Journal of Sound and Vibration (1999) **227**(4), 787–806
Article No. jsvi.1999.2385, available online at http://www.idealibrary.com on IDEAL®

MODAL ANALYSIS OF MULTI-SPAN TIMOSHENKO BEAMS CONNECTED OR SUPPORTED BY RESILIENT JOINTS WITH DAMPING

S.-W. Hong and J.-W. Kim

School of Mechanical Engineering, Kumoh National University of Technology, 188 Shinpyung, Kumi, Kyungbuk, Korea

(*Received* 27 *January* 1999, *and in final form* 8 *May* 1999)

The present paper proposes an exact modelling and modal analysis method for non-uniform, multi-span beam-type structure supported and/or connected by resilient joints with damping. To this end, an exact dynamic matrix for a Timoshenko beam element is derived by means of the spatial domain Laplace transform. A generalized modal analysis method is also proposed and applied to the derivation of frequency response and time response formulas for general beam structures. Three examples are provided for validating and/or illustrating the proposed method. In the first numerical example, the proposed method is compared with FEM. The second example deals with a three-stepped beam structure supported by joints with damping property. In the final example, a dynamic analysis of a multi-span beam under moving load is demonstrated. The numerical study proves that the proposed method is useful for the dynamic analysis of multi-span beam-type structures.

1. INTRODUCTION

Exact solutions for distributed-parameter systems have attracted much attention from many researchers due to the fact that the dynamic analysis of a distributed-parameter system provides physical insight into the system [1–6]. Quite often, there is a need of exact and closed-form solutions for distributed-parameter systems. For example, the dynamic analysis of a multi-span beam under moving loads [6] is straightforward, if exact or closed-form solutions for the distributed parameter systems are available. So far, however, it has not been easy enough to obtain exact or closed-form solutions for non-uniform, multi-span beam systems with damping.

There have been several approaches to gain exact or closed-form solutions for general distributed-parameter beam systems. Yang *et al.* [7, 8] presented an attractive method to obtain exact and closed-form solutions for one-dimensional distributed-parameter systems by using the distributed transfer function synthesis technique. However, the implementation of the method seems not easy enough because of the difficulty in handling distributed transfer function matrices that

0022-460X/99/440787 + 20 $30.00/0

should be integrated over the length of the beam element. Several researchers who were interested in the dynamic behaviour of beams under moving loads have attempted to attain the exact solutions of beam-type structures [6, 9, 10]. These studies are mostly concerned with the dynamic-stiffness-based method, which confines itself to undamped systems. The exact solutions for multi-span beams supported or connected by joints with damping have seldom been discussed. It is believed to be still desirable to develop a more systematic method for attaining exact or closed-form solutions of distributed-parameter beam systems involved with damped joints.

The present paper deals with a new modelling and modal analysis method for multi-span beam structures supported or connected by joints with damping. First of all, a comprehensive modelling procedure to obtain an exact dynamic matrix for a uniform Timoshenko beam element is presented. A spatial state equation for a Timoshenko beam model, after applying Laplace transformation with respect to time, is used for developing the exact dynamic matrix. The state equation is Laplace transformed once more with respect to the spatial co-ordinate. Along with resolving the inverse matrix formula, inverse Laplace transformation for the resulting equation, with respect to the spatial co-ordinate, leads to a kind of exact transfer matrix between the boundary values at one end and the values at an in-between point of a uniform beam element. Substitution of the other boundary values for the beam element into the resulting equation and rearrangement of the variables yields an element dynamic matrix, which can be thought of as an exact dynamic matrix for a uniform Timoshenko beam element. Lumped inertia and joint elements with stiffness and damping are also modelled in the Laplace domain. With the element matrices derived, an actual system can be assembled in the same manner as the finite element method (FEM).

In this paper, a generalized modal analysis method is developed to easily deal with the exact system dynamic matrix that includes transcedental functions of the Laplace variable. Subsequently, any dynamic analysis can be readily accomplished with the help of the modal decomposition method. The most important advantage of the proposed method is that it can deliver exact and closed-form solutions for multi-span, distributed-parameter beam structures. A great reduction for the system matrix size is also expected due to the fact that a uniform beam segment, regardless of the length, can be modelled by a single element. In addition, changing parameters for any uniform beam section can be easily accomplished through the proposed method, different from FEM, which requires re-meshing to adjust the parameters. It is believed that the modelling procedure presented in this paper can also be applied to similar dynamic analysis concerned with distributed-parameter systems [11].

In order to validate the proposed method, three numerical examples are presented. A simple, single-span beam is considered as the first numerical example, in which eigenvalues by the proposed method for two extreme boundary conditions are compared with those from analytical results available. Then, frequency response functions (FRFs) as well as eigenvalues for general boundary conditions are also compared with those from FEM. Another example of application for a multi-span beam supported by resilient joints with damping is also presented. In the final example, a multi-span beam under a moving load is analysed as a rigorous application of the proposed method. The numerical study shows that the proposed method is very useful for the dynamic analysis of general distributed-parameter beam structures involved with lumped elements.

2. MODELLING

2.1. MODELLING OF TIMOSHENKO BEAM ELEMENT

The equations of motion for the Timoshenko beam, which contains shear deformation and rotary inertia, can be written in a spatial state equation form as

$$\begin{aligned} \frac{\partial u}{\partial x} &= \phi - \frac{F}{kAG}, \\ \frac{\partial \phi}{\partial x} &= \frac{M}{EI_d}, \\ \frac{\partial F}{\partial x} &= -\rho A \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial M}{\partial x} &= F + \rho I_d \frac{\partial^2 \phi}{\partial t^2}, \end{aligned} \tag{1}$$

where u and ϕ are the transverse and angular displacements of the beam and F and M are the corresponding force and moment respectively. ρ, G and E are the density, shear modulus and Young's modulus respectively. A and I_d are the area and the diametral moment of inertia respectively, and k is the shape factor that is dependent on the cross-sectional shape.

Laplace transformation of equation (1) with respect to time, with zero initial conditions, leads to

$$\begin{aligned} \frac{\partial u^*}{\partial x} &= \phi^* - \frac{F^*}{kAG}, \\ \frac{\partial \phi^*}{\partial x} &= \frac{M^*}{EI_d}, \\ \frac{\partial F^*}{\partial x} &= -\rho A s^2 u^*, \\ \frac{\partial M^*}{\partial x} &= F^* + \rho I_d s^2 \phi^*. \end{aligned} \tag{2}$$

Here, the asterisk represents the Laplace transform of the corresponding state variable, s being the Laplace variable for time. Equation (2) can be rewritten, in a simple matrix form, as

$$\frac{\partial \Psi(x,s)}{\partial x} = \mathbf{B}(s)\,\Psi(x,s), \tag{3}$$

where

$$\Psi(x,s)=\begin{Bmatrix} u^* \\ \phi^* \\ F^* \\ M^* \end{Bmatrix}, \qquad \mathbf{B}(s)=\begin{bmatrix} 0 & 1 & -d & 0 \\ 0 & 0 & 0 & b \\ -c & 0 & 0 & 0 \\ 0 & a & 1 & 0 \end{bmatrix}.$$

