Journal of Sound and Vibration (1999) **227**(3), 682–684
Article No. jsvi.1999.2373, available online at http://www.idealibrary.com on IDEAL®

FUNDAMENTAL FREQUENCY OF A WAVY NON-HOMOGENEOUS CIRCULAR MEMBRANE

C. Y. Wang

Department of Mathematics, Michigan State University, East Lansing, MI 48824, U.S.A.

(*Received* 13 *May* 1999)

The study of the vibration of membranes is important in the design of drums, speakers and receivers. The vibration of homogeneous membranes have been discussed by Rayleigh [1] and Kuttler and Sigillito [2]. Composite membranes composed of joining many strips of different homogeneous pieces were studied by Vodicka [3], Kato [4], Bhadra [5], Kalotas and Lee [6]. In these cases, the governing equation was solved for each piece, then matched at the interfaces. The continuously non-homogeneous rectangular membrane has been considered by Masad [7], Laura *et al.* [8] and Wang [9]. The last source also reported an exact solution of a continuously non-homogeneous annular membrane.

The present note studies the fundamental frequencies of a continuously non-homogeneous circular membrane. The density (or thickness) is assumed to be a sinusoidal function of radius. This class includes important wavy, ribbed membranes and also convex or concave lens-like membranes.

The equation of motion for a non-homogeneous membrane is

$$\nabla^2 w + k^2 \rho(\mathbf{x}) w = 0, \tag{1}$$

where all lengths are normalized by the dimensional L, w is the displacement, $\rho(\mathbf{x})$ is a density function with a mean of unity, and k is the constant normalized frequency

$$k = (\text{frequency})\, L \sqrt{(\text{mean density})/(\text{tension per length})}. \tag{2}$$

The boundary condition is that $w = 0$ on the perimeter of the domain σ. The density function satisfies

$$\frac{1}{\sigma} \iint \rho(\mathbf{x})\, \mathrm{d}\sigma = 1. \tag{3}$$

Note that the total mass is fixed for all density functions. When the membrane is homogeneous $\rho(\mathbf{x}) = 1$.

If $\rho(\mathbf{x})$ is a function of radius only, equation (1) in polar co-ordinates (r, θ) becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) w + k^2 \rho(r) w = 0. \tag{4}$$

0022-460X/99/430682 + 03 $30.00/0

The separable solution is

$$w = \cos(n\theta)\varphi_n(r), \tag{5}$$

where $\varphi_n(r)$ satisfies

$$\varphi_n''(r) + \frac{\varphi_n'}{r} + \left[k^2\rho(r) - \frac{n^2}{r^2}\right]\varphi_n = 0. \tag{6}$$

If the domain is a circle, the boundary conditions are

$$\varphi_n(0) \text{ bounded}, \quad \varphi_n(1) = 0. \tag{7}$$

In general, the eigenvalue k for each mode can be obtained by numerical integration of equations (6, 7).

Consider the case of a circular membrane with a wavy density (or thickness) variation given by

$$\rho(r) = 1 + e[\cos(\alpha r) + c] > 0, \tag{8}$$

where e, α, c are constants. The constant mass condition (3) gives

$$c = \frac{2}{\alpha}\left[\frac{1-\cos\alpha}{\alpha} - \sin\alpha\right]. \tag{9}$$

For $\alpha \leqslant \pi$ the density would be montonically decreasing from the center if $e > 0$, and increasing if $e < 0$. For large α the density concentrations are periodic concentric rings about the membrane center.

We also know the fundamental mode is axisymmetric, i.e., $n = 0$. Equation (6) gives

$$\varphi_0'' + \frac{\varphi_0'}{r} + k^2[1 + e\langle\cos(\alpha r) + c\rangle]\varphi_0 = 0. \tag{10}$$

Since for linear vibrations the amplitude is arbitrary we can set

$$\varphi_0(0) = 1. \tag{11}$$

Due to symmetry,

$$\varphi_0'(0) = 0. \tag{12}$$

We guess k and integrate equations (10)–(12) by the Runge–Kutta algorithm and check whether $\varphi_0(1) = 0$ for the first time. By one parameter shooting the eigenvalue k can be obtained without difficulty.

The results are shown in Figure 1; there the normalized fundamental frequency k is plotted as a function of α for various constant amplitude e. The cross-section of the membrane (if density is due to thickness) is also shown. The following are observed.

