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LETTERS TO THE EDITOR

CONVERGENCE OF EIGENVALUES OF A CANTILEVER BEAM WITH 8- AND 20-NODE HEXAHEDRAL ELEMENTS

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1. INTRODUCTION

The finite element discretization of a vibration problem replaces the original structure by a mass matrix, **M**, and a stiffness matrix **K**. The accuracy of the solution depends on the quality of both the stiffness and mass matrices. Clear guidelines exist as to how **K** is to be derived using an energy (or variational or virtual work) principle. Early attempts to deal with dynamics took the simplistic course of "lumping" the mass of the element at the element nodes. The lumped mass matrix is diagonal and hence does not involve mass coupling between the degrees of freedom. It was realized [1, 2] that, from Hamilton's principle, a non-diagonal or "consistent" mass matrix could be derived from the kinetic energy by using the same trial functions that were used to determine the stiffness matrix. In a variational sense therefore, the "lumped" mass approaches are "non-consistent"; they conserve mass but not necessarily momentum or kinetic energy of the consistent mass matrix.

Various lumped mass approximations corresponding to different ways of apportioning the total mass to the various nodes are possible. For many problems the lumped mass approaches leads to the same convergence rate of eigenvalues as the consistent mass matrix. However, this trend cannot be extrapolated to all problems as the lumped mass matrices do not conserve kinetic energy. The torsional eigenvalues of a beam is a typical example which is investigated in detail in this paper. In addition to the torsional eigenvalues, convergence of flexural and shear eigenvalues is also investigated.

2. ERROR ESTIMATES FROM THE RAYLEIGH QUOTIENT

The Rayleigh quotient approximation (of which the finite element method is a piecewise version) can be traced to Rayleigh's description of the fundamental frequency as a minimum of the Rayleigh quotient [3]. Considering the discretized

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eigenvalue problem, $\mathbf{K}\boldsymbol{\phi} = \lambda\mathbf{M}\boldsymbol{\phi}$, the eigenvalue may be written in the Rayleigh quotient form as

$$\lambda = (\boldsymbol{\phi}^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi})/(\boldsymbol{\phi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}) \tag{1}$$

where λ is the eigenvalue and $\boldsymbol{\phi}$ the corresponding eigenvector of the discretized system.

The Rayleigh quotient is used here to investigate the error trend of eigenvalues by independently assessing the errors in the numerator and the denominator. Rayleigh quotient, in conjunction with functional analysis concepts, has been used [4] to arrive at error estimates. Recently, a simplified approach using this concept has been used [5] effectively to understand the error trend of bar and beam elements in free vibration.

For simplicity, let us consider the finite element formulation for one-dimensional problems. Using trial functions for displacements complete to x^n, the potential energies (where, for simplicity, the strains are assumed to be first derivatives of the displacements) after discretization are accurate to $O(h^{2n})$. Stress correspondence paradigm [6, 7] provides an effective explanation of this fact. The kinetic energies are, however, accurate to $O(h^{2(n+1)})$ if a consistent mass approach is strictly observed in formulating the mass matrix. From this, we may argue that if λ_e, $\mathbf{K}_e$, and $\mathbf{M}_e$ are the quantities corresponding to the unknown exact analytical solution to the problem, then the finite element approximation of eigenvalue, λ_f, is given [4] by

$$\lambda_f = [\boldsymbol{\phi}_e^{\mathrm{T}}\mathbf{K}_e\boldsymbol{\phi}_e + O(h^{2n})]/[\boldsymbol{\phi}_e^{\mathrm{T}}\mathbf{M}_e\boldsymbol{\phi}_e + O(h^{2(n+1)})] \tag{2}$$

$$= \lambda_e + O(h^{2n}) - O(h^{2(n+1)}). \tag{3}$$

For this case, the dominating error is due to the numerator of equation (2), or, in other words, the error in the formulation of the stiffness matrix. This explains the well-known upper-bound nature of the eigenvalues with consistent mass matrices. In any non-consistent or lumped mass formulation, the order of accuracy associated with the kinetic energy would be poorer than $O(h^{2(n+1)})$. However, if by a judicious choice of lumping scheme it can be maintained at $O(h^{2n})$, which is the accuracy of the potential energy term in the numerator of the Rayleigh quotient, then it will be possible that the rate of convergence of the eigenvalue is no worse than that obtained with a consistent mass formulation. This is the idea behind the success of many lumped mass matrices. On the other hand, if the order of accuracy associated with kinetic energy is poorer than $O(h^{2n})$, then the accuracy of prediction of eigenvalues would be governed by the accuracy of kinetic energy terms, or, equivalently, the mass matrix.

