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PARAMETRIC VARIABLE STRUCTURE CONTROL OF A SPINNING BEAM SYSTEM VIA AXIAL FORCE

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This paper presents a parametric variable structure control (PVSC) for the spinning simple-flexure beam via a control force applied along the axial direction to suppress the transverse vibrations. The partial differential equation (PDE) is regarded as the distributed parameter system (DPS) and is selected as the object to be controlled. The PVSC law designed by Lyapunov's direct method ensures that the system is asymptotically stable and satisfies the reaching condition simultaneously. The finite difference method is employed in the numerical simulations to show the effectiveness of the control laws.

1. INTRODUCTION

Simply supported beams are essential machine elements and are commonly used in many mechanical systems. For instance, marine propulsion, mechanical drive trains, face-milling cutters and drilling of offshore oil, etc., are examples of such systems. Large-amplitude vibrations can cause failure of machines and decrease their accuracy. Hence, how to control these factors which cause system vibrations is an important task. There exists an extensive literature on current research pertaining to the vibration and dynamic stability of the simply supported beam [1]. In these studies, the vibrations of a DPS are governed by one or more coupled partial differential equations whose coefficients or parameters are functions of spatial and time variables.

Recently, the parametric stabilization of vibrations has received a lot of attention. Franke [2, 3] investigated the control of the bilinear DPS, in which the furnace, population dynamics, and the variable-length beam were used as examples. While attention has been directed to instabilities caused by parametrical excitation, parametric stabilization of vibration is a relatively new concept [4]. Axial compression can buckle a beam but parametric control can stabilize the transverse vibration of a beam.

Variable structure control (VSC) was studied in the early 1950s for systems represented by single-input high-order differential equations. For such simple systems, both the sliding mode and reaching mode controls can be designed by trial and error [5–7]. Now it is recognized that robustness and invariance are the most

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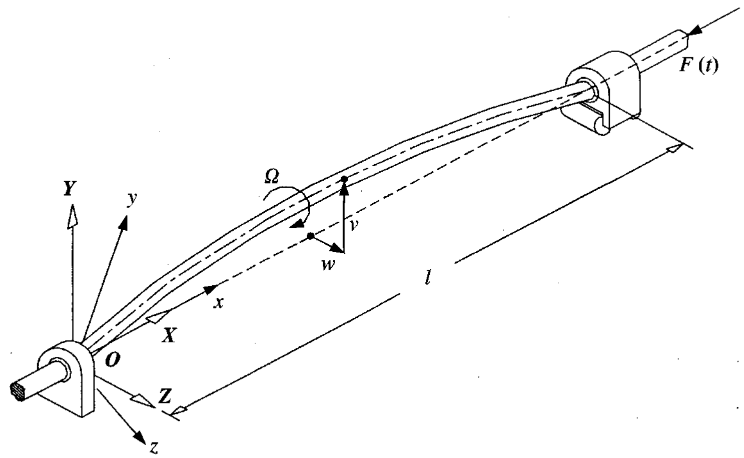

Figure 1. Schematic diagram of a distributed parameter spinning beam system.

important virtues of the VSC systems. Under certain conditions, the sliding mode of a VSC system is invariant, more than just robust, with respect to system perturbations and external disturbances. Recently, Fung and Liao [8] studied the vibration reduction of an axially moving string by use of the VSC scheme, which is based on the independent mode space control method. The sliding mode criterion problem is defined on the associated finite-dimensional approximation of the control system, which is governed by a set of ordinary differential equations (ODEs).

In the present study, vibration control of a spinning simple-flexure beam via a control force applied along its axial direction is investigated. Lyapunov's direct method is employed to develop a new PVSC algorithm, which renders the system asymptotically stable and satisfies the reaching condition simultaneously. Asymptotical stability of the control system is proved. The finite difference method is used for the numerical simulations of the closed-loop control system. Finally, some important conclusions are drawn.

