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ON THE EFFECT OF AN ATTACHED SPRING-MASS SYSTEM ON THE FREQUENCY SPECTRUM OF LONGITUDINALLY VIBRATING ELASTIC RODS CARRYING A TIP MASS

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1. INTRODUCTION

The study in reference [1] was concerned essentially with the determination of the frequency equation and sensitivity of the eigenfrequencies of a fixed–free longitudinally vibrating elastic rod carrying a tip mass (primary system) to which a spring-mass (secondary system) is attached in-span. The present study deals mainly with the same mechanical system described above.

On the other hand, this article focuses on the qualitative investigation of the effect of attaching a secondary system on the frequency spectrum of the primary system, by making use of the frequency equation obtained in reference [1] and its special cases. For the sake of convenience and completeness of the presentation, the simple systems obtained by limit processes and the related frequency equations in reference [1] are included here in addition to the frequency equation of the main combined system.

2. THEORY

The system to be dealt with in present study is shown in Figure 1. It is essentially a fixed–free longitudinally vibrating elastic rod of axial rigidity EA and mass per unit length m carrying a tip mass M (primary system) to which a secondary spring-mass system is attached at a point along the span. The exact characteristic equation of the combined system described above in order to determine their eigenfrequencies from reference [1] is

$$(\cos\bar{\beta} - \beta_M\bar{\beta}\sin\bar{\beta})\left[\frac{\alpha_{me}}{\alpha_{ke}}\bar{\beta}^2 + \frac{\alpha_{me}}{2}\bar{\beta}\sin 2\eta\bar{\beta} - 1\right] + \alpha_{me}\bar{\beta}\,(\sin^2\eta\bar{\beta}\sin\bar{\beta} + \beta_M\bar{\beta}\cos\bar{\beta}) = 0. \tag{1}$$

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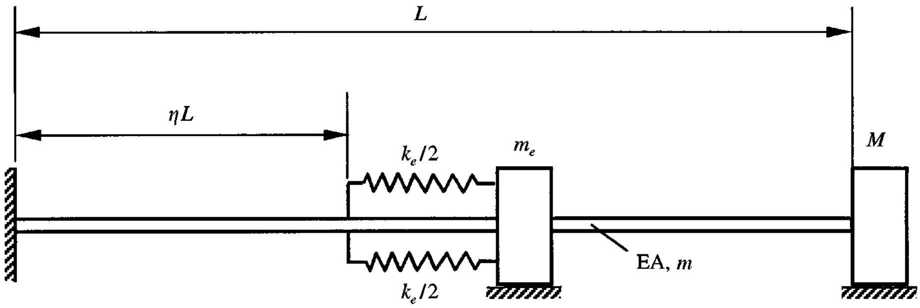

Figure 1. Fixed–free longitudinally vibrating elastic rod with a tip mass and spring-mass attached in-span.

where the following parameters are introduced:

$$\beta^2 = \frac{m\omega^2}{EA}, \quad \bar{\beta} = \beta L, \tag{2}$$

$$\alpha_{me} = \frac{m_e}{mL}, \quad \alpha_{ke} = \frac{k_e}{EA/L}, \quad \beta_M = \frac{M}{mL}$$

The roots of the transcendental equation above give the dimensionless frequency parameters and therefore the eigenfrequencies of the system in Figure 1. Having obtained the frequency equation of the general system in Figure 1, the frequency equations of simpler systems as limiting cases can be obtained from reference [1]. The first limiting case is obtained when α_{ke} is equal to zero. The frequency equation of the system for this case is

$$\beta_M \bar{\beta} \sin^2 \bar{\beta}(\tan \bar{\beta} + \cot \bar{\beta}) - 1 = 0. \tag{3}$$

The second limiting case is obtained when α_{ke} goes to infinity. The frequency equation of the system for this case is

$$(\cos \bar{\beta} - \beta_M \bar{\beta} \sin \bar{\beta})\left[-1 + \frac{\alpha_{me}}{2}\bar{\beta} \sin 2\eta\bar{\beta}\right] + \alpha_{me}\,\bar{\beta} \sin^2 \eta\bar{\beta}(\sin\bar{\beta} + \beta_M \bar{\beta} \cos \bar{\beta}) = 0. \tag{4}$$