Here

$$a=\rho I_d s^2, \qquad b=\frac{1}{EI_d}, \qquad c=\rho A s^2, \qquad d=\frac{1}{kAG}.$$

Laplace transformation of equation (3) for the spatial co-ordinate x, with consideration of boundary values at $x=0$, may yield

$$\tilde{\Psi}(\lambda,s)=[\lambda\mathbf{I}-\mathbf{B}]^{-1}\Psi(0,s). \tag{4}$$

Here, λ is the Laplace variable for the spatial co-ordinate and $\tilde{\Psi}(\lambda, s)$ represents the spatial Laplace transform for $\Psi(x, \lambda)$. One can resolve $[\lambda\mathbf{I}-\mathbf{B}]^{-1}$ in equation (4) as

$$[\lambda\mathbf{I}-\mathbf{B}]^{-1}=\frac{1}{(\lambda^2-\alpha^2)(\lambda^2-\beta^2)}$$

$$\begin{bmatrix} \lambda(\lambda^2-ab) & \lambda^2 & -d\lambda^2+b(ad+1) & b\lambda \\ -bc & \lambda(\lambda^2-cd) & b\lambda & b(\lambda^2-cd) \\ -c(\lambda^2-ab) & -c\lambda & \lambda(\lambda^2-ab) & -bc \\ -c\lambda & a\lambda^2-c(ad+1) & \lambda^2 & \lambda(\lambda^2-cd) \end{bmatrix},$$

where

$$\alpha^2=\tfrac{1}{2}\left\{(ab+cd)+\sqrt{(ab+cd)^2-4(abcd+bc)}\right\},$$

$$\beta^2=\tfrac{1}{2}\left\{(ab+cd)-\sqrt{(ab+cd)^2-4(abcd+bc)}\right\}.$$

The inverse Laplace transformation of equation (4) for x gives the following:

$$\Psi(x,s)=\mathbf{C}(x,s)\Psi(0,s), \tag{5}$$

where

$$\mathbf{C}(x,s)=$$

$$\begin{bmatrix} g_3-abg_1 & g_2 & -dg_2+(abd+b)g_0 & bg_1 \\ -bcg_0 & g_3-cdg_1 & bg_1 & bg_2-bcdg_0 \\ -cg_2+abcg_0 & -cg_1 & g_3-abg_1 & -bcg_0 \\ -cg_1 & ag_2-(acd+c)g_0 & g_2 & g_3-cdg_1 \end{bmatrix},$$

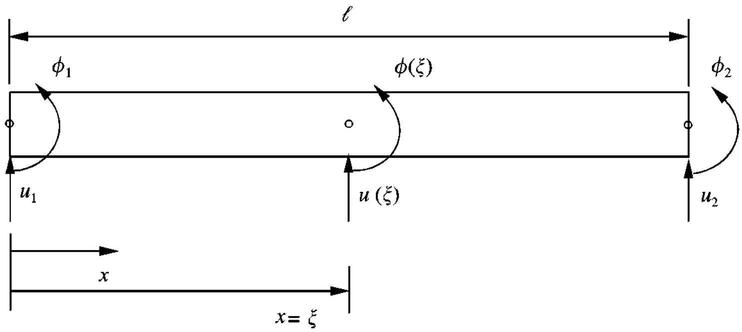

Figure 1. Sign conventions for the beam element.

$$g_0 = \frac{1}{\alpha^2-\beta^2}\left\{\frac{1}{\alpha}\sinh\alpha x - \frac{1}{\beta}\sinh\beta x\right\},\quad g_1 = \frac{1}{\alpha^2-\beta^2}\{\cosh\alpha x - \cosh\beta x\},$$

$$g_2 = \frac{1}{\alpha^2-\beta^2}\{\alpha\sinh\alpha x - \beta\sinh\beta x\},\quad g_3 = \frac{1}{\alpha^2-\beta^2}\{\alpha^2\cosh\alpha x - \beta^2\cosh\beta x\}.$$

Hereafter, a uniform beam as shown in Figure 1 is considered to derive an exact dynamic matrix for a uniform beam element. The sign conventions for the displacements at $x = 0$, ξ and l are also indicated in Figure 1. Substitution of the forces and displacements at $x = 0$ and ξ into equation (5) and rearrangement of the variables in equation (5) yields

$$\begin{Bmatrix} F_1^* \\ M_1^* \\ F^*(\xi) \\ M^*(\xi) \end{Bmatrix} = \mathbf{D}^e(\xi)\begin{Bmatrix} u_1^* \\ \phi_1^* \\ u^*(\xi) \\ \phi^*(\xi) \end{Bmatrix}, \tag{6}$$

where

$$\mathbf{D}^e(\xi) = \begin{bmatrix} \mathbf{D}_{11}(\xi) & \mathbf{D}_{12}(\xi) \\ \mathbf{D}_{21}(\xi) & \mathbf{D}_{22}(\xi) \end{bmatrix} = \begin{bmatrix} d_1(\xi) & d_2(\xi) & d_4(\xi) & d_5(\xi) \\ d_2(\xi) & d_3(\xi) & -d_5(\xi) & d_6(\xi) \\ d_4(\xi) & -d_5(\xi) & d_1(\xi) & -d_2(\xi) \\ d_5(\xi) & d_6(\xi) & -d_2(\xi) & d_3(\xi) \end{bmatrix} \tag{7}$$

and

$$\Delta = \frac{1}{\alpha^2-\beta^2}\left\{2b(1-\cosh\alpha\xi\cosh\beta\xi) + \frac{\alpha\beta}{c}\{\mu^2+v^2\}\sinh\alpha\xi\sinh\beta\xi\right\},$$

$$d_1 = \frac{1}{\Delta}\{-\mu\sinh\alpha\xi\cosh\beta\xi + v\sinh\beta\xi\cosh\alpha\xi\},$$

$$d_2 = \frac{1}{\Delta}\left\{\frac{(\beta\mu+\alpha v)}{\alpha^2-\beta^2}\sinh\alpha\xi\sinh\beta\xi - \frac{(ab-cd)}{\alpha^2-\beta^2}(1-\cosh\alpha\xi\cosh\beta\xi)\right\}.$$

$$d_3 = \frac{1}{\Delta}\frac{\alpha\beta}{bc}\{v \sinh \alpha\xi \cosh \beta\xi - \mu \sinh \beta\xi \cosh \alpha\xi\},$$

$$d_4 = \frac{1}{\Delta}\{\mu \sinh \alpha\xi - v \sinh \beta\xi\},$$

$$d_5 = \frac{1}{\Delta}\{\cosh \alpha\xi - \cosh \beta\xi\},$$

$$d_6 = \frac{1}{\Delta}\frac{\alpha\beta}{bc}\{- v \sinh \alpha\xi + \mu \sinh \beta\xi\},$$

$$\mu = \frac{(cd - \alpha^2)}{\alpha}, \qquad v = \frac{(cd - \beta^2)}{\beta}.$$

Note here that the partitioned matrices, $\mathbf{D}_{ik}(\xi)$, $i, k = 1, 2$, are 2×2 and $\mathbf{D}_{12}(\xi) = \mathbf{D}_{21}^{\mathrm{T}}(\xi)$. By substituting l for ξ, one can have the following equation:

$$\begin{Bmatrix} F_1^* \\ M_1^* \\ F_2^* \\ M_2^* \end{Bmatrix} = \mathbf{D}^e(l) \begin{Bmatrix} u_1^* \\ \phi_1^* \\ u_2^* \\ \phi_2^* \end{Bmatrix}, \tag{8}$$

where $\mathbf{D}^e(l)$ is equivalent to an exact dynamic matrix of an element in the s domain.

