m = 0, 1, \ldots, n; \quad j = 0, 1, \ldots, m. \tag{11}$$

$$(c)_k = \begin{cases} 1 & \text{for } k = 0, \\ c(c+1)(c+2)\ldots(c+k-1) & \text{for } k = 1, 2, \ldots, \end{cases} \tag{12}$$

and $a_k[h]$ is a k-th coefficient of the expansion of function $h(x)$ into a Chebyshev series in relation to 1st kind Chebyshev polynomials.

The proof of this theorem can be found in reference [11, pp. 231–234].

Solutions of partial differential equations (4) and (5) will be sought in the form of the following Chebyshev series:

$$w(x, t) = \sum_{l=0}^{\infty}{}' a_l[w]\, T_l(x) = \sum_{l=0}^{\infty}{}' w_l(t) T_l(x), \tag{13}$$

$$u(x, t) = \sum_{l=0}^{\infty}{}' a_l[u]\, T_l(x) = \sum_{l=0}^{\infty}{}' u_l(t) T_l(x), \tag{14}$$

where

$$\sum_{l=0}^{\infty}{}' a_l[f] = \frac{1}{2} a_0[f] + a_1[f] + a_2[f] + \cdots, \tag{15}$$

and $a_l[w]$, $a_l[u]$ are the sought coefficients of the expansion of displacements functions w and u into Chebyshev series, denoted further as w_l and u_l respectively.

Since the quoted theorem applies to ordinary differential equations in which the unknown function is a one-variable function, time variable t will be treated as a parameter.

One starts solving the problem by solving the 4th order equation (4) which describes the displacement w. In this case, functions $\hat{P}_m$, $\hat{P}$ in equation (8) are defined by the following formulas:

$$\hat{P}_0(x) = EJ(x),$$

$$\hat{P}_1(x) = 2\frac{\partial EJ(x)}{\partial x},$$

$$\hat{P}_2(x) = \frac{\partial^2 EJ(x)}{\partial x^2} - n(N(x) + C(x)),$$

$$\hat{P}_3(x) = -n\frac{\partial N(x)}{\partial x},$$

$$\hat{P}_4(x) = nK(x),$$

$$\hat{P}(x,t) = np(x,t) - g\rho(x)\frac{\partial^2 w}{\partial t^2} = p(x,t) - g\rho(x)\ddot{w}(x,t). \tag{16}$$

Substituting expressions which describe functions $\hat{P}_m$ into formula (9), gives

$$Q_0(x) = EJ(x),$$

$$Q_1(x) = \;-2\frac{\partial EJ(x)}{\partial x},$$

$$Q_2(x) = \frac{\partial^2 EJ(x)}{\partial x^2} - n(N(x) + C(x)),$$

$$Q_3(x) = n\frac{\partial N(x)}{\partial x},$$

$$Q_4(x) = nK(x). \tag{17}$$

When polynomials $b_{nmj}(k)$ are calculated and the relation (see reference [11] 128, (33)) specifying the value of the kth coefficient of the expansion of the product of functions $f(x)$ and $g(x)$ by the following formula:

$$a_k[f(x)\cdot g(x)] = \frac{1}{2}\sum_{l=0}^{\infty}{}' a_l[f](a_{k-l}[g] + a_{k+l}[g]) \tag{18}$$

is used, are obtains equations (10) in the following form:

$$8(k^2-9)(k^2-4)(k^2-1)k\sum_{l=0}^{\infty}{}' a_l[w]\{a_{k-l}[Q_0] + a_{k+l}[Q_0]\}$$

$$+ 4(k^2-9)(k^2-4)(k^2-1)\sum_{l=0}^{\infty}{}' a_l[w]\{a_{k-l-1}[Q_1]$$

$$+ a_{k+l-1}[Q_1] - a_{k-l+1}[Q_1] - a_{k+l+1}[Q_1]\}$$

$$+ 2(k^2-9)(k^2-4)\sum_{l=0}^{\infty}{}' a_l[w]\{(k+1)(a_{k-l-2}[Q_2]+a_{k+l-2}[Q_2])$$

$$-2k(a_{k-1}[Q_2]+a_{k+1}[Q_2])+(k-1)(a_{k-l+2}[Q_2]+a_{k+l+2}[Q_2])\}$$

$$+(k^2-9)\sum_{l=0}^{\infty}{}' a_l[w]\{(k+1)(k+2)(a_{k-l-3}[Q_3]+a_{k+l-3}[Q_3])$$

$$-3k(k-1)(k+2)(a_{k-l-1}[Q_3]+a_{k+l+1}[Q_3]+3(k+1)(k-2)(a_{k-l+1}[Q_3]$$

$$+a_{k+l+1}[Q_3])-3k(k-1)(k-2)(a_{k-l+3}[Q_3]+a_{k+l+3}[Q_3])\}$$

$$+\frac{1}{2}\sum_{l=0}^{\infty}{}' a_l[w]\{(k+1)(k+2)(k+3)(a_{k-l-4}[Q_4]+a_{k+l-4}[Q_4])$$