3. ERROR ESTIMATE FOR TORSIONAL EIGENVALUES

Considering the free torsional vibration of a uniform rod, the eigenvalue may be written in Rayleigh's quotient form as

$$\lambda_{tor} = (\boldsymbol{\phi}_{tor}^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi}_{tor})/(\boldsymbol{\phi}_{tor}^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_{tor}), \tag{4}$$

where ϕ_{tor} is the modeshape vector corresponding to the torsional mode. In general, the accuracy of the strain energy (or, equivalently, the numerator of Rayleigh quotient) depends on the mesh refinement along both the longitudinal and transverse directions of the rod. The higher the number of elements along the longitudinal axis, the better the representation of variation of twist along the length of the rod. Similarly, the higher the number of elements across the cross-section, the better the representation of rotation and warping of sections. In the absence of warping, however, the cross-sections simply rotate as a rigid body about the longitudinal axis, and hence the exact displacement field across the cross-section can be captured even if the displacement model (element shape functions) is complete only upto linear terms.

In this paper, we limit ourselves to the study of the effect of mesh refinement across the cross-section. Hence, we assume that the mesh density along the length is sufficiently high so that it has no effect on our study. Under such conditions, the element with a linear displacement field has no difficulty in representing the cross-sectional rotation, and hence involves no error in the potential energy and hence in the numerator of Rayleigh quotient. The denominator also will not involve any error if the kinetic energy due to the rigid body rotation can be represented accurately by the element. The consistent mass matrix conserves the kinetic energy inherently. Hence, the use of an element with at least a linear displacement model together with the consistent mass matrix should yield exact torsional eigenvalues. The lumped mass matrix does not conserve kinetic energy and the denominator of Rayleigh quotient involves an error. Hence, the lumped mass matrix cannot yield exact eigenvalues. The example problem considered in a later section verifies these predictions.

We now develop an error model to predict the convergence (with respect to the mesh refinement across the cross-section) of torsional eigenvalues for the case of lumped mass matrix. Consider a typical finite element mesh of $n \times n$ elements of a square cross-section of side a as in Figure 1. Considering a lumped mass matrix where the total mass is simply apportioned amongst the nodes, the polar mass moment of inertia of the lumped masses of a typical element (i, j) about its own centroidal axis parallel to the longitudinal axis of the beam is given by

$$I^{ij} = (\text{No. of nodes}) \times (\text{mass associated with each node}) \times (\text{distance of each node from the centroidal axis})^2 \tag{5}$$

$$= \rho l a^4/(2n^4), \tag{6}$$

where l is the length of the element along the longitudinal axis. The polar mass moment of inertia of the element about the centroidal longitudinal axis of the cross-section is given by

$$I_{zz}^{ij} = I^{ij} + (\text{mass of the element}) \times (r^{ij})^2 \tag{7}$$

$$= I^{ij} + (\rho l a^2/n^2) \times (r^{ij})^2, \tag{8}$$

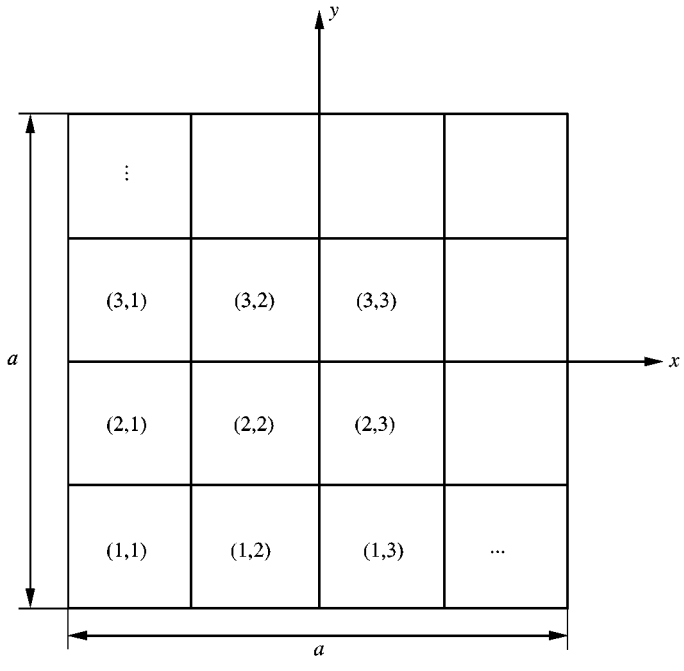

Figure 1. Finite element mesh across the cross-section.