2. EQUATIONS OF MOTION

The physical model shown in Figure 1 is a simply supported spinning beam controlled by an axial force $F(t)$ applied at the right-hand-side boundary. The analyzed spatial length is l, Young's modulus is Y and moment of inertia is I. The governing equations [9] of the transverse vibrations are

$$v{:}\quad \rho A(v_{tt} - 2\Omega w_t - \Omega^2 v) - F v_{xx} + YI v_{xxxx} = 0, \tag{1}$$

$$w{:}\quad \rho A(w_{tt} + 2\Omega v_t - \Omega^2 w) - F w_{xx} + YI w_{xxxx} = 0, \tag{2}$$

and the boundary conditions are

$$v{:}\quad v(0,t) = v(l,t) = 0,\ v_{xx}(0,t) = v_{xx}(l,t) = 0, \tag{3}$$

$$w{:}\quad w(0,t) = w(l,t) = 0,\ w_{xx}(0,t) = w_{xx}(l,t) = 0, \tag{4}$$

where ρ is the mass density and YIv_{xxxx} and YIw_{xxxx} are the terms due to bending rigidity of the simply supported beam. The vibrational behavior of a rotating uniform beam with a constant spinning speed was investigated by Bauer [10] for all possible combinations of the free, clamped, simply supported and guided boundaries. In this paper, the control force $F(t)$ is employed to suppress the transverse vibrations $v(x, t)$ and $w(x, t)$ of the spinning simple-flexure beam system. The control law is designed by Lyapunov's direct method and described in the following section.

3. CONTROL DESIGN BY LYAPUNOV'S METHOD

Orlov [11] applied Lyapunov's method to synthesize the controller of a DPS. Habib and Radcliffe [12] used the energy as Lyapunov's functional to design a stabilizing, axial damper control law for a pinned Euler–Bernoulli beam. In the present study, Lyapunov's method is used to design the general form of control law for the spinning simple-flexure beam system. We propose to take the total energy of the uncontrol beam system as Lyapunov's functional candidate:

$$E(t) = \frac{1}{2}\int_0^l [\rho A(v_t - \Omega w)^2 + YIv_{xx}^2]\,\mathrm{d}x + \frac{1}{2}\int_0^l [\rho A(w_t - \Omega v)^2 + YIw_{xx}^2]\,\mathrm{d}x + \frac{1}{2}J\Omega^2. \tag{5}$$

The total derivative $(\mathrm{d}/\mathrm{d}t)(\ )$ is distinct from the partial derivative $(\partial/\partial t)(\ )$. The displacements $v(x, t)$ and $w(x, t)$ are measured in the rotating co-ordinate system Oxyz. Their total derivatives with respect to time are $\mathrm{d}v/\mathrm{d}t = \partial v/\partial t - \Omega w$ and $\mathrm{d}w/\mathrm{d}t = \partial w/\partial t + \Omega v$ respectively. By virtue of equations (1) and (2), the time derivative of equation (5) in v function is

$$\dot{E}_v(t) = \int_0^l [(v_t - \Omega w)(Fv_{xx} - YIv_{xxxx})]\,\mathrm{d}x + \int_0^l [YIv_{xx}(v_l - \Omega w)_{xx}]\,\mathrm{d}x \tag{6a}$$

Integrating by parts and using the boundary conditions (3) and (4), we have

$$\int_0^l [(v_t - \Omega w)(-YIv_{xxxx})]\,\mathrm{d}x = -\int_0^l [YIv_{xx}(v_t - \Omega w)_{xx}]\,\mathrm{d}x. \tag{6b}$$

Taking time derivative of the total energy (5) in w function, we obtain $\dot{E}_w(t)$ which is similar to equation (6a). Finally, the time derivative of the Lyapunov functional is

$$\dot{E}(t) = F(t)S(t), \tag{7}$$

where $S(t) = \int_0^l [(v_l - \Omega w)v_{xx} + (w_t + \Omega v)w_{xx}]\,\mathrm{d}x$.

If we choose the control law as

$$F(t) = -P(t)\,\mathrm{sgn}(S), \tag{8}$$

where $P(t)$ is an arbitrary positive function of time, the control law (8) results in the negativeness of $\dot{E}(t)$ in equation (7).