$$-4(k+3)(k^2-4)(a_{k-l-2}[Q_4]+a_{k+l-2}[Q_4])+6k(k^2-9)(a_{k-l}[Q_4]$$

$$+a_{k+1}[Q_4])-4(k-3)(k^2-4)(a_{k-l+2}[Q_4]+a_{k+l+2}[Q_4])$$

$$+(k-1)(k-2)(k-3)(a_{k-l+4}[Q_4]+a_{k+l+4}[Q_4])\}$$

$$=n\{(k+1)(k+2)(k+3)a_{k-4}[p]-4(k+3)(k^2-4)a_{k-2}[p]$$

$$+6k(k^2-9)a_k[p]-4(k-3)(k^2-4)a_{k+2}[p]+(k-1)(k-2)(k-3)a_{k+4}[p]\}$$

$$-\frac{1}{2}g\sum_{l=0}^{\infty}{}' a_l[\ddot{w}]\{(k+1)(k+2)(k+3)(a_{k-l-4}[\rho]+a_{k+l-4}[\rho])$$

$$-4(k+3)(k^2-4)(a_{k-l-2}[\rho]+a_{k+l-2}[\rho])+6k(k^2-9)(a_{k-l}[\rho]+a_{k+l}[\rho])$$

$$-4(k-3)(k^2-4)(a_{k-l+2}[\rho]+a_{k+l+2}[\rho])$$

$$+(k-1)(k-2)(k-3)(a_{k-l+4}[\rho]+a_{k+l+4}[\rho])\},\quad k=0,1,2,3,\ldots. \tag{19}$$

$a_l[w]$ are the sought coefficients of the expansion of the displacement function w, and $a_l[\ddot{w}]$ are coefficients of the expansion of function $\ddot{w}$

$$\ddot{w}(x,t)=\frac{\partial^2 w(x,t)}{\partial t^2}=\sum_{l=0}^{\infty}{}' a_l[\ddot{w}]T_l(x)=\sum_{l=0}^{\infty}{}' \ddot{w}_l(t)T_l(x), \tag{20}$$

further denoted as $\ddot{w}_l$.

If the expansions of the functional coefficients occurring in differential equation (4) are denoted, respectively, by

$$EJ(x)=\sum_{l=0}^{\infty}{}' e_l T_l(x),\quad N(x)=\sum_{l=0}^{\infty}{}' n_l T_l(x),$$

$$C(x)=\sum_{l=0}^{\infty}{}' c_l T_l(x),\quad K(x)=\sum_{l=0}^{\infty}{}' k_l T_l(x),$$

$$\rho(x)=\sum_{l=0}^{\infty}{}' g_l T_l(x),\quad p(x,t)=\sum_{l=0}^{\infty}{}' p_l(t) T_l(x), \tag{21}$$

then the coefficients of the expansion of function $Q_i(x)$ into a Chebyshev series will have this form

$$a_l[Q_0] = a_l[EJ] = e_l,$$

$$a_l[Q_1] = a_l\left[-2\frac{\partial EJ}{\partial x}\right] = -2e_l',$$

$$a_l[Q_2] = a_l\left[\frac{\partial^2 EJ}{\partial x^2} - n(N + C)\right] = e'' - n(n_l + c_l),$$

$$a_l[Q_3] = a_l\left[n\frac{\partial N}{\partial x}\right] = nn_l',$$

$$a_l[Q_4] = a_l[K] = k_l, \tag{22}$$

where the following coefficient notation convention was adopted: if the lth coefficient of the expansion of function f is denoted by a_l, i.e., $a_l = a_l[f]$, then a_l' stands for the lth coefficient of the expansion of function f', i.e., $a_l' = a_l[f']$.

If the following relation ([11], 124, (17))

$$a_l = \frac{1}{2l}(a_{l-1}' - a_{l+1}'), \quad l \neq 0, \tag{23}$$

is used and this identity (neglecting the proof)

$$\begin{aligned}(k+1)(k-l-1)e_{k-l-1}' + (k+1)(k+l-1)e_{k+l-1}' \\ - (k-1)(k-l+1)e_{k-l+1}' - (k-1)(k+l+1)e_{k+l+1}' \\ = 2(k+1)(k-l-1)(k-l)e_{k-l} - 4l\sum_{j=1}^{l-1}(k-l+2j)e_{k-l+2j} \\ + 2(k-1)(k+l+1)(k+l)e_{k+l},\end{aligned} \tag{24}$$

one obtains an infinite system of ordinary differential equations which can be used to calculate coefficients w_l of the expansion of the displacement function w, given by formula (13):

$$\begin{aligned}\sum_{l=0}^{\infty}{}'\{8(k^2-9)(k^2-1)l[k+1)(l-1)e_{k-l} - 2\sum_{j=1}^{l-1}(k-l+2j)e_{k-l+2j} \\ + (k-1)(l-1)e_{k+l}] - 2n(k^2-9)l[(k+1)(k+2)(n_{k-l-2} - n_{k+l-2}) \\ - 2(k^2-4)(n_{k-l} + n_{k+l}) + (k-1)(k-2)(n_{k-l+2} - n_{k+l+2}) \\ - 2n(k^2-9)(k^2-4)[(k+1)(c_{k-l-2} + c_{k+l-2}) - 2k(c_{k-l} + c_{k+l})\end{aligned}$$