(1) The frequency may be increased or decreased from the homogeneous frequency of $k_0 = 2{\cdot}4048$, the larger the amplitude e the larger the changes. Changing the sign of e does not reflect k about k_0.
(2) The increase or decrease in k depends on the periodicity parameter α. There are certain α values where the frequency is insensitive to e and density variations, thus close to k_0.

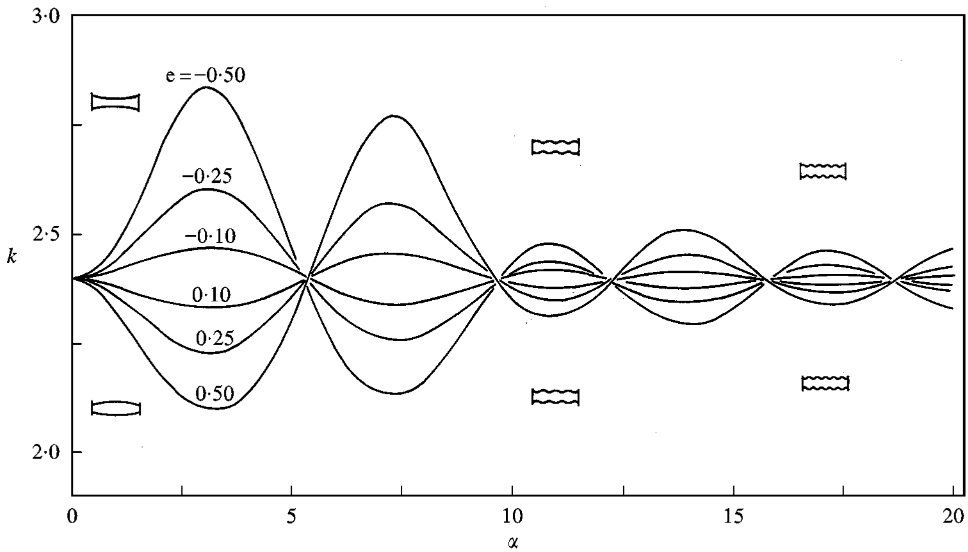

Figure 1. The fundamental frequency for various amplitude e and frequency α. Exaggerated cross-sections are shown.

We find that the fundamental frequency for a circular membrane with periodic density (or thickness) variations may be increased or decreased. By considering $n \neq 0$ and looking for eigenvalues of k close to $k_j = j$th zero of Bessel function J_0, the higher modes and frequencies can be obtained.

For an annulus with boundaries at $r = a$ and $r = 1$, the method is slightly modified. Set $\varphi_n(a) = 0$ and $\varphi_n'(a) = 1$, guess k and integrate equation (6) as an initial value problem. The eigenvalue k is obtained when $\varphi_n(1) = 0$ is satisfied.

REFERENCES

1. Rayleigh, 1945. *The Theory of Sound*, Vol. 1. New York: Dover, Chapter 9, second edition.
2. J. R. Kuttler and V. G. Sigillito, 1984 *SIAM Review* **26**, 163–193. Eigenvalues of the Laplacian in two dimensions.
3. V. Vodicka 1962 *Journal of Physics Society of Japan* **17**, 698–702. Free vibration of a composite circular membrane.
4. K. Kato 1976 *Journal of Sound and Vibration* **49**, 535–540. Free vibration analysis of a composite rectangular membrane consisting of strips.
5. B. C. Bhadra 1979 *Indian Journal of Pure and Applied Mathematics* **10**, 223–229. Symmetrical vibration of a composite circular membrane fastened at the circumference.
6. T. M. Kalotas and A. R. Lee 1993 *Acustica* **78**, 220–225. The modes of a circular membrane of non-uniform radial density.
7. J. A. Masad 1996 *Journal of Sound and Vibration* **195**, 674–678. Free vibrations of a non-homogeneous rectangular membrane.
8. P. A. A. Laura, R. E. Rossi and R. H. Gutierrez 1997 *Journal of Sound and Vibration* **204**, 373–376. The fundamental frequency of non-homogeneous rectangular membranes.
9. C. Y. Wang 1998 *Journal of Sound and Vibration* **210**, 555–558. Some exact solutions of the vibration of non-homogeneous membranes.