We define the state space $\mathscr{H}$ of the system as follows:

$$\mathscr{H} := \{(v, w, v_t, w_t)^T \mid v, w \in H_0^3 \cap H_l^3, v_t, w_t \in L^2\}. \tag{9}$$

The spaces L^n and H_l^k are defined respectively as

$$L^n := \left\{ f : [0, 1] \to \mathbf{R} \,\middle|\, \int_0^l f^n \,\mathrm{d}x < \infty \right\}, \tag{10}$$

$$H_l^k := \{ f | f, f', f'', \ldots, f^{(k)} \in L^2, f(l) = 0 \}. \tag{11}$$

We set $\eta = [\dot{v}\ \dot{w}\ v_{xx}\ w_{xx}]^T$ as the generalized co-ordinate vector. It is known that the zero co-ordinate vector $\eta = 0$ corresponds to the equilibrium state of the system. From equations (4) and (5), as $\eta = 0$, we have the transverse displacements, $v(x, t) = w(x, t) = 0$ and $v_{xx}(x, t) = w_{xx}(x, t) = 0$. These conditions in conjunction with $v_t(x, t) = w_t(x, t) = 0$ at $x = 0$ and l yield $\dot{v} = v_t - \Omega w = 0$ and $\dot{w} = w_t + \Omega v = 0$. Thus, it is obvious that the equilibrium state is achieved as $\eta = 0$.

To prove the stability of the equilibrium state $\eta = 0$, we extend the proof process of Habib and Radchiffe [12], in which the beam does not spin. The auxiliary vectors, $\zeta = [\zeta_1\ \zeta_2\ \zeta_3\ \zeta_4]^T$ and $\xi = [\xi_1\ \xi_2\ \xi_3\ \xi_4]^T$, are defined so that we can introduce the metric $\sigma_1(\zeta, \xi)$ as

$$\sigma_1(\zeta, \xi) = \left\{ \frac{1}{2} \int_0^l [\rho A[(\zeta_1 - \xi_1)^2 + (\zeta_2 - \xi_2)^2] + YI[(\zeta_3 - \xi_3)^2 + (\zeta_4 - \xi_4)^2]] \,\mathrm{d}x \right\}^{1/2}. \tag{12}$$

It follows that

$$\sigma_1(\eta, 0) = \left\{ \frac{1}{2} \int_0^l [\rho A[\dot{v}^2 + \dot{w}^2] + YI[v_{xx}^2 + w_{xx}^2]] \,\mathrm{d}x \right\}^{1/2}. \tag{13}$$

Thus, $\sigma_1(\eta, 0)$ is a measure of distance between the equilibrium state, $\eta = 0$, and the deformed state, $\eta \neq 0$. Therefore, the metric $\sigma_1(\zeta, \xi)$ is positive definite and symmetric.

Comparing equations (5) and (13) yields $E(t) \geqslant \sigma_1^2(\eta, 0)$ for $\eta \neq 0$; therefore $E(t)$ is positive definite. To prove that $E(t)$ admits an infinitely small upper bound in the neighborhood of $\eta = 0$, it is required that $E(t) \leqslant \lambda \sigma_0^2(\eta, 0)$, where $\lambda \geqslant 1$ is a positive constant and σ_0 is the metric defined by

$$\sigma_0^2(\zeta, \xi) = \sigma_1^2(\zeta, \xi) + \frac{1}{2} \int_0^l F(v_x^2 + w_x^2) \,\mathrm{d}x + \frac{1}{2} J \Omega^2. \tag{14}$$

The control force (8) ensures that the spinning simple-flexure beam is asymptotically stable. However, the problem of mathematical description of the sliding mode does not arise until now. The original point $v(x, t) = w(x, t) = 0$ of the sliding mode is the control goal.