$$+ (k-1)(c_{k-l+2} + c_{k+l+2})] + \frac{1}{2} n[(k+1)(k+2)(k+3)(k_{k-l-4} + k_{k+l-4})$$

$$- 4(k+3)(k^2-4)(k_{k-l-2} + k_{k+l-2}) + 6k(k^2-9)(k_{k-l} - k_{k+l})$$

$$- 4(k-3)(k^2-4)(k_{k-l+2} + k_{k+l+2}) + (k+1)(k+2)(k+3)(k_{k-l+4} + k_{k+l+4})]\} w_l$$

$$+ \frac{1}{2} g \{(k+1)(k+2)(k+3)(g_{k-l-4} + g_{k+l-4}) - 4(k+3)(k^2-4)(g_{k-l-2} + g_{k+l-2})$$

$$+ 6k(k^2-9)(g_{k-l} - g_{k+l}) - 4(k-3)(k^2-4)(g_{k-l+2} + g_{k+l+2})$$

$$+ (k+1)(k+2)(k+3)(g_{k-l+4} + g_{k+l+4})\} \ddot{w}_l$$

$$= n\{(k+1)(k+2)(k+3)p_{k-4} - 4(k+3)(k^2-4)p_{k-2} + 6k(k^2-9)p_k$$

$$- 4(k-3)(k^2-4)p_{k+2} + (k+1)(k+2)(k+3)p_{k+4}, \quad k = 0, 1, 2, 3, \ldots. \tag{25}$$

To determine the displacement u, differential equation (5) should be solved. To solve it, we shall use the theorem presented at the beginning of section 3 (see formulas (8)–(12)). Functions $\hat{P}_m$, $\hat{P}$ in formula (8) and related to them by functions Q_m (9) are defined in this case by the following formulas:

$$\hat{P}_0(x) = -\, d\, EA(x),$$

$$\hat{P}_1(x) = -\, d \frac{\partial EA(x)}{\partial x},$$

$$\hat{P}_2(x) = nF(x),$$

$$\hat{P}(x,t) = nr(x,t) - g \frac{\partial^2 u}{\partial t^2} = nr(x,t) - g\ddot{u}, \tag{26}$$

$$Q_0(x) = -\, d\, EA(x),$$

$$Q_1(x) = d \frac{\partial EA(x)}{\partial x},$$

$$Q_2(x) = nF(x). \tag{27}$$

The function of the displacement u will be sought in the form of a Chebyshev series defined by formula (14). If one treats variable t as a parameter, having calculated coefficients $b_{nmj}(k)$ defined by formula (11) and applied relation (18), one gets

$$2(k^2-1)k \sum_{l=0}^{\infty}{}' a_l[u] \{a_{k-1}[Q_0] + a_{k+1}[Q_0]\}$$

$$+ (k^2-1) \sum_{l=0}^{\infty}{}' a_l[u] \{a_{k-l-1}[Q_1] + a_{k+l-1}[Q_1] + a_{k-l+1}[Q_1] + a_{k+l+1}[Q_1]\}$$

$$+\frac{1}{2}\sum_{l=0}^{\infty}{}' a_l[u]\{(k+1)(a_{k-l-2}[Q_2]+a_{k+l-2}[Q_2])-2k(a_{k-l}[Q_2]+a_{k+l}[Q_2])$$

$$+(k-1)(a_{k-l+2}[Q_2]+a_{k+l+2}[Q_2])\}=n\{(k+1)a_{k-2}[r]-2ka_k[r]$$

$$+(k-1)a_{k+2}[r]-\frac{1}{2}g\sum_{l=0}^{\infty}{}' a_l[\ddot{u}]\{(k+1)(a_{k-l-2}[\rho]+a_{k+l-2}[\rho])$$

$$-2k(a_{k-l}[\rho]+a_{k+l}[\rho])+(k-1)(a_{k-l+2}[\rho]+a_{k+l+2}[\rho])\}. \tag{28}$$

If functions $EA(x)$, $F(x)$, $\rho(x)$, $r(x, t)$ are expanded into Chebyshev series

$$EA(x)=\sum_{l=0}^{\infty}{}' d_l T_l(x), \quad F(x)=\sum_{l=0}^{\infty}{}' f_l T_l(x),$$

$$\rho(x)=\sum_{l=0}^{\infty}{}' g_l T_l(x), \quad r(x,t)=\sum_{l=0}^{\infty}{}' r_l(t)\,T_l(x) \tag{29}$$

and similar transformation as in the case of equation (4) are applied, the following infinite system of ordinary differential equations, enabling the determination of coefficients u_l, will be obtained:

$$\sum_{l=0}^{\infty}{}'\{-2d(k^2-1)k(d_{k-l}+d_{k+l})-2d(k^2-1)\,l\,(d_{k-l}+d_{k+l})$$

$$+\frac{1}{2}n[(k+1)(f_{k-l-2}+f_{k+l-2})-2k(f_{k-l}+f_{k+l})+(k-1)(f_{k-l+2}+f_{k+l+2})]\}u_l$$

$$+\frac{1}{2}g\sum_{l=0}^{\infty}{}'\{(k+1)(g_{k-l-2}+g_{k+l-2})-2k(g_{k-l}+g_{k+l})+(k-1)(g_{k-l+2}+g_{k+l+2})\}\ddot{u}_l$$