where r^{ij} is the distance of the centroid of the element (i, j) from the centroidal longitudinal axis of the cross-section given by

$$(r^{ij})^2 = [a(2i - 1 - n)/2n]^2 + [a(2j - 1 - n)/2n]^2. \tag{9}$$

The sum of polar mass moment of inertia of all the elements in the cross-section is given by

$$I_{zz} = \sum_i \sum_j I_{zz}^{ij}. \tag{10}$$

After substitution for I_{zz}^{ij} from equation (8), and considerable algebraic simplification, the expression for I_{zz} is obtained as

$$I_{zz} = \rho l a^4 (1 + 2/n^2)/6. \tag{11}$$

This is the polar mass moment of inertia based on lumped mass approximation. The exact polar mass moment of inertia of a square cross-section rod, I_e, is given by

$$I_e = \rho l a^4/6. \tag{12}$$

Therefore, the error in lumped mass eigenvalues is given by

$$(\lambda_{tor}^l/\lambda_{tor}^e - 1) = (I_e/I_{zz} - 1) = -2/(2 + n^2), \tag{13}$$

where λ^l_{tor} and λ^e_{tor} are the lumped mass approximation and the exact eigenvalues respectively. Equation (13) may be written in terms of the size of the element, h, as

$$(\lambda^l_{tor}/\lambda^e_{tor} - 1) = -2/[2 + (a/h)^2]. \tag{14}$$

It may be recalled that this error estimate is based on the assumptions that the mesh refinement along the length is sufficiently high, and there is no warping of cross-sections.

4. NUMERICAL EXAMPLE

This section deals with a numerical example to verify our prediction of error model. A cantilever beam of 10 units length and 1×1 cross-section is modelled with an 8-node hexahedral element. All the nodes are so constrained that no axial motion and hence no warping is possible. All the nodes of the fixed end are restricted in all the three degrees of freedom. The material properties, Young's modulus, density and the Poisson ratio, are taken as $2{\cdot}1 \times 10^{11}$, 7860 and 0·3 respectively. The first two torsional eigenvalues computed using the lumped and consistent mass matrices are listed in Table 1 for various mesh refinements. The lumping scheme used for the present investigation is that proposed by Hinton *et al.* [8]. For a square element as used in our investigation, this scheme simply

TABLE 1

Eigenvalues for the first two torsional modes without warping effects — 8-node hexahedral element

	Mesh across cross-section	Eigenvalues with consistent mass matrix for 20 and 40 number of elements along length			Eigenvalues with lumped mass matrix for 20 and 40 number of elements along length		
		20	40	% diff.	20	40	% diff.
First mode	1×1	253679·9	253584·9	−0·037	84472·4	84505·2	0·039
	2×2	253679·9	253584·9	−0·037	168948·5	169010·5	0·037
	3×3	253679·9	253584·9	−0·037	207342·5	207422·6	0·039
	4×4	253679·9	253584·9	−0·037	225263·4	225346·9	0·037
	5×5	253679·9	253584·9	−0·037	234645·5	234736·8	0·039
	6×6	253679·9	253584·9	−0·037	240082·0	240174·4	0·038
	Theory	253549·6					
Second mode	1×1	2292565·4	2284581·0	−0·349	757175·5	7598021·1	0·346
	2×2	2292565·4	2284581·0	−0·349	1514282·4	1519544·7	0·346
	3×3	2292565·4	2284581·0	−0·349	1858313·9	1864829·2	0·349
	4×4	2292565·4	2284581·0	−0·349	2018898·6	2026047·1	0·353
	5×5	2292565·4	2284581·0	−0·349	2103143·9	2110439·8	0·346
	6×6	2292565·4	2284581·0	−0·349	2151899·1	2159279·0	0·342
	Theory	2281922·7					

apportions the total mass equally amongst the nodes. The first observation from Table 1 is that mesh refinement along the length from 20 to 40 elements results in marginal improvement in eigenvalues, typically 0·037% for the first eigenvalue, 0·349% for the second, indicating sufficient convergence has already been reached with respect to mesh refinement along the length of the beam. Thus, we can now use the finite element results to verify our error prediction given by equation (14).

Next, the predicted error in eigenvalues is calculated using equation (14), and the actual error observed in finite element is calculated using the eigenvalues listed in Table 1 (corresponding to 40 elements along the length). The results are summarized in Table 2. Good agreement is seen between the predicted and observed errors thus validating the error model developed.

Interestingly, we observe from equation (13) that for reducing the error to about 1%, a 14×14 mesh is required for the case of lumped mass matrix whereas a 1×1 mesh produces near exact eigenvalues in the case of consistent mass matrix. As pointed out in section 3, this difference is due to the inability of the lumped mass matrix to conserve the polar mass moment of inertia of the cross-section. This clearly demonstrates the need for conserving the polar mass moment of inertia of cross-sections in order to have faster convergence.