4. CONTROL DESIGN BY REACHING CONDITION

The condition under which the state will move towards and reach a sliding surface is called the reaching condition. The system trajectory under the reaching condition is called the reaching mode or reaching phase. It is noticed that the control law (8) does not satisfy the reaching condition. In order to own the property of the reaching condition, the sufficient condition $\dot{S}(t)\cdot S(t) < 0$ must be satisfied [13, 14]. Taking the time derivative of switching function $S(t)$, using equations (1) and (2) and integrating by parts, we have

$$\rho A\dot{S}(t) = F\int_0^l (v_{xx}^2 + w_{xx}^2)\,\mathrm{d}x + \int_0^l \{YI(v_{xxx}^2 + w_{xxx}^2) + \rho A[(v_t - \Omega w)_x^2 + (w_t + \Omega v)_x^2]\}\,\mathrm{d}x. \tag{15}$$

In order to satisfy the sufficient condition $\dot{S}(t)\cdot S(t) < 0$, the control law can be expressed as

$$F(t) = -\frac{r_1\int_0^l (v_{xxx}^2 + w_{xxx}^2)\,\mathrm{d}x + r_2\int_0^l [v_t - \Omega w)_x^2 + (w_l + \Omega v)_x^2]\mathrm{d}x}{\int_0^l (v_{xx}^2 + w_{xx}^2)\,\mathrm{d}x}\cdot \mathrm{sgn}(S)$$

$$r_1 > YI, r_2 > \rho A, \tag{16}$$

which is the same as equation (8) if we take

$$P(t) = -\frac{r_1\int_0^l (v_{xxx}^2 + w_{xxx}^2)\,\mathrm{d}x + r_2\int_0^l [(v_t - \Omega w)_x^2 + (w_l + \Omega v)_x^2]\mathrm{d}x}{\int_0^l (v_{xx}^2 + w_{xx}^2)\,\mathrm{d}x} > 0. \tag{17}$$

From equations (8), (16) and (17), some observations can be obtained as follows.

(i) Lyapunov's direct method in the above section is employed to find the switching function $S(t)$ only. This process is different from that of Orlov [11] by setting a switching function in the Lyapunov functional candidate.

(ii) In equation (16), we must notice the situation when denominator $\int_0^l (v_{xx}^2 + w_{xx}^2)\,\mathrm{d}x$ is close to zero, where the control force will become unbounded. In order to avoid the unboundness, the control law can be modified as

$$F(t) = -\frac{r_1\int_0^l (v_{xxx}^2 + w_{xxx}^2)\,\mathrm{d}x + r_2\int_0^l [(v_t - \Omega w)_x^2 + (w_l + \Omega v)_x^2]\mathrm{d}x}{\int_0^l (v_{xx}^2 + w_{xx}^2)\,\mathrm{d}x + \varepsilon}\cdot\frac{S}{|S| + \mu}$$

$$r_1 > YI, r_2 > \rho A, \tag{18a}$$

where ε and μ are small positive constants. The main purpose of adding an ε in the denominator of the control law is to bound the control force. The switching function sgn(S) in equation (16) replaced by $S/(|S| + \mu)$ is to smooth the discontinuous function and erase the chattering motions.

(iii) Another approximation, called the saturation controller, to obviate the unbounded control input is proposed as follows:

$$F(t) = \begin{cases} \bar{F}\,\mathrm{sgn}(F_1) & \text{if } |F| > \bar{F} \\ F & \text{if } |F| < \bar{F}, \end{cases} \tag{18b}$$

where $\bar{F}$ is an upper bounded control force. Due to the limitation of $\bar{F}$, the unbounded problem will not occur. However, the chattering phenomenon will not be improved.

(iv) When using the two controllers (18a) and (18b) are used, the parametric control system described by equations (1) and (2) becomes non-linear.

5. EXISTENCE OF SLIDING MODE

It can be shown that the switching function $S(t) = \int_0^l [(v_t - \Omega w)v_{xx} + (w_t + \Omega v)w_{xx}]\,\mathrm{d}x$ is part of the time derivative of $\int_0^l (vv_{xx} + ww_{xx})\,\mathrm{d}x$. From this viewpoint, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l (vv_{xx} + ww_{xx})\,\mathrm{d}x$$

$$= \int_0^l [(v_t - \Omega w)v_{xx} + (w_t + \Omega v)w_{xx}]\,\mathrm{d}x + \int_0^l [(v(v_t - \Omega w)_{xx} + w(w_t + \Omega v)_{xx}]\,\mathrm{d}x. \tag{19}$$