2.2. MODELLING OF LUMPED INERTIA ELEMENT

The equations of motion for a concentrated inertia element can be written as

$$m^c \ddot{u} = F, \qquad J^c \ddot{\phi} = M, \tag{9}$$

where m^c and J^c denote the mass and the mass moment of inertia respectively. Applying Laplace transformation to equations (9) provides the following equation of motion, in the s domain, for the concentrated inertia element:

$$\begin{Bmatrix} F^* \\ M^* \end{Bmatrix} = \begin{bmatrix} m^c s^2 & 0 \\ 0 & J^c s^2 \end{bmatrix} \begin{Bmatrix} u^* \\ \phi^* \end{Bmatrix}. \tag{10}$$

2.3. MODELLING OF SUPPORTING/CONNECTING ELEMENTS

There are two kinds of joints that are essential for beam structures: connecting and supporting joints. The equation of motion for a supporting joint element can be written as

$$\begin{aligned} F &= c^t \dot{u} + k^t u, \\ M &= c^r \dot{\phi} + k^r \phi, \end{aligned} \tag{11}$$

where c and k are the damping and stiffness coefficients respectively, and the superscripts t and r represent "transverse" and "rotational" respectively. On the other hand, the equation of motion for a connecting joint element can be written as

$$\begin{aligned} F_i &= -F_j = c^t(\dot{u}_t - \dot{u}_j) + k^t(u_i - u_j), \\ M_i &= -M_j = c^r(\dot{\phi}_t - \dot{\phi}_j) + k^r(\phi_i - \phi_j). \end{aligned} \tag{12}$$

where the subscripts i and j denote two connecting nodal points. Taking Laplace transformation for equations (11) and (12),

$$\begin{Bmatrix} F^* \\ M^* \end{Bmatrix} = \begin{bmatrix} sc^t + k^t & 0 \\ 0 & sc^r + k^r \end{bmatrix} \begin{Bmatrix} u^* \\ \phi^* \end{Bmatrix} \tag{13}$$

and

$$\begin{Bmatrix} F_i^* \\ M_i^* \\ F_j^* \\ M_j^* \end{Bmatrix} = \begin{bmatrix} sc^t + k^t & 0 & -(sc^t + k^t) & 0 \\ 0 & sc^r + k^r & 0 & -(sc^r + k^r) \\ -(sc^t + k^t) & 0 & sc^t + k^t & 0 \\ 0 & -(sc^r + k^r) & 0 & sc^r + k^r \end{bmatrix} \begin{Bmatrix} u_i^* \\ \phi_i^* \\ u_j^* \\ \phi_j^* \end{Bmatrix}. \tag{14}$$

2.4. ASSEMBLING PROCEDURE FOR THE GLOBAL SYSTEM DYNAMIC MATRIX

The assembling procedure for the global system dynamic matrix can be accomplished in the same manner as FEM. The first step is to decompose the entire structure into uniform, distributed parameter beam elements, and lumped inertia and joint elements. The next step is to make an element dynamic matrix for each of the beam elements and lumped elements with equations given in the previous sections. The final step is to assemble the element dynamic matrices in the same manner that the global matrices are constructed in FEM. This assembling procedure may result in the following system matrix equation:

$$F^*(s) = \mathbf{D}(s)\mathbf{q}^*(s), \tag{15}$$

where $\mathbf{q}^*$ and $\mathbf{F}^*$ are the Laplace transforms of the global displacement and force vectors.

From equation (15), the transfer function and frequency response function matrices can be written as

$$\mathbf{H}(s) = \mathbf{D}^{-1}(s) \tag{16}$$

and

$$\mathbf{H}(\mathrm{j}\omega) = \mathbf{D}^{-1}(s)|_{s=\mathrm{j}\omega} \tag{17}$$

Exact transfer function and frequency response function matrices can be obtained through direct computation of equations (16) and (17). However, the direct computational method requires repetitive inversion of complex matrices, the size of which tends to be larger in the case of complicated structures. The following section presents a generalized modal analysis scheme, which can provide a response formula based on modal summation.

3. GENERALIZED MODAL ANALYSIS FOR GENERAL BEAM STRUCTURE

Since the system dynamic matrix contains transcendental functions, conventional modal analysis schemes are not appropriate. A generalized modal analysis method will be presented in this section to make the dynamic analysis straightforward.

3.1. EIGENVALUE AND EIGENVECTOR

The eigenvalue problem associated with equation (15) is written as

$$\mathbf{D}(s)\,\mathbf{q}^*(s) = \mathbf{0}. \tag{18}$$

Thus, the eigenvalues associated with equation (15) can be attained from the non-trivial solution condition, i.e.,

$$\det\{\mathbf{D}(s)\} = 0. \tag{19}$$

Equation (19), unlike conventional eigenvalue problems, necessitates a special algorithm for solving non-linear equations. In this paper, a modified bisection method is adopted which is suitable for general complex equations. It is obvious that the number of eigenvalues is infinite because $\mathbf{D}(s)$ contains transcendental functions. The corresponding eigenvectors can be readily obtained by using equation (18), once equation (19) is solved. The eigenvalues and the corresponding eigenvectors satisfy the following lemma, an extended version from a matrix polynomial theorem [12].

Lemma 1. *For all, distinct eigenvalues, s_i, $i = 1, 2, \ldots, \infty$, from equation* (18), *the corresponding eigenvectors, q_i, $i = 1, 2, \ldots, \infty$, satisfy the orthogonality condition such as*

$$\mathbf{q}_i^t\{\mathbf{D}(s_i) - \mathbf{D}(s_k)\}\,q_k = 0, \qquad i \neq k, \qquad i, k = 1, 2, \ldots, \infty\,. \tag{20}$$

The eigenvectors can conveniently be normalized so as to satisfy

$$\mathbf{q}_i^t\left\{\frac{\mathrm{d}\mathbf{D}(s)}{\mathrm{d}s}\right\}_{s=s_i}\mathbf{q}_i = 1, \quad i = 1, 2, \ldots, \infty\,. \tag{21}$$

It should be noted here that the vector set, $\mathbf{q}_i$, $i = 1, 2, \ldots, \infty$, is linearly dependent. Analytical derivatives for element matrices are required for making use of equation (21) in eigenvector normalization.

3.2. MODAL DECOMPOSITION

Since the system dynamic matrix can be expressed as a matrix polynomial of infinite order by using Taylor's series expansion, the inverse of system dynamic matrix can be written by a series of partial fractions in the same way as the inverse of a matrix polynomial [12].