$$=(k+1)\,r_{k-2}-2k\,r_k+(k-1)\,r_{k+2}. \tag{30}$$

So far no boundary conditions have been taken into account. The conditions follow from the type of support at points ± 1 (beam-ends) and for basic modes of support they are described in Table 1. To describe them, formulas (6) will be used, expansions of functions $EJ(x)$, $N(x)$, $EA(x)$ into Chebyshev series (formulas (21) and (29)) and the following relations ([11] 48, (14), (16)):

$$T_n^{(m)}(1)=\begin{cases}1 & \text{for } m=0,\\ \dfrac{n}{(2m-1)!!}\displaystyle\prod_{k=-m+1}^{m-1}(n+k) & \text{for } m>0.\end{cases} \tag{31}$$

$$T_n^{(m)}(-1)=(-1)^{n-m}\,T_n^{(m)}(1). \tag{32}$$

To determine the values of the cross-sectional forces defined in formula (6), at points ± 1, it is necessary to calculate functions EJ, $\partial EJ/\partial x$, N, EA at these points.

The values of Chebyshev polynomials and their derivatives ($m = 0, 1, 2, 3$) at points ± 1, calculated from formulas (28) and (29) are

$$
\begin{aligned}
&T_n(1) = 1, && T_n(-1) = (-1)^n,\\
&T'_n(1) = n^2, && T'_n(-1) = -(-1)^n n^2,\\
&T''_n(1) = n^2(n^2-1)/3, && T''_n(-1) = (-1)^n n^2(n^2-1)/3,\\
&T'''_n(1) = n^2(n^2-1)(n^2-4)/15, && T'''_n(-1) = -(-1)^n n^2(n^2-1)(n^2-4)/15.
\end{aligned}
\tag{33}
$$

If one substitutes them into the expansions of the functions EJ, $\partial EJ/\partial x$, N, EA, one obtains

$$
\begin{aligned}
EJ(+1) &= EJ_+ = \sum_{l=0}^{\infty}{}' e_l T_l(1) = \sum_{l=0}^{\infty}{}' e_l,\\
EJ(-1) &= EJ_- = \sum_{l=0}^{\infty}{}' e_l T_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l e_l,\\
N(+1) &= N_+ = \sum_{l=0}^{\infty}{}' n_l T_l(1) = \sum_{l=0}^{\infty}{}' n_l,\\
N(-1) &= N_- = \sum_{l=0}^{\infty}{}' n_l T_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l m_l,\\
\left.\frac{\partial EJ}{\partial x}\right|_{x=+1} &= EJ'_+ = \sum_{l=0}^{\infty}{}' e_l T_l(1) = \sum_{l=0}^{\infty}{}' l^2 e_l,\\
\left.\frac{\partial EJ}{\partial x}\right|_{x=-1} &= EJ'_- = \sum_{l=0}^{\infty}{}' e_l T'_l(-1) = -\sum_{l=0}^{\infty}{}' (-1)^l l^2 e_l,\\
EA(+1) &= EA_+ = \sum_{l=0}^{\infty}{}' d_l T_l(1) = \sum_{l=0}^{\infty}{}' d_l,\\
\mathrm{EA}(-1) &= \mathrm{EA}_- = \sum_{l=0}^{\infty}{}' d_l T_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l d_l.
\end{aligned}
\tag{34}
$$

If formulas (6) and the calculated values of the functions (33) and (34) are used, relations required to define the boundary conditions will be obtained. These relations bound with the transverse vibration problem and the longitudinal vibration problem, respectively, have the following form:

$$
\begin{aligned}
w(+1, t) &= \frac{W(+a, t)}{a} = w_+ = \sum_{l=0}^{\infty}{}' w_l,\\
w(-1, t) &= \frac{W(-a, t)}{a} = w_- = \sum_{l=0}^{\infty}{}' (-1)^l w_l,\\
\phi(+1, t) &= \Phi(+a, t) = \phi_+ = \sum_{l=0}^{\infty}{}' l^2 w_l,
\end{aligned}
$$

$$\phi(-1,t)=\Phi(-a,t)=\phi_-=-\sum_{l=0}^{\infty}{}'(-1)^l l^2 w_l,$$

$$m(+1,t)=\frac{M(+a,t)a}{EJ_0}=m_+=-EJ_+\frac{1}{3}\sum_{l=0}^{\infty}{}' l^2(l^2-1)w_l, \tag{35}$$

$$m(-1,t)=\frac{M(-a,t)a}{EJ_0}=m_-=-EJ_-\frac{1}{3}\sum_{l=0}^{\infty}{}'(-1)^l l^2(l^2-1)w_l,$$

$$t(+1,t)=\frac{T(+a,t)a^2}{EJ_0}=t_+=-\sum_{l=0}^{\infty}{}' l^2\left[\frac{1}{3}(l^2-1)EJ'_+ + \frac{1}{15}(l^2-1)(l^2-4)EJ_+ + nN_+\right]w_l,$$

$$t(-1,t)=\frac{T(-a,t)a^2}{EJ_0}=t_-=\sum_{l=0}^{\infty}{}'(-1)^l l^2\left[-\frac{1}{3}(l^2-1)EJ'_- + \frac{1}{15}(l^2-1)(l^2-4)EJ_- + nN_-\right]w_l$$