Integrating by parts for the last term in equation (19), applying the boundary conditions (3) and (4) using $v_t = w_t = 0$ at $x = 0$ and l, we obtain

$$\int_0^l [v(v_t - \Omega w)_{xx} + w(w_t + \Omega v)_{xx}]\,\mathrm{d}x$$

$$= v(v_t - \Omega w)_x|_0^l - v_x(v_t - \Omega w)|_0^l + \int_0^l (v_t - \Omega w)v_{xx}\,\mathrm{d}x$$

$$+ w(w_t + \Omega v)_x|_0^l - w_x(w_t + \Omega v)|_0^l + \int_0^l (w_t + \Omega v)w_{xx}\,\mathrm{d}x$$

$$= \int_0^l [(v_t - \Omega w)v_{xx} + (w_t + \Omega v)w_{xx}]\,\mathrm{d}x. \tag{20}$$

Substituting equation (20) into equation (19), we have

$$S = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l (vv_{xx} + ww_{xx})\,\mathrm{d}x. \tag{21a}$$

Integrating by parts with respect to equation (21a), we have

$$S = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l (v_x^2 + w_x^2)\,\mathrm{d}x. \tag{21b}$$

On the switching hypersurface $S = 0$, the value $(\mathrm{d}/\mathrm{d}t)\int_0^l (v_x^2 + w_x^2)\,\mathrm{d}x$ is zero. Then, $\int_0^l (v_x^2 + w_x^2)\,\mathrm{d}x$ is a constant. The original point $v(x, t) = w(x, t) = 0$ is a special case such that $\int_0^l (v_x^2 + w_x^2)\,\mathrm{d}x = 0$.

6. NUMERICAL RESULTS AND DISCUSSION

The numerical results are provided to verify the availability of the proposed PVSC algorithms. The system parameters are $\rho A = 0{\cdot}7$ kg/m, $YI = 38$ N m^2 and $l = 1$ m. By use of the finite difference method, the numerical solutions of these control systems are obtained directly. An explicit central finite difference scheme with a mesh of 10 elements along the beam length is chosen to approximate the transverse vibrations and the convergence criterion is chosen as $\Delta x/\Delta t^2 < 0{\cdot}5$ [15].

6.1. STATIONARY MODEL

Figures 2(a–e) show the control results of the simple-flexure beam in stationary situation $\Omega = 0$. The initial conditions of the transverse displacements are specified by $v(x, 0) = 0{\cdot}03 \sin(x\pi/l)$ and $w(x, 0) = 0$. The initial velocities are zero. The