Next, the error in the lumped and consistent mass eigenvalues is investigated in the presence of warping of cross-sections. The eigenvalues are re-computed without restraining the cross-sectional planes to move in a plane. The results are summarized in Table 3. The first observation from Table 3 is that for the meshes $40 \times 1 \times 1$ as well as $40 \times 2 \times 2$, the eigenvalues are no different from those listed in Table 1. This only suggests that 1×1 and 2×2 meshes across the cross-section are not able to pick the warping deformations. For computing the error in eigenvalues, the theoretical values of eigenvalues considering the warping effects are not available. Hence, alternatively, the average of the consistent and lumped mass

TABLE 2

Validation of error model for lumped mass eigenvalues of torsional mode—8-node hexahedral element

Mesh	Error in eigenvalues		
	Predicted (r.h.s of equation (14))	Observed (finite element results)	
		First eigenvalue	Second eigenvalue
1×1	−0·6667	−0·6667	−0·6670
2×2	−0·3333	−0·3334	−0·3341
3×3	−0·1818	−0·1819	−0·1828
4×4	−0·1111	−0·1112	−0·1121
5×5	−0·0741	−0·0742	−0·0752
6×6	−0·0526	−0·0527	−0·0538

*Calculated as $(\lambda^l_{tor}/\lambda^e_{tor} - 1)$ using the lumped mass eigenvalues (λ^l_{tor}) and the theoretical value (λ^e_{tor}) listed in Table 1.

eigenvalues corresponding to the finest mesh, $40 \times 15 \times 15$, is taken here as the reference for computing the error. The errors (excluding the case of $40 \times 1 \times 1$ and $40 \times 2 \times 2$ mesh) are plotted in Figure 2 against the square of the element size, h^2. For both consistent and lumped mass matrices, the plot is seen to be essentially straight lines, which suggests that the errors in both cases follow a similar trend, i.e., $O(h^2)$ although they are of opposite signs.

TABLE 3

Eigenvalues for the first two torsional modes considering warping effects—8-node hexahedral element

	First eigenvalue		Second eigenvalue	
Mesh	Consistent mass matrix	Lumped mass matrix	Consistent mass matrix	Lumped mass matrix
$40 \times 1 \times 1$	**253584·9**	**84505·2**	**2284581·0**	**759802·1**
$40 \times 2 \times 2$	**253584·9**	**169010·5**	**2284581·0**	**1519544·7**
$40 \times 3 \times 3$	233696·8	191151·9	2105878·3	1718356·4
$40 \times 4 \times 4$	225806·5	200656·6	2034641·9	1803900·7
$40 \times 5 \times 5$	221935·8	205441·3	1999661·0	1846854·2
$40 \times 6 \times 6$	219768·5	208144·3	1980161·7	1871355·9
$40 \times 10 \times 10$	216522·8	212223·2	1950740·9	1907977·1
$40 \times 15 \times 15$	215477·4	213521·9	1941274·7	1919624·6

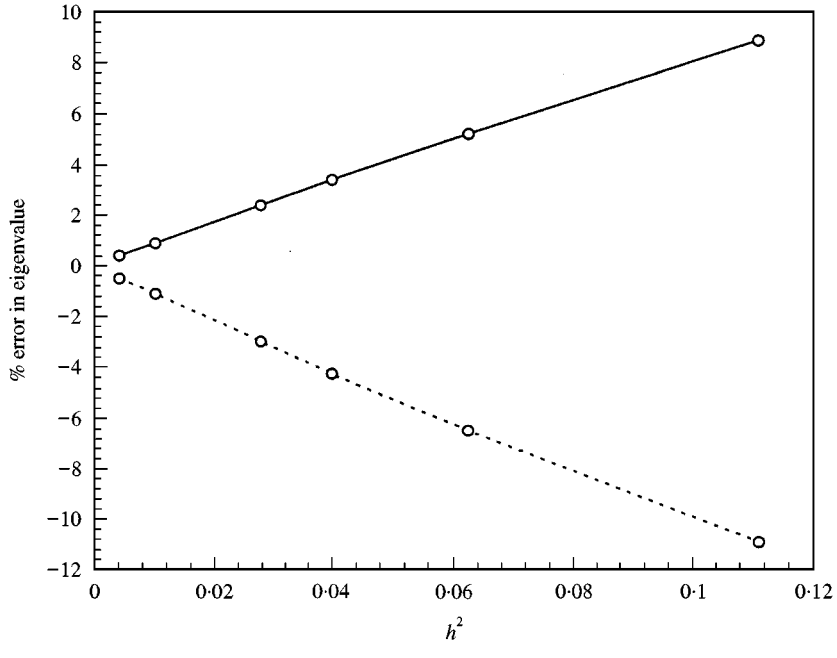

Figure 2. Convergence of eigenvalue with square of element size—8-node hexahedral element. (———) % error in first eigenvalue with consistent mass matrix; (– – – –) % error in first eigenvalue with lumped mass matrix.