The third limiting case is obtained when α_{me} goes to infinity. The frequency equation of the system for this case is

$$(\cos \bar{\beta} - \beta_M \bar{\beta} \sin \bar{\beta})\left(\frac{\bar{\beta}^2}{\alpha_{ke}} + \frac{\bar{\beta}}{2} \sin 2\eta\bar{\beta}\right) + \bar{\beta} \sin^2 \eta\bar{\beta}(\sin \bar{\beta} + \beta_M \bar{\beta} \cos \bar{\beta}) = 0. \tag{5}$$

The fourth limiting case is obtained when α_{me} is equal to zero. The frequency equation of the system for this case is obtained from equation (3).

The fifth limiting case is obtained when β_M is equal to zero. The frequency equation of the system for this case is

$$\cos\bar{\beta}\left[-1+\frac{\alpha_{me}}{\alpha_{ke}}\bar{\beta}^2+\frac{2\alpha_{me}}{\bar{\beta}\sin 2\eta\bar{\beta}}\right]+\alpha_{me}\,\bar{\beta}\sin^2\eta\bar{\beta}\sin\bar{\beta}=0. \tag{6}$$

The sixth limiting case is obtained when β_M goes to infinity. The frequency equation of the system for this case is

$$\alpha_{me}\bar{\beta}^2\sin^2\eta\bar{\beta}\cos\bar{\beta}-\bar{\beta}\sin\bar{\beta}\left[-1+\frac{\alpha_{me}}{\alpha_{ke}}\bar{\beta}^2+\frac{\alpha_{me}}{2}\bar{\beta}\sin 2\eta\bar{\beta}\right]=0. \tag{7}$$

Finally, the seventh limiting case is obtained when η is equal to zero. The frequency equation of the system for this case is obtained from equation (3).

3. NUMERICAL EVALUATIONS AND DISCUSSION

The effect of the attachment of the spring-mass system on the eigenfrequencies of the primary system is given in the form of figures. In this context, the effect of the variation of the position of the attachment point and the parameters of the spring-mass system on the frequency spectrum of the combined system are investigated. The numerical solutions of the roots of the transendental frequency equations (1)–(7) are carried out by MATHEMATICA.