Lemma 2. *Provided* s_i, $i = 1, 2, \ldots, \infty$, *are all distinct and the corresponding eigenvectors are normalized by equation* (21), *the inverse of system dynamic matrix* (*or transfer function matrix*) *can be expressed as an infinite series of partial fractions, i.e.,*

$$\mathbf{H}(s) = \{\mathbf{D}(s)\}^{-1} = \sum_{i=1}^{\infty} \frac{\mathbf{P}_i}{s - s_i}, \qquad \mathbf{P}_i = \mathbf{q}_i \mathbf{q}_i^t, \quad i = 1, 2, \ldots, \infty. \tag{22}$$

From Lemma 2, it is easy to show that, if the system is stable, the frequency response function matrix can also be written simply as

$$\mathbf{H}(\mathrm{j}\omega) = \{\mathbf{D}(\mathrm{j}\omega)\}^{-1} = \sum_{i=1}^{\infty} \frac{\mathbf{P}_i}{\mathrm{j}\omega - s_i}. \tag{23}$$

On the other hand, since the exact system dynamic matrix includes Laplace variable s in itself, it is impossible to directly calculate time domain responses. However, inverse Laplace transformation for the transfer matrix can yield the impulse response function matrix as

$$\mathbf{G}(t) = L^{-1}\{\mathbf{H}(s)\} = \begin{cases} \sum_{i=1}^{\infty} \mathbf{P}_i \mathrm{e}^{s_i t}, & t \geqslant 0, \\ \mathbf{0} & t < 0. \end{cases} \tag{24}$$

Therefore, the time response when a general force is applied to the system can be obtained by means of the convolution integral, i.e.,

$$\mathbf{q}(t) = \int_{-\infty}^{\infty} \mathbf{G}(t - \tau)\mathbf{F}(\tau)\,\mathrm{d}\tau. \tag{25}$$

3.3. REPRESENTATION OF RESPONSE AT AN INTERIOR POINT OF BEAM ELEMENT

It is interesting to know that a finite matrix can represent a distributed-parameter system without causing any error. However, additional formulae are required to elicit full solutions for a distributed-parameter system. Once the responses at nodal points are computed, the responses at interior points between two nodal points can be obtained from the relation as

$$\begin{Bmatrix} u^*(\xi) \\ \phi^*(\xi) \end{Bmatrix} = \mathbf{D}_{12}^{-1}(\xi)[\mathbf{D}_{11}(l) - \mathbf{D}_{11}(\xi) \quad \mathbf{D}_{12}(l)] \begin{Bmatrix} u_1^* \\ \phi_1^* \\ u_2^* \\ \phi_2^* \end{Bmatrix}. \tag{26}$$

It is easy to obtain responses at an in-between point of interest by using equation (26) together with equations (22)–(24).

4. NUMERICAL EXAMPLES

Three examples are provided here to validate the proposed method. In the first example, a simple beam is taken to compare the proposed method with existing methods. In the second example, a modal analysis procedure is demonstrated by using a three-stepped beam supported by two resilient joints. The final example illustrates the proposed method with a multi-span beam under a moving load.

4.1. NUMERICAL EXAMPLE 1

The numerical model, which is composed of a uniform beam and two identical supporting joints at both ends of the beam is shown in Figure 2. The detailed specifications of the beam and joints are given in Table 1. The shape factor formula is adopted from reference [13]. To show the exactness of the proposed method, natural frequencies are examined for the beam in the absence of joints, but under hinged–hinged and free–free boundary conditions. In Table 2, the natural frequencies obtained by the proposed method are compared with those by analytical formulas available in reference [13]. It can be seen that the natural frequencies obtained by the proposed method are identical to those obtained by analytical formulas.

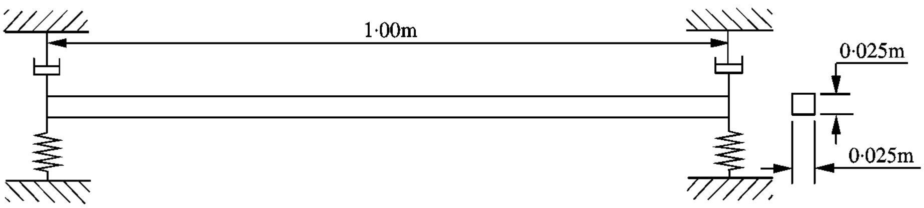

Figure 2. Numerical model 1.

TABLE 1

Specifications of numerical model 1

Elements	Properties	Data
Beam	Length (m)	1·0
	Width (cm)	2·5
	Depth (cm)	2·5
	Young's modulus (GN/m^2)	200
	Shear modulus (GN/m^2)	80
	Poisson's ratio ν	0·3
	Shape factor κ	$10(1+\nu)/(12+11\nu)$
	Density (kg/m^3)	8000
Supporting joints (2 identical)	Stiffness (MN/m)	2
	Damping (Ns/m)	20

TABLE 2

Comparison of natural frequencies from analytical formulas in reference [13] *and the proposed method for the uniform beam under hinged–hinged and free–free boundary conditions*

	Natural frequency (ω_n), rad/s			
	Hinged–hinged		Free–free*	
Mode No.	Analytical formula†	Proposed method	Analytical formula‡	Proposed method
1	3·55778498e + 002	3·55778498e + 002	8·05532939e + 002	8·05532939e + 002
2	1·41882448e + 003	1·41882448e + 003	2·21142738e + 003	2·21142738e + 003
3	3·17650875e + 003	3·17650875e + 003	4·30966485e + 003	4·30966485e + 003
4	5·60854995e + 003	5·60854995e + 003	7·06898477e + 003	7·06898477e + 003
5	8·68806614e + 003	8·68806614e + 003	1·04602176e + 004	1·04602176e + 004

* Rigid body modes (0 rad/s) are excluded.

† Hinged–hinged [13]:

$$\omega_n^4 r_y^2 - \left[\frac{n^2\pi^2}{l^2}\frac{EI}{\bar{\rho}} + \frac{n^2\pi^2}{l^2} r_y^2 \frac{kAG}{\bar{\rho}} + \frac{kAG}{\bar{\rho}}\right]\omega_n^2 + \frac{n^4\pi^4}{l^4}\frac{kAGEI}{\bar{\rho}^2} = 0.$$

‡ Free–free [13]:

$$\cosh\beta_1 l\cos\beta_2 l - 1 + \tfrac{1}{2}c_1\sinh\beta_1 l\sin\beta_2 l = 0,$$

where

$$c_1 = \frac{\beta_2}{\beta_1}\left(\frac{\bar{\rho}\omega_n^2 - kAG\beta_2^2}{\bar{\rho}\omega_n^2 + kAG\beta_1^2}\right) - \frac{\beta_1}{\beta_2}\left(\frac{\bar{\rho}\omega_n^2 + kAG\beta_1^2}{\bar{\rho}\omega_n^2 - kAG\beta_2^2}\right),$$

$$\beta_1^2 = \frac{1}{2}\left\{-\bar{\rho}\omega_n^2\left(\frac{1}{kAG} + \frac{r_y^2}{EI}\right) + \sqrt{\left(\bar{\rho}\omega_n^2\left(\frac{1}{kAG} + \frac{r_y^2}{EI}\right)\right)^2 + \frac{4\bar{\rho}\omega_n^2}{EI}\left(1 - \frac{\bar{\rho}\omega_n^2 r_y^2}{kAG}\right)}\right\},$$

$$\beta_2^2 = \frac{1}{2}\left\{\bar{\rho}\omega_n^2\left(\frac{1}{kAG} + \frac{r_y^2}{EI}\right) + \sqrt{\left(\bar{\rho}\omega_n^2\left(\frac{1}{kAG} + \frac{r_y^2}{EI}\right)\right)^2 + \frac{4\bar{\rho}\omega_n^2}{EI}\left(1 - \frac{\bar{\rho}\omega_n^2 r_y^2}{kAG}\right)}\right\},$$

Here r_y is the radius of gyration of the beam and $\bar{\rho} = \rho A$.