and

$$u(+1,t)=\frac{U(+a,t)}{a}=u_+=\sum_{l=0}^{\infty}{}' u_l,$$

$$u(-1,t)=\frac{U(-a,t)}{a}=u_-=\sum_{l=0}^{\infty}{}'(-1)^l u_l,$$

$$s(+1,t)=\frac{S(+a,t)}{EA_0}=s_+=EA_+\sum_{l=0}^{\infty}{}' l^2 u_l,$$

$$s(-1,t)=\frac{S(-a,t)}{EA_0}=s_-=-EA_-\sum_{l=0}^{\infty}{}'(-1)^l l^2 u_l. \tag{36}$$

In infinite systems of equations (25) and (30), depending on the order of differential equation n to which they apply, the first n equations for $k=0,1,\dots,n-1$ are satisfied as regards the identity. These equations are replaced by the boundary conditions defining equations.

This method, consisting in the search for a displacement function in the form of a Chebyshev series and its application to the solution of the partial differential equations of the presented theorem, leads to an infinite system of ordinary differential equations and in the case of stationary problems, to an infinite system of algebraic equations.

Infinite system of equations can be presented in the following matrix form:

$$\begin{bmatrix}\mathbf{A}_{pp} & \mathbf{A}_{pr}\\ \mathbf{A}_{rp} & \mathbf{A}_{rr}\end{bmatrix}\begin{bmatrix}\mathbf{w}_p\\ \mathbf{w}_r\end{bmatrix}+\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{B}_{rp} & \mathbf{B}_{rr}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{w}}_p\\ \ddot{\mathbf{w}}_r\end{bmatrix}=\begin{bmatrix}\mathbf{C}_p\\ \mathbf{C}_r\end{bmatrix}, \tag{37}$$

or after multiplication, in the form

$$\mathbf{A}_{pp}\mathbf{w}_p + \mathbf{A}_{pr}\mathbf{w}_r = \mathbf{C}_p,$$

$$\mathbf{A}_{rp}\mathbf{w}_p + \mathbf{A}_{rr}\mathbf{w}_r + \mathbf{B}_{rp}\ddot{\mathbf{w}}_p + \mathbf{B}_{rr}\ddot{\mathbf{w}}_r = \mathbf{C}_r, \tag{38}$$

where submatrices $\mathbf{A}_{pp}$, $\mathbf{A}_{pr}$ have dimensions $n \times n$ and $n \times \infty\, (n = 4$ or $2)$, respectively, and their elements are coefficients related to the boundary conditions; submatrices $\mathbf{A}_{rp}$, $\mathbf{A}_{rr}$ and $\mathbf{B}_{rp}$, $\mathbf{B}_{rr}$ are matrices of the coefficients occurring in equations (25) or (30), $\mathbf{w}_p = [w_0, \ldots, w_{n-1}]^{\mathrm{T}}, \mathbf{w}_r = [w_n,\ w_{n+1},\ w_{n+2}, \ldots]^{\mathrm{T}}$; and vectors $\mathbf{C}_p$, $\mathbf{C}_r$ define the boundary conditions and the coefficients associated with an external load.

From equation (38), if $\det \mathbf{A}_{pp} \neq 0$ ($\det \mathbf{A}_{pp} = 0$ for a geometrically variable system), one obtains

$$\mathbf{w}_p = -\,\mathbf{A}_{pp}^{-1}\mathbf{A}_{pr}\mathbf{w}_r + \mathbf{A}_{pp}^{-1}\mathbf{C}_p, \tag{39}$$

If equation (39) is substituted into equation $(38)_2$, then

$$(\mathbf{A}_{rr} - \mathbf{A}_{rp}\mathbf{A}_{pp}^{-1}\mathbf{A}_{pr})\mathbf{w}_r + (\mathbf{B}_{rr} - \mathbf{B}_{rp}\mathbf{A}_{pp}^{-1}\mathbf{A}_{pr})\ddot{\mathbf{w}}_r = \mathbf{C}_r - (\mathbf{A}_{rp} + \mathbf{B}_{rp})\mathbf{A}_{pp}^{-1}\mathbf{C}_p. \tag{40}$$

In the case of the eigenproblem when $\mathbf{C}_r = \mathbf{C}_p = \mathbf{0}$ and $\ddot{\mathbf{w}}_r = -\,\omega^2\mathbf{w}_r$, matrix equation (37) assumes this form

$$[(\mathbf{A}_{rr} - \mathbf{A}_{rp}\mathbf{A}_{pp}^{-1}\mathbf{A}_{pr}) - \omega^2(\mathbf{B}_{rr} - \mathbf{B}_{rp}\mathbf{A}_{pp}^{-1}\mathbf{A}_{pr})]\mathbf{w}_r = 0, \tag{41}$$

4. NUMERICAL EXAMPLES

To illustrate the method better, consider the eigenproblem for bars shown in Figures 2 and 3. The presented examples come from references [2–4].