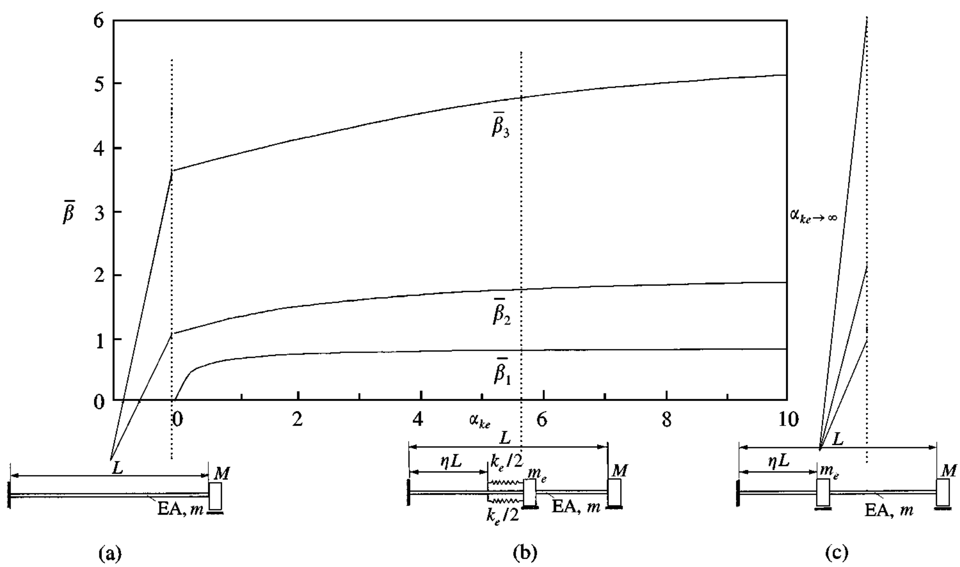

Figure 2. Plots of the first three eigenfrequency parameters of the system in Figure 1 as functions of the stiffness parameter α_{ke} ($\eta = 0{\cdot}4$, $\alpha_{me} = 1$, $\beta_M = 0{\cdot}5$), together with systems obtained through limiting cases: (a) Fixed–free longitudinally vibrating elastic rod with a tip mass [equation (3)], (b) Fixed–free longitudinally vibrating elastic rod with a tip mass and spring-mass attached in-span [equation (1)], (c) Fixed–free longitudinally vibrating elastic rod with a tip mass and in-span mass [equation (4)].

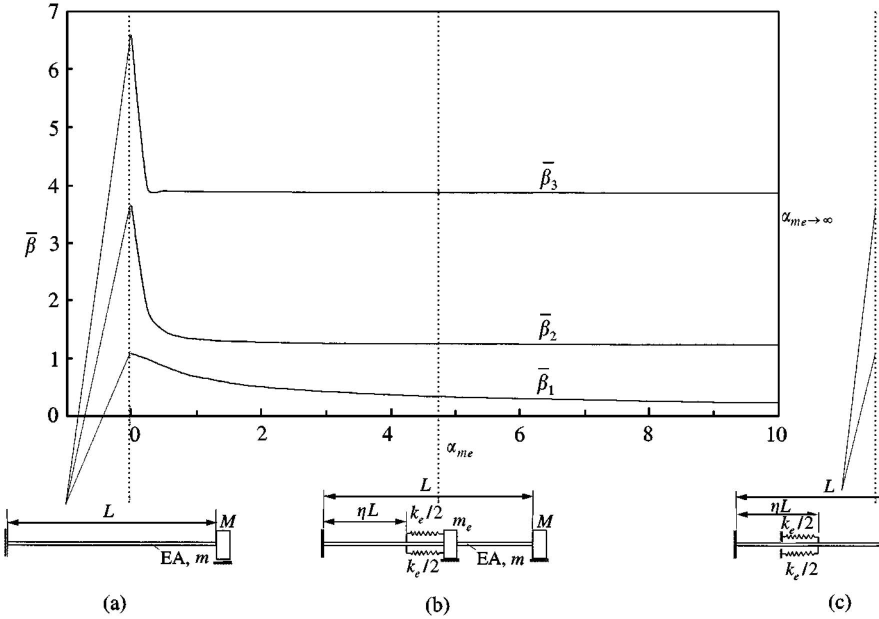

Figure 3. Plots of the first three eigenfrequency parameters of the system in Figure 1 as functions of the mass parameter $\alpha_{me}(\eta = 0{\cdot}4,\ \alpha_{ke} = 1,\ \beta_M = 0{\cdot}5)$, together with systems obtained through limiting cases: (a) Fixed–free longitudinally vibrating elastic rod with a tip mass [equation (3)]. (b) Fixed–free longitudinally vibrating elastic rod with a tip mass and spring-mass attached in-span [equation (1)]. (c) Fixed–free longitudinally vibrating elastic rod with a tip mass restrained by springs in-span [equation (5)].