The proposed method is compared with the FEM, which also includes rotary inertia and shear deformation effect. Only one element is taken to model the beam in the proposed method, while the number of elements is varied in the FEM to demonstrate the difference. For increasing the number of elements, new nodes are generated without relocating the previous nodes, in order to ensure the monotonic convergence of eigenvalues with the number of elements. In Figure 3, FRFs obtained by the proposed method are compared with those by the FEM. The peak frequencies of FRFs by the FEM approach those by the proposed method, as the number of elements is increased. Eigenvalues, computed both from the proposed method and the FEM, are compared as well in Table 3. The eigenvalues obtained by the FEM also converge to those by the proposed method. These results imply that the proposed method provides exact solutions. Therefore, it is shown

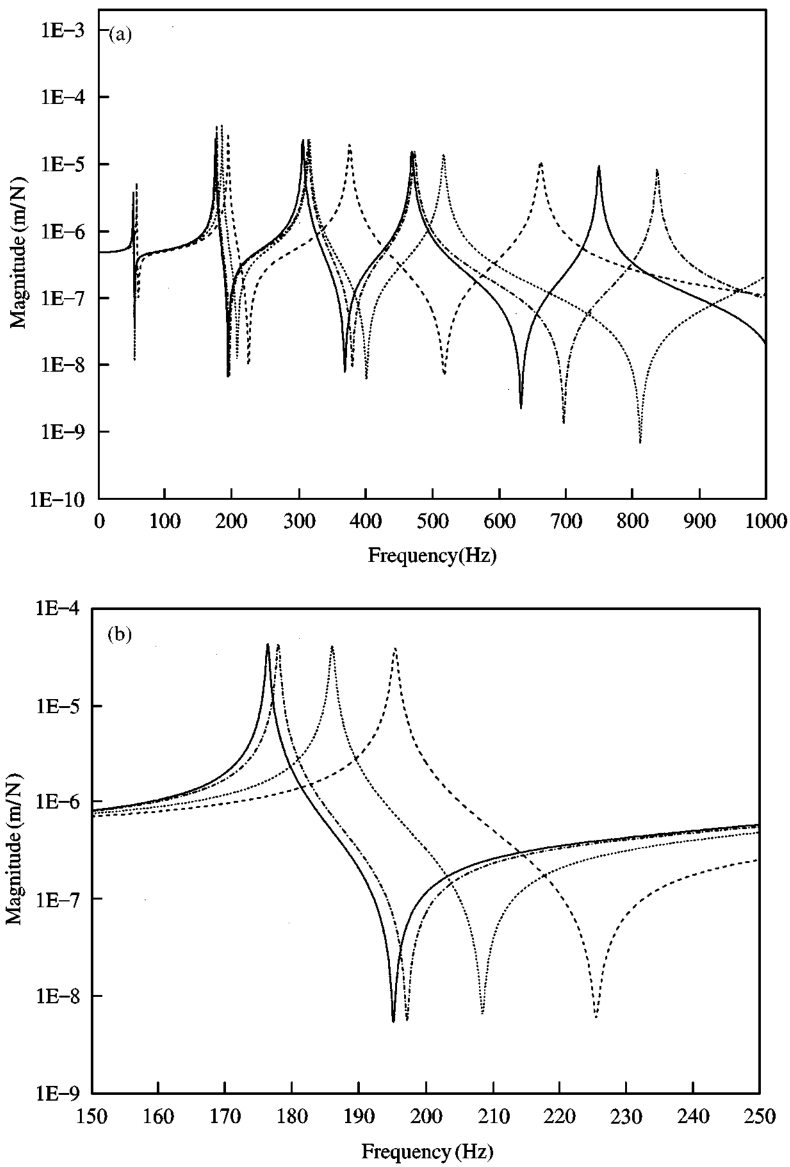

Figure 3. Comparison of FRFs computed by FEM and the proposed method: excited and measured at the left end of the beam. (a) wide range of frequency; (b) zoomed around the second mode. —— proposed method; ----- FEM 2 element; · · · · · FEM 4 element; .-.-.-.- FEM 16 element.

that a single exact element matrix proposed here can model uniform beam without causing any error. On the other hand, it is observed in Table 3 that higher natural frequencies from the FEM have larger errors than lower natural frequencies. The reason is believed to be that constraints tend to increase the system stiffness and, conceptually, discretization with fewer degrees of freedom implies imposing more constraints on the system [14].

TABLE 3

Comparison of eigenvalues from FEM and the proposed method for numerical model 1. Real part of eigenvalue/imaginary part of eigenvalue (eigenvalue $\lambda_k = \sigma_k + j\omega_k$, rad/s)

Mode No.	FEM 2 elements	FEM 4 elements	FEM 16 elements	Proposed method 1 element
1	− 6·7545e − 002 + j3·3568e + 002	− 6·6720e − 002 + j3·3453e + 002	− 6·6652e − 002 + j3·3444e + 002	− 6·6651e − 002 + j3·3444e + 002
2	− 3·5023e + 000 + j1·1683e + 003	− 2·7644e + 000 + j1·1111e + 003	− 2·7331e + 000 + j1·1079e + 003	− 2·7327e + 000 + j1·1079e + 003
3	− 1·4203e + 001 + j1·9860e + 003	− 1·2563e + 001 + j1·9429e + 003	− 1·2138e + 001 + j1·9273e + 003	− 1·2133e + 001 + j1·9271e + 003
4	− 2·2424e + 001 + j3·2481e + 003	− 2·1488e + 001 + j2·9958e + 003	− 2·0125e + 001 + j2·9549e + 003	− 2·0106e + 001 + j2·9542e + 003
5	− 3·7002e + 001 + j6·8026e + 003	− 2·0213e + 001 + j4·7456e + 003	− 2·0182e + 001 + j4·7139e + 003	− 2·0135e + 001 + j4·7111e + 003

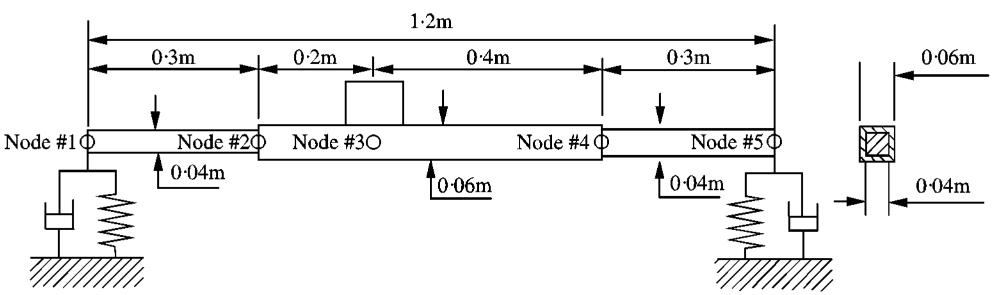

Figure 4. Numerical model 2.