The other parameters of the problem (ρ_V–is mass per unit of volume) are

$$E = 2{\cdot}068929 \times 10^{11}\ \mathrm{N/m^2}(30 \times 10^6\ \mathrm{lb/in^2}),$$

$$\rho_V = 7845{\cdot}4494\ \mathrm{kg/m^3}(0{\cdot}00073386\ \mathrm{lb\,s^2/in^4}).$$

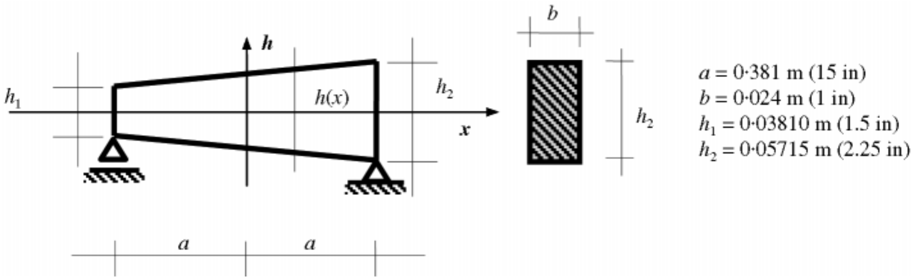

Figure 2. A simply supported beam — example 1.

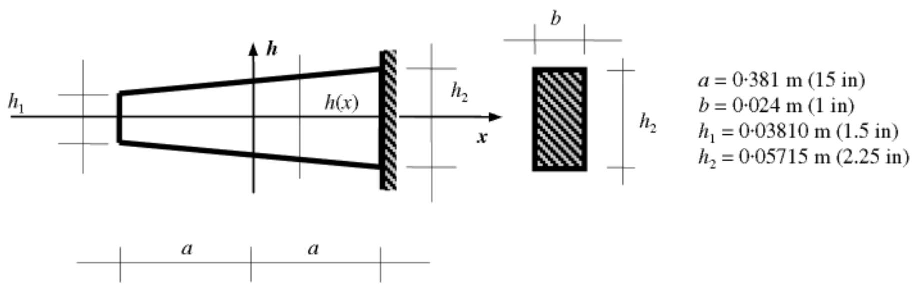

Figure 3. A cantilever beam — example 2.

If the following relations ([11] 25, (13))

$$T_0(x) = 1,$$

$$T_m(x) \cdot T_n(x) = \frac{1}{2}[T_{m-n}(x) + T_{m+n}(x)], \tag{42}$$

are used to expand the beam's geometrical and strength characteristics into Chebyshev series, which give

$$h(x) = c_0 T_0(x) + c_1 T_1(x),$$

$$A(x) = bc_0 T_0(x) + bc_1 T_1(x),$$

$$J(x) = \frac{1}{12} bh^3 = \frac{1}{12} b\left[\left(c_0^3 + \frac{3}{2} c_0 c_1^2\right)T_0(x) + \left(3c_0^2 c_1 + \frac{3}{4} c_1^3\right)T_1(x) + \frac{3}{2} c_0 c_1^2 T_2(x) + \frac{1}{4} c_1^3 T_3(x)\right], \tag{43}$$

where $c_0 = (h_1 + h_2)/2 = 4{\cdot}7625$, $c_1 = (h_2 - h_1)/2 = 0{\cdot}9525$. After substituting the numerical values and applying transformations one obtains the following Chebyshev expansions:

$$EJ(x) = \left[\frac{1}{2} \cdot 1002{\cdot}8567098\, T_0 + 286{\cdot}66564444\, T_1 + 28{\cdot}38273707\, T_2 + 0{\cdot}9460912335\, T_3\right] \times 10^2\ \mathrm{N\,m}^2,$$

$$EA(x) = \left[\frac{1}{2} \cdot 500{\cdot}5463376\, T_0 + 50{\cdot}05463376\, T_1\right] \times 10^6\ \mathrm{N},$$

$$\rho(x) = \left[\frac{1}{2} \cdot 18{\cdot}980888801\, T_0 + 1{\cdot}898088801\, T_1\right] \mathrm{kg/m}. \tag{44}$$

The boundary conditions required to solve the problem are as follows:
(a) for the beam in Figure 2

- transverse vibration problem

$$w(-1,t)=0,$$
$$m(-1,t)=0,$$
$$w(+1,t)=0,$$
$$m(+1,t)=0,$$

- longitudinal vibration problem

$$s(-1,t)=0,$$
$$u(+1,t)=0,$$

(b) for the beam in Figure 3

- transverse vibration problem

$$t(-1,t)=0,$$
$$m(-1,t)=0,$$
$$w(+1,t)=0,$$
$$\phi(+1,t)=0.$$

- longitudinal vibration problem

$$s(-1,t)=0,$$
$$u(+1,t)=0.$$

As it has already been mentioned, the infinite system of differential equations for the eigenproblem becomes a system of algebraic equations. To solve it, it will be limited to a finite system. Then the displacement functions are defined by finite sums of the Chebyshev series

$$w=\sum_{l=0}^{lw}{}' w_l T_l(x),$$
$$u=\sum_{l=0}^{lw}{}' u_l T_l(x) \tag{45}$$

Testing the convergence of the solutions, the system was solved for ever larger dimension lw of the approximation base (formula (45)).