In Figure 2, the variation of the first three eigenfrequency parameters of the combined system in Figure 1 is depicted as a function of the stiffness parameter α_{ke} in the range of 0–10, where the mass parameter is kept constant as $\alpha_{me} = 1{\cdot}0$, $\beta_M = 0{\cdot}5$ and η are taken equal to 0·4. The second and third curves originate on the vertical axis at the points $\bar{\beta} = 1{\cdot}076874$ and 3·643597, which correspond to the first two eigenfrequency parameters of a fixed–free longitudinally vibrating elastic rod with a tip mass M, such that $M = 0{\cdot}5$ mL, as shown on the left hand side of Figure 2 at the bottom. The first curve which can be attributed to the appended spring-mass system takes on very small values in the vicinity of the origin. After a rapid increase at the beginning, the values on this curve become slightly larger as the stiffness increases. It can be shown that this curve approaches the value $\bar{\beta} = 0{\cdot}943222$ as α_{ke} reaches infinity. This corresponds to the first eigenfrequency parameter of a fixed–free longitudinally vibrating elastic rod with a tip mass $M = 0{\cdot}5$ mL, to which a mass m_e is attached at the location $\eta = 0{\cdot}4$, where $m_e = mL$, as shown on the right-hand side of Figure 2. The second curve increases in a uniform manner as the stiffness parameter gets larger and reaches $\bar{\beta} = 2{\cdot}234041$, which represents the second eigenfrequency parameter of the elastic rod described above.

The third curve, originating at the second eigenfrequency parameter of a fixed–free longitudinally vibrating elastic rod with a tip mass M, such that $M = 0{\cdot}5$ mL, increases and reaches, for $\alpha_{ke} \to \infty$, the value $\bar{\beta} = 6{\cdot}004996$, which is

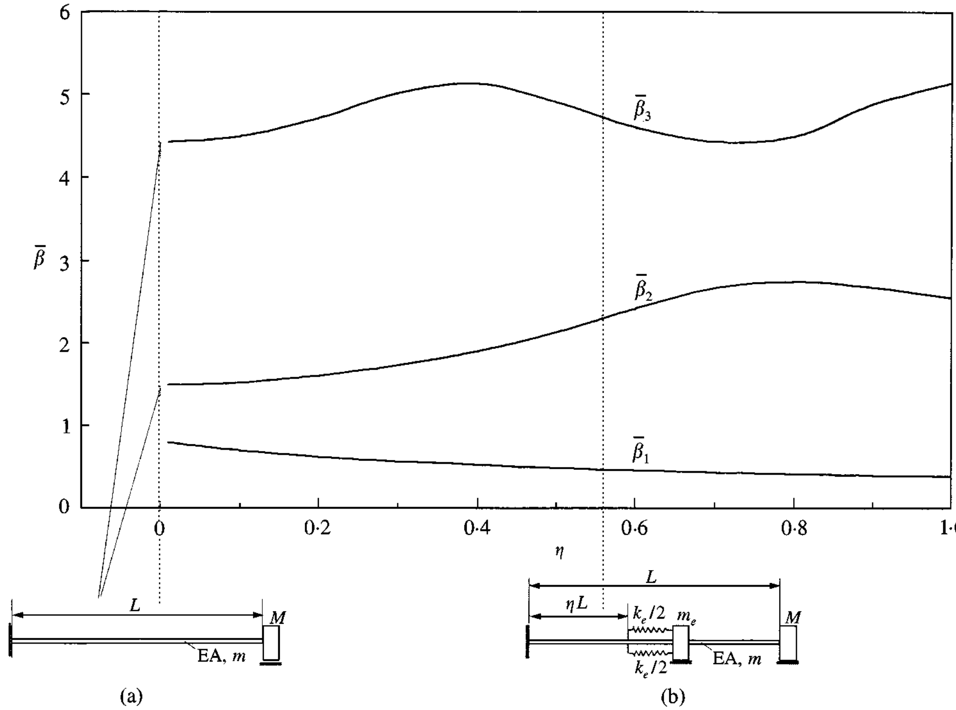

Figure 4. Plots of the first three eigenfrequency parameters of the system in Figure 1 as functions of the location parameter of the spring-mass system η; $\alpha_{me} = 5$, $\alpha_{ke} = 5$, and $\beta_M = 0{\cdot}1$ are taken, (a) Fixed–free longitudinally vibrating elastic rod with a tip mass [equation (3)]. (b) Fixed–free longitudinally vibrating elastic rod with a tip mass and spring-mass attached in-span [equation (1)].

the third eigenfrequency of the elastic rod above, with $\beta_M = M/mL = 0{\cdot}5$, $\eta = 0{\cdot}4$, $\alpha_{me} = 1{\cdot}0$.