4.2. NUMERICAL EXAMPLE 2

In this example, a three-stepped beam with a lumped mass is considered to show the modal analysis procedure by the proposed method. The schematic drawing for the beam considered in this example is shown in Figure 4. The detailed specifications of the beam structure are given in Table 4. Since the beam has three different sections and a lumped mass in the middle segment, four elements are used for modelling, as shown in Figure 4. In Table 5, eigenvalues computed from the proposed method are compared with those from the FEM by increasing the number of elements. It is clearly observed that the eigenvalues from the FEM converge to those from the proposed method as the number of elements is increased. The first three mode shapes with neglecting damping in supporting joints are presented in Figure 5. Unlike other discretization methods, the present method gives continuous-mode shape functions. The essence of the modal analysis is to be able to compute frequency and time response by means of modal summation. Typical frequency response functions by the modal expansion method and the

TABLE 4

Structural properties for numerical model 2

Elements	Properties	Data
Beam	Young's modulus (GN/m^2)	200
	Shear modulus (GN/m^2)	80
	Poisson's ratio ν	0·3
	Shape factor κ	$10(1+\nu)/(12+11\nu)$
	Density (kg/m^3)	8000
Concentrated mass	Mass (kg)	20
	Mass moment of inertia (kg m^2)	$3{\cdot}333\times10^{-4}$
Supporting joints	Stiffness (MN/m)	2
(2 identical)	Damping (Ns/m)	10

direct computational method are compared in Figure 6. Seven modes are included in the modal expansion method. Figure 6 ascertains that the modal expansion method gives accurate results, as the direct method does. Figure 7 shows a typical impulse response function computed by equation (24).

4.3. NUMERICAL EXAMPLE 3

The present example deals with a three-span beam structure under a moving load, in order to show the applicability of the proposed method to general beam systems. This example is adopted from reference [6] except the Timoshenko beam model is substituted in place of the Euler–Bernoulli beam model. Figure 8 shows a schematic diagram for the system, and the specifications are described in Table 6. In this case, one element per span is taken to model the multi-span beam system. The first three eigenvalues and the corresponding shape functions are given in Figure 9. A constant moving load with constant speed, v, can be modelled as

$$f(x,t) = f_0\delta(x - vt). \tag{27}$$

Then the modal force for the ith mode can be written as

$$p_i(t) = \int_0^L \varphi_i(x) f(x,t)\,\mathrm{d}x = \varphi_i(vt) f_0 \tag{28}$$

where $\varphi_i(x)$ is the ith mode shape function synthesized with nodal eigenvectors and equation (26). In this case, the summation for the ith modal response and the conjugate modal response is given by

$$r_i(t) = 2\mathrm{Re}\left\{\int_0^t p_i(\tau)\mathrm{e}^{s_i(t-\tau)}\,\mathrm{d}\tau\right\}. \tag{29}$$

TABLE 5

Comparison of eigenvalues from FEM and the proposed method for numerical model 2. Real part of eigenvalue/imaginary part of eigenvalue (eigenvalue $\lambda_k = \sigma_k + j\omega_k$, rad/s)

Mode No.	FEM 4 elements	FEM 8 elements	FEM 12 elements	FEM 24 elements	Proposed method 4 elements
1	− 6.8404e − 02	− 6.8391e − 02	− 6.8390e − 02	− 6.8390e − 02	− 6.8390e − 02
	+ j2.2242e + 02	+ j2.2241e + 02	+ j2.2241e + 02	+ j2.2241e + 02	+ j2.2241e + 02
2	− 1.1084e + 00	− 1.1042e + 00	− 1.1039e + 00	− 1.1037e + 00	− 1.1037e + 00
	+ j7.3318e + 02	+ j7.3250e + 02	+ j7.3244e + 02	+ j7.3242e + 02	+ j7.3241e + 02
3	− 3.0276e + 00	− 2.9707e + 00	− 2.9659e + 00	− 2.9640e + 00	− 2.9635e + 00
	+ j1.7444e + 03	+ j1.7367e + 03	+ j1.7360e + 03	+ j1.7358e + 03	+ j1.7357e + 03
4	− 3.7744e + 00	− 3.6310e + 00	− 3.6128e + 00	− 3.6052e + 00	− 3.6033e + 00
	+ j3.0528e + 03	+ j3.0256e + 03	+ j3.0217e + 03	+ j3.0201e + 03	+ j3.0197e + 03
5	− 2.8250e + 00	− 2.9164e + 00	− 2.8858e + 00	− 2.8716e + 00	− 2.8679e + 00
	+ j5.1947e + 03	+ j5.1567e + 03	+ j5.1435e + 03	+ j5.1375e + 03	+ j5.1358e + 03

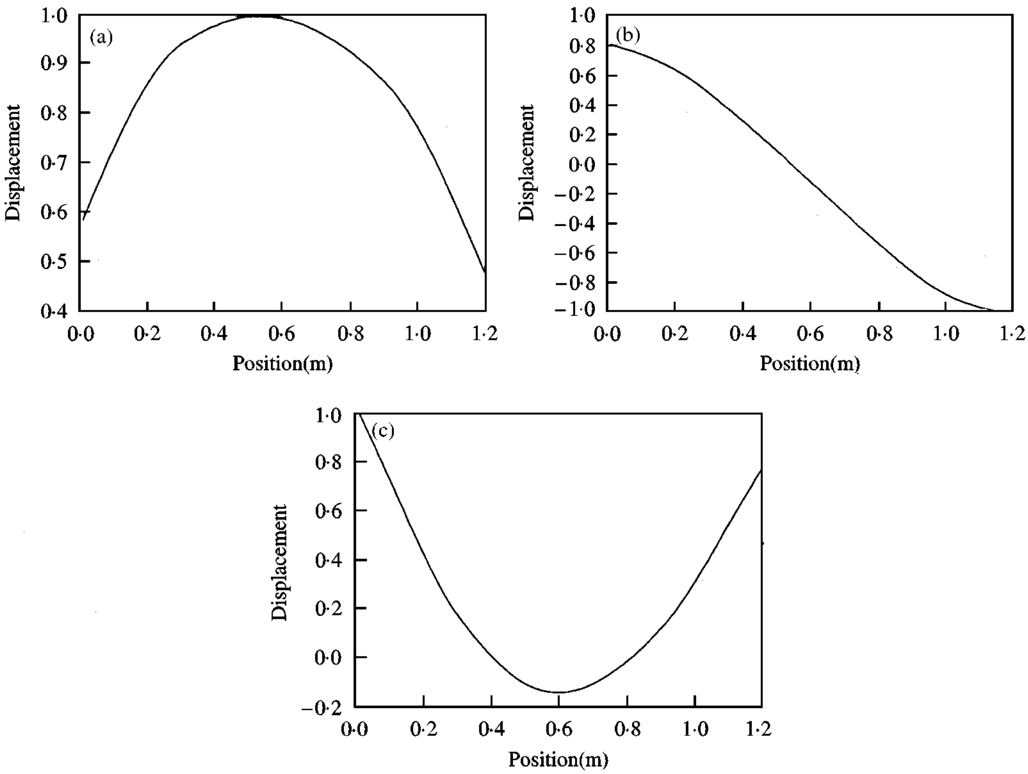

Figure 5. Mode shapes of first three modes for numerical model 2: (a) 1st mode; (b) 2nd mode; (c) 3rd mode.