The obtained eigenfrequencies for transverse and longitudinal vibrations ω_i and the first six eigenfunctions $\tilde{w}_i$ are presented for the simply supported beam in Tables 2 and 3 and Figures 4 and 5, and for the cantilever beam, in Table 4 and Figure 6 (for cantilever beam only eigenfrequencies and eigenfunctions for transverse vibration are presented since the longitudinal eigenfrequencies and the eigenfunctions for the cantilever beam are the same as for the simply supported beam).

The occurrence of eigenfrequencies with complex values may be a little surprising. One should bear in mind, however, that the matrix of the system-of-equations coefficients is not symmetric (symmetry of matrix is a sufficient condition for obtaining real eigenvalues). Such a matrix can be obtained by other solution methods, e.g., the finite element method. One should also bear in mind that although higher eigenfrequencies are real numbers, the values calculated at a limited approximation base can differ significantly from the actual values.

The transverse eigenfrequency values calculated for the system shown in Figure 2 have been compared with the results reported in references [2, 4] and with the

TABLE 2

Eigenfrequencies for simply supported beam — transverse vibration

lw	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
10	188,45	757,90	1731,62	3020,90	4261,61 − 948,149 i	4261,61 + 948,149 i	
11	188,44	758,47	1701,19	3246,87	4228,18	5900,53 − 1929,02 i	5900,53 + 1929,02 i
12	188,44	757,97	1706,07	3005,56	5316,36 − 862,489 i	5316,36 + 862,489 i	7946,19 − 3203,25 i
13	188,44	757,97	1703,86	3029,82	4626,08	7142,18 − 1823,16 i	7142,18 + 1823,16 i
14	188,44	757,99	1703,59	3027,54	4724,00	6473,88	9265,44 − 2995,34 i
15	188,44	757,99	1703,98	3025,60	4729,70	6809,61	8423,55
16	188,44	757,99	1703,99	3027,90	4722,50	6806,43	9655,41
17	188,44	757,99	1703,97	3028,01	4730,26	6793,73	9219,68
18	188,44	757,99	1703,97	3027,74	4731,03	6811,75	9247,68

lw	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}
12	7946,19 + 3203,25 i						
13	10440,9 − 4833,81 i	10440,9 + 4833,81 i					
14	9265,44 + 2995,34 i	13448,5 − 6884,73 i	13448,5 + 6884,73 i				
15	11751,2 − 4475,23 i	11751,2 + 4475,23 i	17042,1 − 9418,89 i	17042,1 + 9418,89 i			
16	10038,00	14662,6 − 6314,38 i	14662,6 + 6314,38 i	21300,3 − 12501,3 i	21300,3 + 12501,3 i		
17	12289,3 − 1278,51 i	12289,3 + 1278,51 i	18059,1 − 8557,13 i	18059,1 + 8557,13 i	26306,1 − 16200,9 i	26306,1 + 16200,9 i	
18	11836,30	15001,3 − 2341,42 i	15001,3 + 2341,42 i	21999,9 − 11249,3 i	21999,9 + 11249,3 i	32147,4 − 20590,6 i	32147,4 + 20590,6 i

frequency values obtained by the finite element method for the division of the system into 6 Euler-type bar elements (elements with four degrees-of-freedom). The values have been compiled in Table 5.

5. RECAPITULATION

The obtained results prove that the method is correct and useful for the solution of dynamic problems of non-prismatic beams. The presented numerical example shows agreement with results reported by other authors [2, 4] and the obtained results are satisfactorily accurate. One should note, however, that the accuracy decreases here for higher frequencies and eigenforms. One of the consequences of decreased accuracy is that the boundary conditions are not fulfilled in the case of

TABLE 3

Eigenfrequencies for simply supported beam — longitudinal vibration

lw	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
10	1824,94	5105,73	8449,31	11876,50	16204,00	17541,40	20737,40
11	1824,94	5105,83	8453,83	11787,60	15299,80	20460,2 − 1269,71 i	20460,2 + 1269,71 i
12	1824,94	5105,82	8455,38	11809,10	15097,10	18727,80	24200,3 − 2297,12 i
13	1824,94	5105,82	8455,19	11818,00	15159,00	18365,30	22148,10
14	1824,94	5105,82	8455,14	11816,40	15190,20	18495,90	21590,80
15	1824,94	5105,82	8455,14	11815,80	15182,80	18578,30	21813,30
16	1824,94	5105,82	8455,14	11815,90	15179,90	18554,20	21996,10

lw	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}
10	30268,30						
11	24170,70	35941,80					
12	24200,3 + 2297,12 i	27842,90	42121,80				
13	28125,3 − 3349,27 i	28125,3 + 3349,27 i	31824,70	48811,20			
14	25569,70	32269,9 − 4468,27 i	32269,9 + 4468,27 i	36141,80	56011,40		
15	24781,60	29022,40	36661,6 − 5653,71 i	36661,6 + 5653,71 i	40794,10	63722,90	
16	25105,10	27950,20	32544,90	41317,7 − 6903,58 i	41317,7 + 6903,58 i	45775,90	71945,80

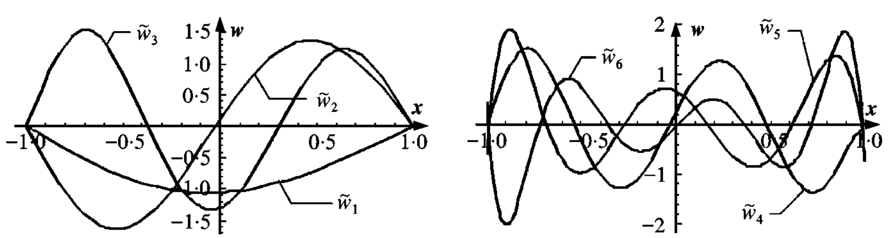

Figure 4. Eigenforms of simply supported beam — transverse vibration.