The reason for the difference in performance of the lumped mass matrix in warping and non-warping situations is not straighforward and hence needs some explanation. For the simpler case of non-warping situation, the lumped mass matrix shows a poorer performance than the consistent mass matrix, whereas for the more complex case of warping situation, its performance (convergence rate) is as good as the consistent mass matrix. This appears rather paradoxical. Based on the error estimation approach discussed in section 3, a qualitative explanation to the apparent paradox is straightforward: in a non-warping situation, an element having linear displacement terms such as the 8-node hexahedral element does not lead to any error in the numerator of Rayleigh quotient. However, the case with the denominator is different; the consistent mass matrix does not lead to any significant error in the denominator whereas the lumped mass matrix does. The error caused by the lumped mass matrix is due to the inability of the lumped mass matrix in representing the cross-sectional mass moment of inertia accurately, and this leads to its poor performance. In a warping situation, however, in addition to the error in the denominator, the numerator also has error due to incomplete representation of warping deformations. If the error in the numerator dominates, then the error in eigenvalues is entirely governed by the error in the numerator. Under such conditions, the type of mass matrix used, consistent or lumped, does not influence the convergence rate of eigenvalues. In other words, the eigenvalue convergence rate becomes "insensitive" to the type of mass matrix used. This appears to be the case in the present example as both the consistent and lumped mass matrices lead to the same order of error, $O(h^2)$ (Figure 2) under the warping situation.

4.1. PERFORMANCE OF 20-NODE HEXAHEDRAL ELEMENT UNDER TORSIONAL VIBRATION

The results presented so far are for an 8-node hexahedral element. Following a similar approach, the performance of a 20-node hexahedral element has been studied. The eigenvalues have been computed for the same cantilever beam as in the case of the 8-node hexahedral element. Tables 4 and 5 list the eigenvalues for this element for non-warping and warping cases of torsional vibration respectively. A scrutiny of Tables 4 and 5 suggests the following conclusions:

(a) For non-warping case (Table 4), the lumped mass eigenvalues involve significant errors which can be traced back to the inability of lumped mass matrix to conserve the cross-sectional polar mass moment of inertia. The consistent mass matrix, however, conserves this inertia and hence even one element across the cross-section is able to yield theoretically correct (upto three decimal places) eigenvalues.

(b) A comparison of Tables 4 and 5 for $40 \times 1 \times 1$ mesh shows that the eigenvalues with and without warping effects are the same, which would only suggest that the warping deformations are not sensed by one 20-node hexahedral element across the cross-section.

TABLE 4

Eigenvalues for the first two torsional modes without warping effects—20-node hexahedral element

Mesh	First eigenvalue		Second eigenvalue	
	Consistent mass matrix	Lumped mass matrix	Consistent mass matrix	Lumped mass matrix
$40 \times 1 \times 1$	253549·6	112305·4	2281922·7	1008501·0
$40 \times 2 \times 2$	253549·6	192847·8	2281922·7	1730402·6
$40 \times 3 \times 3$	253549·6	222433·4	2281922·7	1998772·5
$40 \times 4 \times 4$	253549·6	235047·4	2281922·7	2113179·0
$40 \times 5 \times 5$	253549·6	241389·1	2281922·7	2170742·9
$40 \times 6 \times 6$	253549·6	244977·1	2281922·7	2203264·3
Theory	253549·6		2281922·7	

TABLE 5

Eigenvalues for the first two torsional modes considering warping effects—20-node hexahedral element

Mesh	First eigenvalue		Second eigenvalue	
	Consistent mass matrix	Lumped mass matrix	Consistent mass matrix	Lumped mass matrix
$40 \times 1 \times 1$	**253549·6**	**112305·4**	**2281922·7**	**1008501·0**
$40 \times 2 \times 2$	217993·0	165812·4	1962342·0	1488414·9
$40 \times 3 \times 3$	215471·6	189026·1	1939349·2	1698482·2
$40 \times 4 \times 4$	214952·7	199268·7	1934452·3	1791432·7
$40 \times 5 \times 5$	214783·8	204474·2	1932879·6	1838666·1
$40 \times 6 \times 6$	214708·1	207445·5	1932355·5	1865687·3

(c) For other meshes of the warping case (Table 5), the errors for consistent and lumped mass eigenvalues are plotted in Figures 3 and 4 against h^2 and h^4 respectively. It is seen from these figures that the error in consistent mass eigenvalues is proportional to h^4 whereas that in lumped mass eigenvalue is roughly proportional to h^2.

In the case of consistent mass eigenvalues, the denominator of Rayleigh quotient involves little error because the polar mass moment of inertia is conserved by a consistent mass matrix. Hence, the numerator is the main source of error which arises due to incomplete representation of warping deformations by the shape functions. Since the 20-node hexahedral element is a quadratic element, the error in strain values is of the order of h^2, and hence the error in the strain energy and hence the numerator of Rayleigh quotient is proportional to h^4.