In Figure 3, the first three eigenfrequency parameters of the system in Figure 1 are shown as functions of the dimensionless mass parameter α_{me}, where the stiffness parameter is kept constant at $\alpha_{ke} = 1{\cdot}0$; a tip mass exists such that $\beta_M = M/mL = 0{\cdot}5$ and $\eta = 0{\cdot}4$. The three curves originate at the first three eigenfrequency parameters $\bar{\beta}_1 = 1{\cdot}076874$, $\bar{\beta}_2 = 3{\cdot}643597$ and $\bar{\beta}_3 = 6{\cdot}578334$ of the elastic rod as shown on the left-hand side of the bottom of Figure 3 where $\alpha_{ke} = 1{\cdot}0$ and $\beta_M = 0{\cdot}5$. The values on the first curve diminish continuously and vanish for $\alpha_{me} \to \infty$. The second and third curves vary slowly after a sudden decrease at the beginning. As α_{me} goes to infinity, the second curve approaches $\bar{\beta}_1 = 1{\cdot}170129$, which represents the first eigenfrequency parameter of the fixed–free longitudinally vibrating elastic rod with a tip mass and restrained by springs in-span, as shown on the right-hand side of Figure 3, where $\beta_M = 0{\cdot}5$, $\alpha_{ke} = 1{\cdot}0$ and $\eta = 0{\cdot}4$. The third curve remains practically constant over the whole range considered. Its limit value for $\alpha_{me} \to \infty$ can be shown to be $\bar{\beta}_3 = 3{\cdot}869921$ which is the second eigenfrequency parameter of the elastic rod mentioned above.

So far, the effect of the variation of the mass and stiffness parameters of the secondary system on the eigenfrequencies of the combined system have been investigated. As a third application, one can consider how the position of the

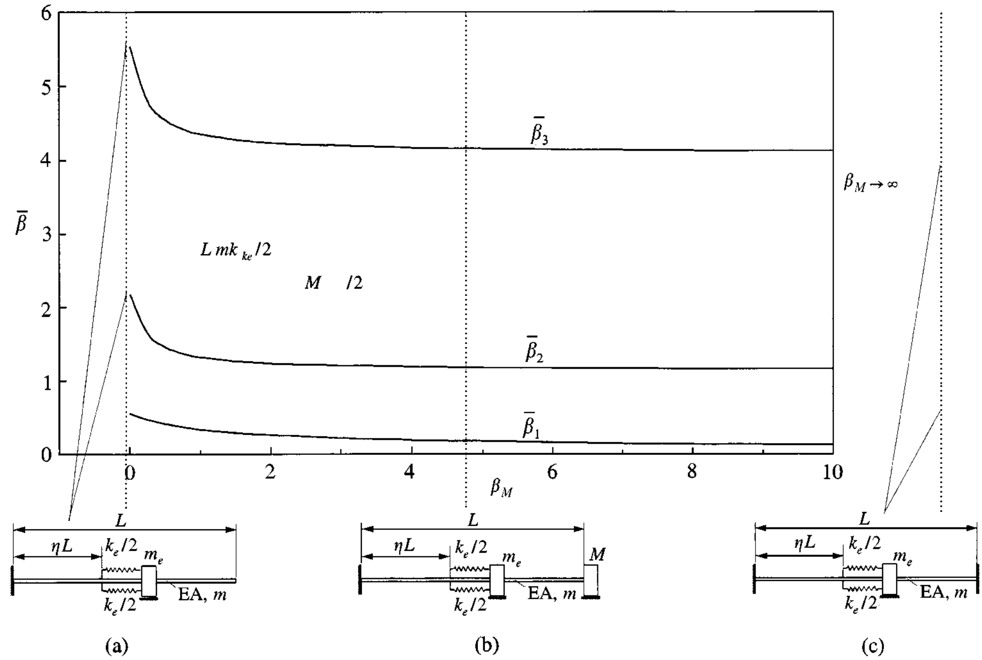

Figure 5. Plots of the first three eigenfrequency parameters of the system in Figure 1 as functions of the tip mass parameter of the system β_M; $\alpha_{me} = 5$, and $\eta = 0{\cdot}4$ are taken. (a) Fixed–free longitudinally vibrating elastic rod carrying a spring mass system attached in-span [equation (6)]. (b) Fixed–free longitudinally vibrating elastic rod with a tip mass and spring-mass attached in-span [equation (1)]. (c) Fixed–fixed longitudinally vibrating elastic rod with a tip mass restrained by springs in-span [equation (7)].