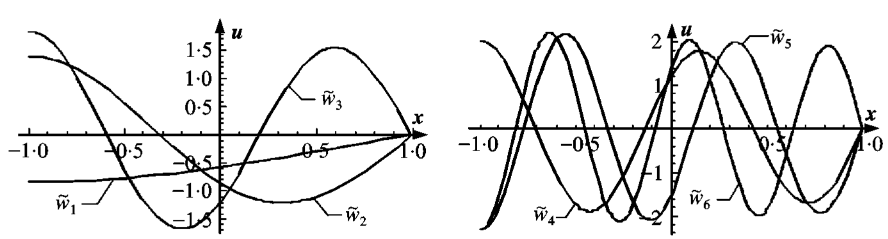

Figure 5. Eigenforms of simply supported beam — longitudinal vibration.

TABLE 4

Eigenfrequencies for cantilever beam — transverse vibration

lw	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
10	85,72	456,16	1179,56	2149,57	4536,25	5728,92	
11	85,67	456,30	1220,33	2195,59	3373,98	7316,91 − 818,159 i	7316,91 + 818,159 i
12	85,66	455,97	1222,09	2406,90	3386,53	4827,23	10019,4 − 1831,19 i
13	85,66	455,84	1216,82	2392,63	4325,42 − 354,473 i	4325,42 + 354,473 i	6532,21
14	85,66	455,81	1215,52	2353,13	4041,66	5912,29 − 1150,24 i	5912,29 + 1150,24 i
15	85,66	455,81	1215,51	2347,21	3858,32	6288,82	7584,03 − 2072,03 i
16	85,66	455,80	1215,55	2349,51	3841,73	5704,38	9070,20
17	85,66	455,80	1215,55	2350,27	3859,93	5665,95	7856,17
18	85,66	455,80	1215,55	2350,21	3864,60	5746,44	7757,53

lw.	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}
12	10019,4 + 1831,19 i						
13	13329,6 − 3012,78 i	13329,6 + 3012,78 i					
14	8528,15	17347,6 − 4466,76 i	17347,6 + 4466,76 i				
15	7584,03 + 2072,03 i	10824,10	22180,5 − 6250,55 i	22180,5 + 6250,55 i			
16	9403,2 − 3355,44 i	9403,2 + 3355,44 i	13395,80	27941,8 − 8415,1 i	27941,8 + 8415,1 i		
17	12255,60	11504,7 − 5020,26 i	11504,7 + 5020,26 i	16195,80	34750,7 − 11011,6 i	34750,7 + 11011,6 i	
18	10296,60	13917,4 − 7057,82 i	13917,4 + 7057,82 i	16031,90	19107,00	42732,4 − 14093,2 i	42732,4 + 14093,2 i

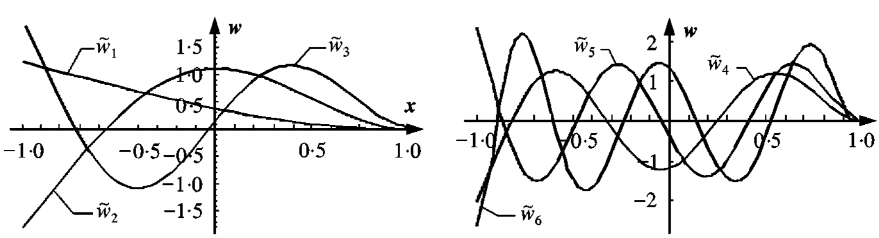

Figure 6. Eigenforms of cantilever beam — transverse vibration.

higher eigenforms. Although the example is limited to a simple scheme, the proposed method can be used to solve more complex systems such as beams with more complicated geometry and any distribution of mass and strength parameters, resting on a two-parameter heterogeneous elastic foundation. Since Chebyshev

TABLE 5

Comparison of results

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
This paper	188,44	757,99	1703,97	3027,75	4729,39	6808,80	9286,04	12103,70
[2] Exact	188,45	758,08	1704,90					
[4]	189,12	843,64	2116,50					
FEM	189,19	760,55	1715,09	3076,50	4873,81	7607,60	10357,42	14175,95

polynomials have some of the best approximation properties, it seems that the proposed method will be particularly useful for solving beams that are complex in shape. If methods involving expansions into power series (references [7–9]) are applied to such beams, this may lead, because of these methods' inferior approximation properties, to considerable inaccuracies when the number of terms in a series is limited. The formulas derived by the author enable the direct solution of such complex problems.

The application of the proposed method to the determination of exact shape functions for non-prismatic finite elements is particularly interesting since this is only one step away from building up a library of such elements.

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