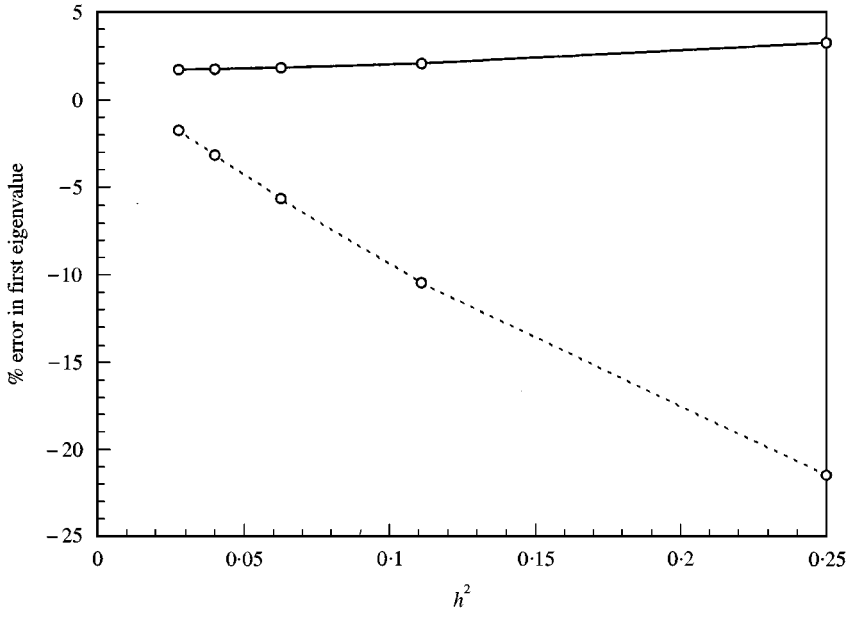

Figure 3. Convergence of eigenvalue with square of element size—20-node hexahedral element. (———) % error in first eigenvalue with consistent mass matrix; (– – – –) % error in first eigenvalue with lumped mass matrix.

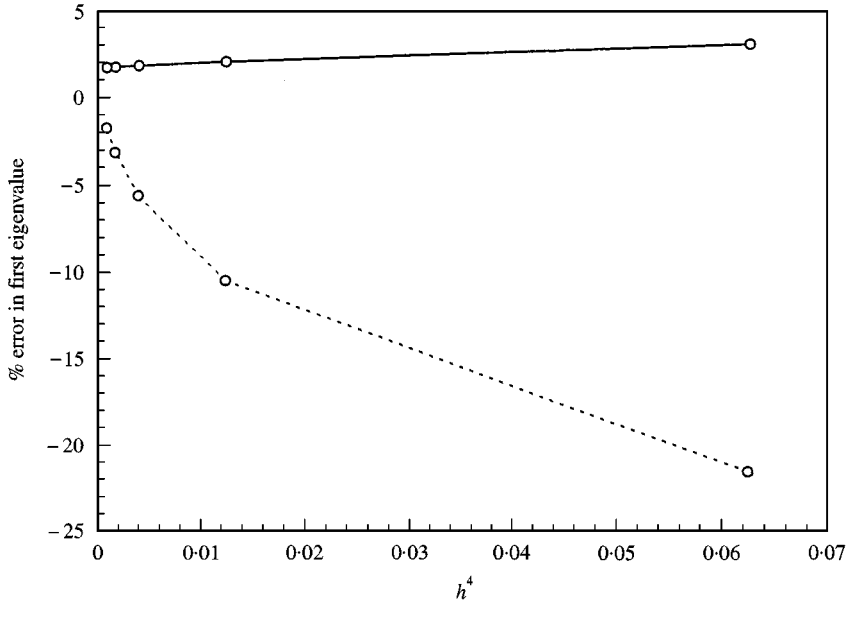

Figure 4. Convergence of eigenvalue with fourth power of element size—20-node hexahedral element. (———) % error in first eigenvalue with consistent mass matrix; (– – – –) % error in first eigenvalue with lumped mass matrix.

Similarly in the case of lumped mass eigenvalues, the error in the numerator is proportional to h^4. However, the error in the denominator due to non-conservation of polar mass moment of inertia over-rides this error and hence the resulting error in eigenvalue is predominantly proportional to h^2.