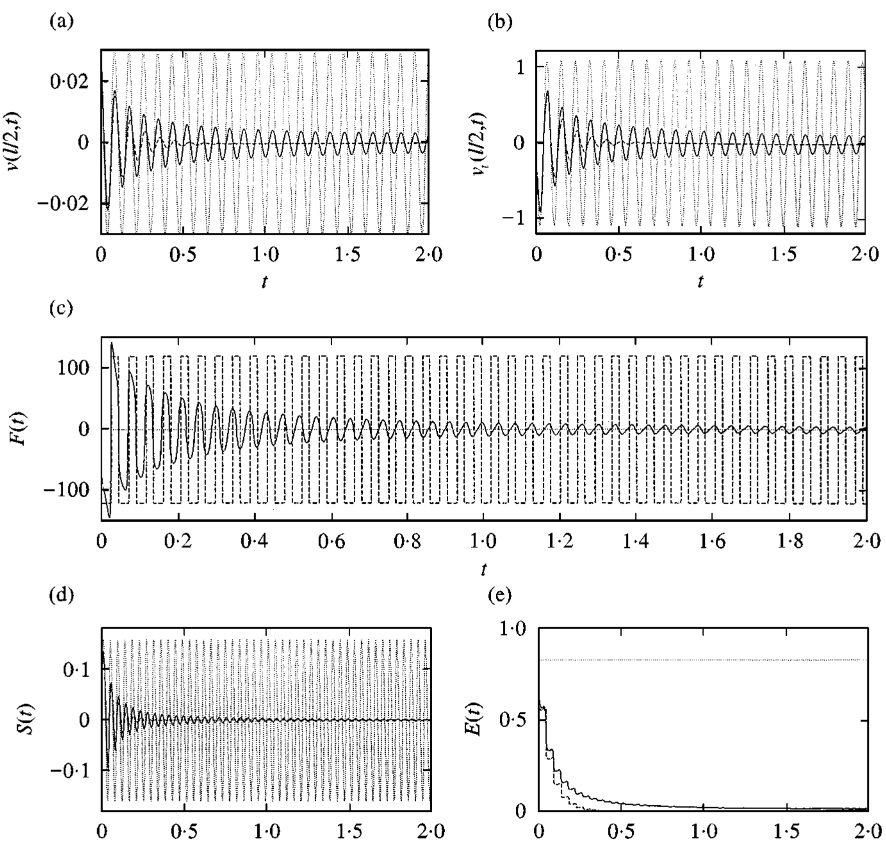

Figure 2. The transient responses of parametric variable structure control in stationary situation. (a) The transverse amplitudes $v(l/2, t)$, (b) the velocity $v_t(l/2, t)$, (c) the control input $F(t)$, (d) the time histories of switching function $S(t)$, (e) the total energy $E(t)$ (—: $\varepsilon = 0{\cdot}05$, $\mu = 0{\cdot}01$ in equation (18a), – – –: saturation controller (18b); $\cdots$: uncontrol).

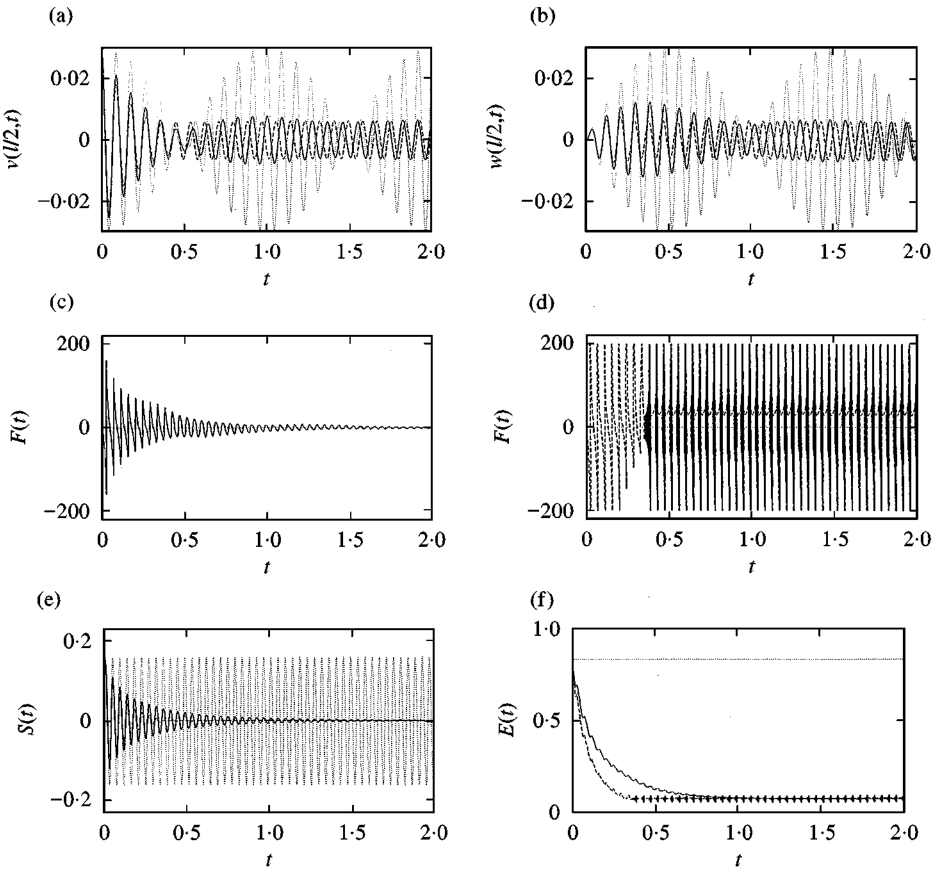

Figure 3. The transient responses of parametric variable structure control in spinning situation. (a) The transverse amplitudes $v(l/2, t)$, (b) the transverse amplitudes $w(l/2, t)$, (c) the control $F(t)$ of (18a), (d) the control $F(t)$ of (18b), (e) the time histories of switching function $S(t)$, (f) the total energy $E(t)$ (—: $\varepsilon = 0{\cdot}05$, $\mu = 0{\cdot}01$ in equation (18a); – – –: saturation controller (18b); ···: uncontrol).