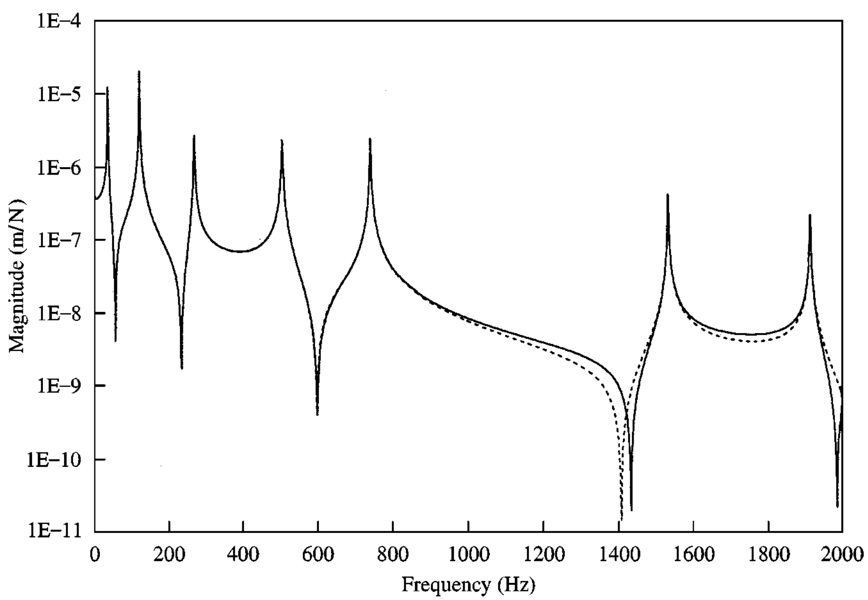

Figure 6. Comparison of FRFs computed by the modal summation and the direct computational method: excited and measured at the 1st and 3rd nodes of the system, respectively, for numerical model 2. —— direct inversion; - - - - mode synthesis.

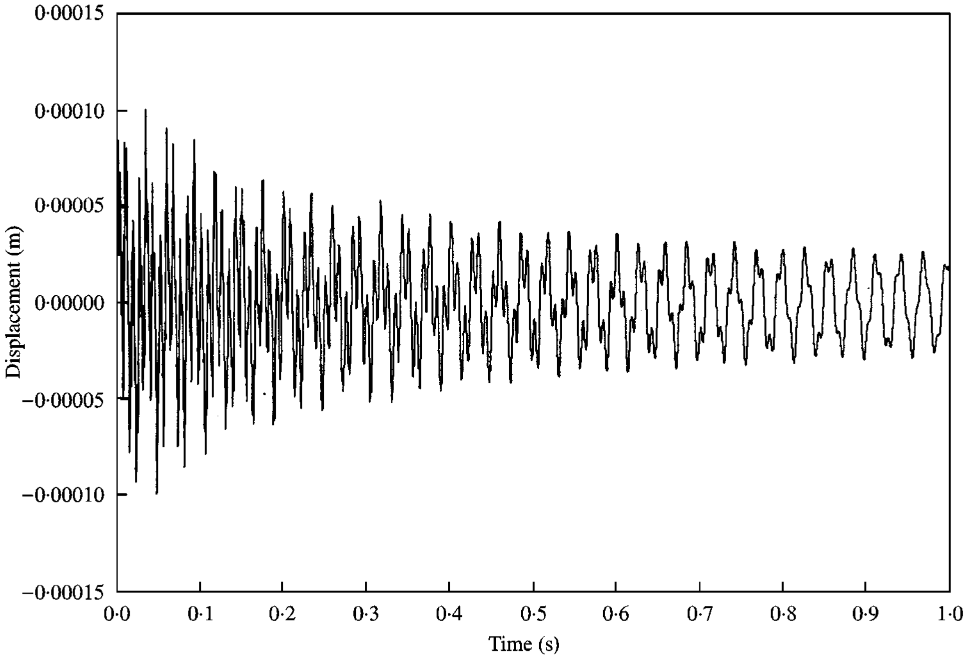

Figure 7. Impulse response function by the modal summation: excited and measured at the 1st and 3rd nodes of the system, respectively, for numerical model 2.

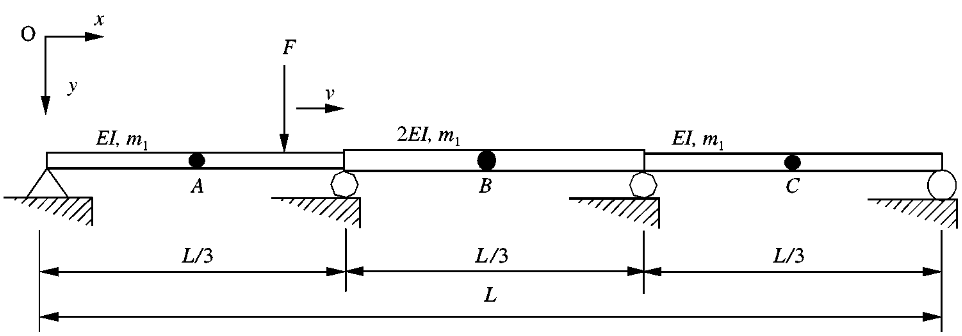

Figure 8. Numerical model 3 [6].

Consequently, the time response can be written as

$$u(x, t) = \sum_{i=1}^{\infty} \varphi_i(x) r_i(t). \tag{30}$$

Figure 10 shows time responses at the center of each beam (indicated in Figure 8 by A, B and C) under a moving load with $v = 35{\cdot}57$ m/s: 10 modes are used in computation. It can be seen that the curves in Figure 10 are in good agreement with those in reference [6].