attachment point of the spring-mass system affects the frequency spectrum of the system. In Figure 4, the first three eigenfrequency parameters of the system shown in Figure 1 are shown as functions of the location of the spring-mass system, where the stiffness parameter is kept constant at $\alpha_{ke} = 5{\cdot}0$; the dimensionless mass parameter $\alpha_{me} = 5{\cdot}0$ and a tip mass exists such that $\beta_M = 0{\cdot}1$. The second and third curves originate on the vertical axis at the points $\bar{\beta} = 1{\cdot}428870$ and $4{\cdot}305801$, which correspond to the first two eigenfrequency parameters of a fixed–free longitudinally vibrating elastic rod with a tip mass M, such that $M = 0{\cdot}5$ mL, as shown on the left-hand side of Figure 4 at the bottom. As the attachment point of the spring-mass approaches the free end of the elastic rod, the fundamental frequency parameter of the combined system decreases and reaches, for $\eta = 1{\cdot}0$, the value $\bar{\beta}_1 = 0{\cdot}384027$. The second curve increases in a waving manner as the location parameter gets larger and reaches $\bar{\beta}_2 = 2{\cdot}545090$. The third curve, originating at the second eigenfrequency of a fixed-free longitudinally vibrating elastic rod with a tip mass M, such that $M = 0{\cdot}5$ mL, again increases in a waving manner as the location parameter gets larger and reaches, for $\eta = 1{\cdot}0$, the value $\bar{\beta}_2 = 5{\cdot}140315$, which is the third eigenfrequency of the elastic rod above, with $\beta_M = M/mL = 0{\cdot}1$, $\alpha_{ke} = 5{\cdot}0$, $\alpha_{me} = 5{\cdot}0$.

In Figure 5, the first three eigenfrequency parameters of the system in Figure 1 are shown as functions of the tip mass parameter β_M, where the stiffness parameter is kept constant at $\alpha_{ke} = 5{\cdot}0$; the mass parameter $\alpha_{me} = 5{\cdot}0$ and the location of the spring-mass system $\eta = 0{\cdot}4$. The three curves originate at the first three eigenfrequency parameters $\bar{\beta}_1 = 2{\cdot}243445$, $\bar{\beta}_2 = 5{\cdot}350751$ and $\bar{\beta}_3 = 7{\cdot}827556$ of the elastic rod as shown on the left-hand side of the bottom of Figure 5. The values on the first curve diminish continuously and vanish for $\beta_M \to \infty$. The second and third curves vary slowly after a sudden decrease at the beginning. As β_M goes to infinity, the second curve approaches $\bar{\beta}_1 = 0{\cdot}667448$, which represents the first eigenfrequency parameter of the fixed–fixed longitudinally vibrating elastic rod carrying a spring-mass system attached in-span, as shown on the right-hand side of Figure 5. The third curve remains practically constant over the whole range considered. Its limit value for $\beta_M \to \infty$ can be shown to be $\bar{\beta}_3 = 4{\cdot}097681$ which is the second eigenfrequency parameter of the elastic rod mentioned above.

4. CONCLUSIONS

The present study has dealt with a fixed–free, longitudinally elastic rod carrying a tip mass to which spring-mass system is attached in-span. The main concern has been the qualitative investigation of the frequency spectrum of the combined system, including systems obtained through limiting processes.

REFERENCES

1. M. GÜRGÖZE 1998 *Journal of Sound and Vibration* **216**, 295–308. On the eigenfrequencies of longitudinally vibrating rods carrying a tip mass and spring-mass in span.