control gains $r_1 = 300$ and $r_2 = 0{\cdot}8$ are used in equation (16). The solid lines are used for the bounded and smoothed control (18a) with $\varepsilon = 0{\cdot}05$ and $\mu = 0{\cdot}01$ while the dash lines are used for the saturation controller (18b). In Figures 2(a, b), the transverse amplitudes $v(l/2, t)$ and velocity $v_t(l/2, t)$ are illustrated respectively. The amplitudes are obviously suppressed via the PVSC laws in comparison with the uncontrol system (dotted lines). The time histories of the control inputs $F(t)$ are shown in Figure 2(c). It is observed that the controller (18a) is bounded and smoothed. As the value of denominator $\int_0^l v_{xx}^2\,\mathrm{d}x$ becomes smaller, the control force is determined by the small positive constant ε. For the saturation controller (18b), the value of $F(t)$ is limited to the effective range $|F(t)| \leqslant \bar{F} = 120$ N, which avoids the extremely large control input. The switching function $S(t) = \int_0^l v_t v_{xx}\,\mathrm{d}x$ and total mechanical energy via the PVSC laws are shown in Figures 2(d, e) respectively. It is seen that the system trajectories reach the switching hypersurface $S(t) = 0$ in Figure 2(d), and the total energies $E(t)$ approach zero in Figure 2(e).

6.2. SPINNING MODEL

Figures 3(a–f) show the control results of the spinning simple-flexure beam with $\Omega = 30$ rpm. The initial conditions are the same as those in the stationary model. The control gains $r_1 = 40$ and $r_2 = 0{\cdot}8$ are used. The transverse amplitudes $v(l/2, t)$ and $w(l/2, t)$ are illustrated in Figures 3(a, b) respectively. The coupling effect between the rigid-body motion and flexible vibrations can be seen clearly in the uncontrol system (dotted lines). The amplitudes of the control system decrease and reach the stable condition as the time increases. Figures 3(c, d) show the time histories of the control forces. The upper bound $\bar{F} = 200$ N is chosen (dash line) for the saturation control (18b), and the constants $\varepsilon = 0{\cdot}05$ and $\mu = 0{\cdot}01$ are specified in the controller (18a) (solid line). It is noticed that the control forces in Figure 3(c) are bounded and smoothed, while the controller in Figure 3(d) is bounded but has a chattering phenomenon. The switching functions $S(t) = \int_0^l [(v_t - \Omega w)v_{xx} + (w_t + \Omega v)w_{xx}]\,\mathrm{d}x$ and total mechanical energy $E(t)$ in equation (5) via the PVSC laws are shown in Figures 3(e, f) respectively. However, $E(t)$ has the impulse phenomenon for the saturation controller (18b).

It is well known that the higher values of the control gains r_1 and r_2 have the quicker convergence and more rapid decay. By decreasing the constants ε and μ in equation (18a) and increasing the upper bound $\bar{F}$ in equation (18b), the control inputs (18a) and (18b) will truly approximate to the controller (16), and the system will get a better performance.

7. CONCLUSIONS

In conclusion, the PVSC laws designed by Lyapunov's direct method are successfully applied in stationary and spinning flexible beams. The proposed control laws ensure that the system is asymptotically stable and satisfies the reaching condition simultaneously. Two approximate controllers are proposed to avoid the unbounded and unsmoothed control inputs. In simulations, a convenient explicit central finite difference scheme is used to solve the numerical solutions.

From the theoretical analysis and numerical results, we draw the conclusions as follows:

1. For a stationary model, the PVSC causes the total mechanical energy and the transient amplitudes to decay to zero, and the system is asymptotically stable.
2. For a spinning model, the PVSC causes the total mechanical energy and the transient amplitudes to decay to the stable state, but not to reach zero, because the coupling effect between the rigid-body motion and flexible vibration exists in the coupled governing equations.

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