TABLE 6
Structural properties for numerical model 3

Properties	Data
Length (m)	60
Sectional area A (m^2)	$0{\cdot}51 \times 10^{-2}$
Young's modulus E (GN/m^2)	104·8
Shear modulus G (GN/m^2)	40·3
Poisson's ratio v	0·3
Shape factor κ	$10(1+v)/(12+11v)$
Rigidity EI (Nm^2)	$1{\cdot}96 \times 10^9$
Mass per unit length m_l (kg/m)	1000
Force f_o (N)	$9{\cdot}48 \times 10^3$

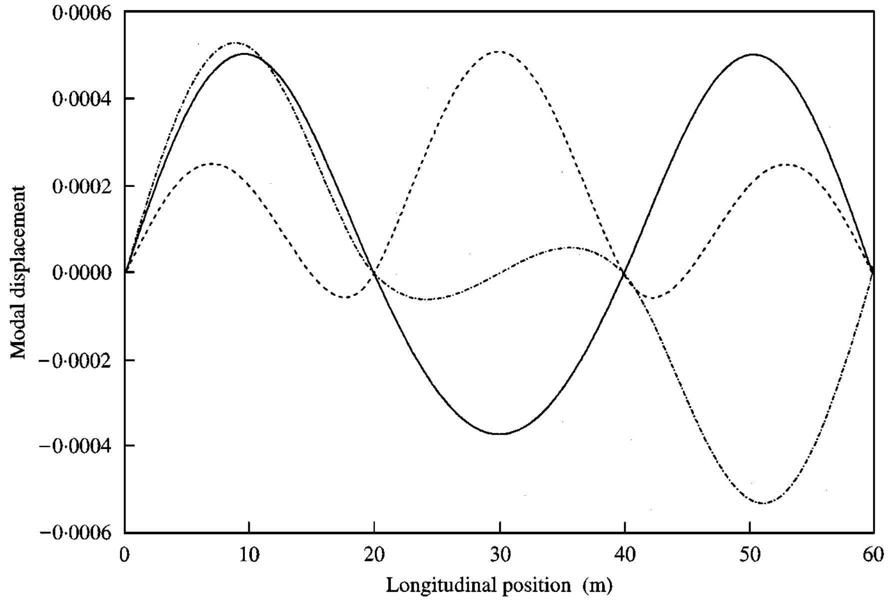

Figure 9. Mode shapes of first three modes for numerical model 3 ($\omega_1 = 6{\cdot}190$ Hz, $\omega_2 = 7{\cdot}550$ Hz, $\omega_3 = 11{\cdot}882$ Hz). —— 1st mode; .-.-.- 2nd mode; - - - - - - 3rd mode.

5. CONCLUDING REMARKS

In this study, an exact dynamic matrix in the Laplace domain for a Timoshenko beam element is derived. The importance of the derivation procedure is in applying Laplace transformation to a spatial state equation of the Timoshenko beam model, twice with regard to time and also spatial co-ordinate. The application of inverse Laplace transformation of the resulting equation with respect to the spatial co-ordinate and the application of the boundary values will come up with the exact dynamic matrix for a uniform Timoshenko beam element. The exact dynamic

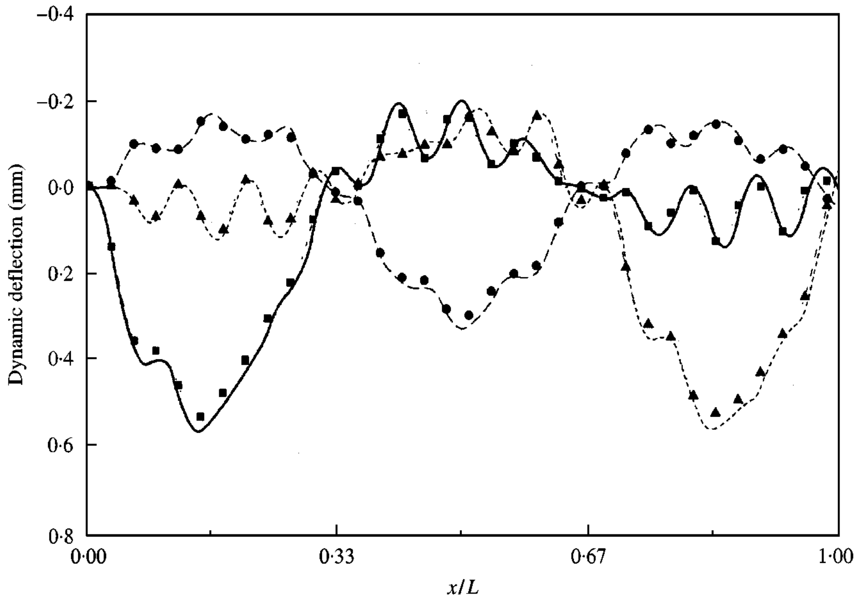

Figure 10. Dynamic response at the center of each span of the beam structure due to a moving load: ——, reference [6]-A; - · - · -, reference [6]-B; - - - - -, reference [6]-C ■, proposed-A; ●, proposed-B; ▲, proposed-C.

matrix for the beam element, together with the other two element matrices for lumped inertia and joint elements, is used to make the global system dynamic matrix of beam structures. A generalized modal analysis procedures is proposed to make the system analysis straightforward. Three numerical examples are provided to show the adequacy and applicability of the proposed method.

The proposed method provides an exact model with finite matrix size and a modal analysis method for multi-span, distributed-parameter beam systems supported and/or connected by resilient joints with damping. In addition, the matrix size of the model is anticipated to be small due to the fact that any uniform segment of the beam can be modelled by a beam element without causing any error. The proposed method also allows dynamic analysis of the system without any re-meshing process, even in the case when beam dimensions are changed. This feature is believed to be useful for design and reanalysis of beam structures.

ACKNOWLEDGMENT

The authors wish to acknowledge the financial support of the Korea Research Foundation given in the program year of 1998.

REFERENCES

1. S. P. Timoshenko, D. H. Young and W. Weaver 1974 *Vibration Problems in Engineering*. New York: Wiley.
2. F. M. Dimentberg 1961 *Flexural Vibrations of Rotating Shaft*. London: Butterworth.

3. C. W. LEE 1993 *Vibration Analysis of Rotors*. Dordrecht: Kluwer Publishing Co.
4. R. KATZ, C. W. LEE, A. G. ULSOY and R. A. SCOTT 1988 *Journal of Sound and Vibration* **122**, 131–148. The dynamic response of a rotating shaft subject to a moving load.
5. A. Y. T. LEUNG and S. P. ZENG 1994 *Journal of Sound and Vibration* **177**, 555–564. Analytical formulation of dynamic stiffness.
6. K. HENCHI, M. FAFARD, G. DHATT and M. TALBOT 1997 *Journal of Sound and Vibration* **199**, 33–50. Dynamic behaviour of multi-span beams under moving loads.
7. B. YANG and C. A. TAN 1991 *ASME Journal of Applied Mechanics* **59**, 1009–1014. Transfer functions of one-dimensional distributed parameter systems.
8. B. YANG and H. FANG 1994 *ASME Journal of Vibration and Acoustics* **116**, 426–432. Transfer function formulation of non-uniformly distributed parameter systems.
9. R.-T. WANG and J.-S. LIN 1998 *Journal of Sound and Vibration* **212**, 417–434. Vibration of multi-span Timoshenko frames due to a moving loads.
10. R.-T. WANG and T.-Y. LIN 1998 *Journal of Sound and Vibration* **213**, 127–138. Random vibration of multi-span Timoshenko beam due to a moving load.
11. S. W. HONG and J. H. PARK 1999 *Journal of Sound and Vibration* submitted. Dynamic analysis of multi-stepped, distributed parameter rotor bearing systems.
12. P. LANCASTER 1996 *Lambda-Matrices and Vibrating Systems*. Oxford: Pergamon Press.
13. W. D. PILKEY 1994 *Formulas for Stress, Strain, and Structural Matrices*. New York: Wiley & Sons, Inc.
14. L. MEIROVITCH 1997 *Principles and Techniques of Vibrations*. Englewood Cliffs, NJ: Prentice-Hall International, Inc.