4.1.1. *Derivation of error estimate*

For the 20-node element, the expression for the error estimate of lumped mass eigenvalues can be derived following a procedure similar to that of the 8-node element. Consider a cube modelled by one 20-node element as shown in Figure 5. The lumped masses at the corner and middle nodes are $M/40$ and $M/15$, respectively, where M is the mass of the element. We make the assumption that, for

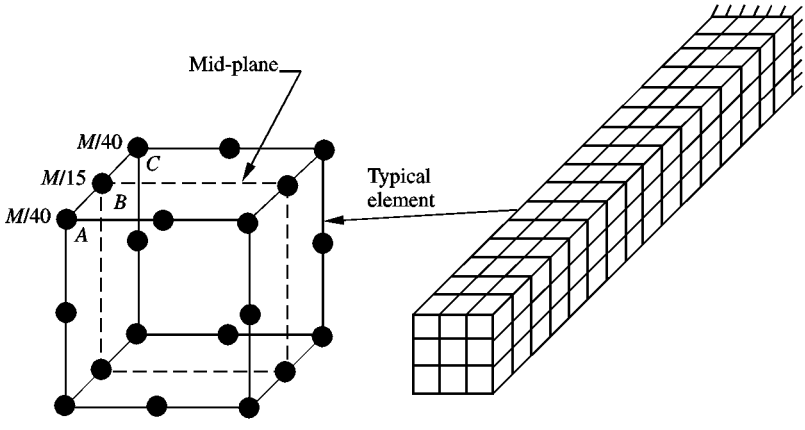

Figure 5. Mass lumping of 20-node hexahedral element. For a sufficiently refined mesh along the length of the cantilever beam, the rotational motion of nodes, e.g. those marked A, B and C in (a), about the axis of the beam are nearly equal, and hence the lumped masses of these nodes can be combined and re-lumped at the mid-plane of the element as shown in (b).

a sufficiently dense mesh along the length of the beam, all the lumped masses of an element along an element edge parallel to the longitudinal axis of the beam undergo the same angle of rotation about the longitudinal axis (Figure 5). For the lumped masses of the 20-node element, the expression for I^{ij} (equation (6)) takes the form

$$I^{ij} = 11\,\rho l a^4/(30n^4). \tag{15}$$

Using this expression in the rest of the derivation of section 3, the expression for the error estimate can be shown to be

$$(\lambda^l_{tor}/\lambda^e_{tor} - 1) = -\,6/[6 + 5(a/h)^2]. \tag{16}$$

Table 6 shows the errors predicted by equation (16) and the errors observed in finite element results. We observe that the agreement between the predicted and observed errors is quite good although it is not so close as in the case of the 8-node element, the reason for which is not obvious.

4.2. PERFORMANCE OF CONSISTENT AND LUMPED MASS MATRICES UNDER BENDING VIBRATIONS

The analysis presented so far is for torsional vibrations. A similar approach is followed for studying the convergence of eigenvalues under bending vibrations. Following equation (4), we may write

$$\lambda_{ben} = (\boldsymbol{\phi}^{\mathrm{T}}_{ben}\mathbf{K}\boldsymbol{\phi}_{ben})/(\boldsymbol{\phi}^{\mathrm{T}}_{ben}\mathbf{M}\boldsymbol{\phi}_{ben}). \tag{17}$$

As in the case of torsional vibrations, we limit ourselves to the study of the effect of mesh refinement across the cross-section. Hence, we assume that the mesh density

TABLE 6

Validation of error model for lumped mass eigenvalues of torsional mode—20-node hexahedral element

Mesh	Error in eigenvalues		
	Predicted (r.h.s of equation (16))	*Observed (finite element results)	
		First eigenvalue	Second eigenvalue
1×1	−0·5454	−0·5571	−0·5580
2×2	−0·2308	−0·2394	−0·2417
3×3	−0·1176	−0·1227	−0·1241
4×4	−0·0698	−0·0730	−0·0739
5×5	−0·0458	−0·0480	−0·0487
6×6	−0·0323	−0·0338	−0·0345

*Calculated as $(\lambda^l_{tor}/\lambda^e_{tor} - 1)$ using the lumped mass eigenvalues (λ^l_{tor}) and the theoretical value (λ^e_{tor}) listed in Table 4.

along the length is sufficiently high. Further, we assume, for simplicity, that the shear deformation is negligible. Under such conditions, the sections undergo simple rotations. As a result, an element with linear displacement field, such as an 8-node hexahedral element, has no difficulty in representing the displacement across the cross-section, and hence involves no error in the strain energy and therefore in the numerator of Rayleigh quotient. Hence, the error in eigenvalue is entirely dependent on the error in the denominator, i.e., the mass matrix. For lower modes, the rotation of cross-section is negligible and hence the kinetic energy is mainly due to transverse motion of the beam. The translational kinetic energy due to transverse motion can be accurately captured if the mass matrix employed conserves at least the mass of the element. Both lumped mass matrix and consistent mass matrix conserve the mass of the element, and hence involve no error in representing the translational kinetic energy or in other words the denominator of Rayleigh quotient. Thus, for lower modes, both the mass matrices are expected to yield comparable estimates of eigenvalues. This is reflected in the results shown in Table 7; the difference in consistent and lumped mass eigenvalues is only 0·8% for the first mode. We observe a difference of 5 and 9·7% between consistent and lumped mass eigenvalues for second and third modes respectively. However, for higher modes, the rotation of cross-section is not negligible, and hence rotational kinetic energy of the element needs to be represented accurately by the mass matrix. The consistent mass matrix conserves the rotational kinetic energy as it conserves the mass moment of inertia of the element. Hence, even one element across the cross-section is able to yield accurate eigenvalues. It can be seen from Table 7 that the consistent mass eigenvalues do not vary appreciably with mesh refinement. However, the lumped mass matrix does not conserve the mass matrix of inertia or the rotational kinetic energy. Hence, one element across the cross-section is not able to produce accurate eigenvalues (Table 7). As the mesh is refined, Table 7 shows that the difference between consistent and lumped mass eigenvalues decreases fast.

Developing an error model for bending vibration similar to that of torsional vibration would be difficult as the kinetic energy and hence the denominator of

TABLE 7

Eigenvalues for the first three bending modes—8-node hexahedral element

	First eigenvalue			Second eigenvalue			Third eigenvalue		
Mesh	Consis. mass matrix	Lumped mass matrix	% diff.	Consis. mass matrix	Lumped mass matrix	% diff.	Consis. mass matrix	Lumped mass matrix	% diff.
$40 \times 1 \times 1$	2734·1	2712·5	0·8	99487·2	94559·6	5·0	700751·1	632742·7	9·7
$40 \times 3 \times 3$	2744·0	2740·7	0·1	99392·1	98538·2	0·9	696130·2	683397·7	1·8
$40 \times 6 \times 6$	2744·0	2742·0	0·1	99055·6	98601·3	0·5	691002·2	683917·2	1·0
Theory (Timoshenko)	2719·0			100066·7			677520·2		

TABLE 8

Eigenvalues for the first three simple shear modes—8-node hexahedral element

Mesh	First eigenvalue		Second eigenvalue	
	Consistent mass matrix	Lumped mass matrix	Consistent mass matrix	Lumped mass matrix
$40 \times 1 \times 1$	253584·9	253515·3	228458·1	2279265·8
$40 \times 3 \times 3$	253584·9	253515·3	228458·1	2279265·8
$40 \times 6 \times 6$	253584·9	253515·3	228458·1	2279265·8
Theory	253584·9		2251922·7	

Rayleigh quotient involves both translational and rotational motion of lumped masses, the relative proportion of which is difficult to assess.

4.3. PERFORMANCE OF CONSISTENT AND LUMPED MASS MATRICES UNDER SIMPLE SHEAR VIBRATIONS

Under simple shear vibrations, the dynamic equilibrium is maintained between strain energy due to shear deformation and kinetic energy due to transverse motion without any rotation of cross-section. This is simulated in finite element analysis by restraining the axial motion of all nodes of the cantilever. The strain energy due to shear deformation is represented exactly (by 8- or 20-node hexahedral element), and hence there is no error in the numerator of Rayleigh quotient. The kinetic energy due to transverse motion is also represented exactly by lumped as well as consistent mass matrices. Hence, under simple shear vibrations, both lumped and consistent mass matrices would be able to yield accurate eigenvalues even with one element across the cross-section. This is confirmed by the numerical results listed in Table 8.

5. CONCLUDING REMARKS

The convergence of eigenvalues of a cantilever beam modelled with 8- and 20-node hexahedral element with respect to mesh refinement across the cross-section has been investigated. The following are some of the important conclusions:

1. When warping effects are suppressed, the consistent mass matrix leads to near exact torsional eigenvalues even for one element across the cross-section whereas the lumped mass matrix leads to significant errors. The reason for this has been observed to be the inability of the lumped mass matrix in representing exactly the polar mass moment of inertia of the cross-section of the beam.
2. For the 8-node hexahedron element, the expression for the relative error in lumped mass torsional eigenvalues has been derived as $-2/[2 + n^2]$ for an

$n \times n$ mesh; typically, for obtaining the eigenvalues within 1% error with lumped mass matrix, a 15×15 mesh is required. For the 20-node hexahedron element, the expression for error has been derived as $-6/[6 + 5n^2]$. The error predicted by this expression tallies closely with the error observed in actual finite element calculation for the cantilever problem considered.

3. When warping deformations are present, both consistent and lumped mass matrices lead to the same order of error in eigenvalues, $O(h^2)$.
4. For lower modes of bending vibrations, both consistent and lumped mass matrices lead to comparable eigenvalues. For higher modes, however, the consistent mass matrix yields more accurate eigenvalues as the mass moment of inertia of the elements, and hence the rotational kinetic energy of the elements, is represented accurately.
5. In the case of simple shear vibrations, there is no rotation of cross-sections of the beam and hence the consistent and lumped mass matrices lead to nearly exact eigenvalues.

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