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BROADBAND DISTURBANCE ATTENUATION OVER AN ENTIRE BEAM

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This paper considers a problem of disturbance attenuation for a pinned–pinned flexible beam. The beam is modelled as a distributed parameter system and a procedure is developed to solve the corresponding disturbance attenuation problem for such a distributed parameter system. A controller is designed in a way that minimizes the effect of disturbances over the entire beam. The controller design approach of this paper is applicable to a large number of active noise and vibration control problems.

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1. INTRODUCTION

The acoustic noise spectrum covers a range of 50 Hz to 20 kHz. In the majority of noise control problems disturbances above 1000 Hz can generally be reduced by means of passive techniques. However, successful reduction of disturbances in the 50–1000 Hz range demands more sophisticated methods. This has motivated much research in the field of active control of noise and vibrations in recent years [1].

The problem of active control of a vibrating finite beam is of importance since it captures many aspects of the practical problems in flexible structure vibrations and control. As an example, in reference [2] an aircraft wing is modelled as a cantilever beam to study the flutter modes excited by a constant velocity airflow.

The control problem considered in this paper is shown in Figure 1. The beam is hinged at both ends, x_1 is the location of a disturbance input, x_2 is a point at which the disturbance is to be minimized and x_3 is the location of a control input.

In active noise and vibration control problems, the nature of the disturbance is of particular importance. In a number of industrial applications, a tonal disturbance may consist of a number of tones (or even a single tone). For such tonal disturbances, a number of algorithms have already been developed, some of which

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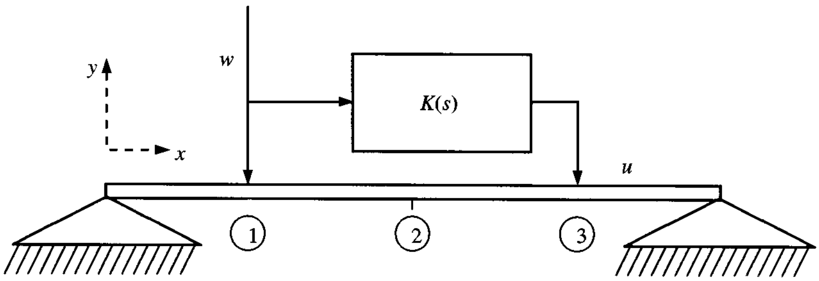

Figure 1. Typical implementation of feed-forward control.

rely on feedforward control techniques; e.g., see references [3, 1] and references therein. These feedforward controllers invert the system dynamics. In acoustic systems, as in the distributed parameter systems, there is an inherent time delay between the disturbance and the output. This implies that a feedforward controller for distributed parameter systems has to anticipate the disturbance and apply a corrective control action. For an arbitrary disturbance, this results in a non-causal controller but for a tonal (periodic) disturbance the prediction of the disturbance ahead of time is easy and hence tonal disturbances pose little theoretical problems in the design of feedforward controllers. However, when the disturbance consists of broadband noise, causality becomes a major issue and the majority of these techniques fail to perform satisfactorily [4].

Let the flexible structure transfer function between the force at location a and displacement at location b be given by

$$G_{ab}(s) = \frac{n_{ab}(s)}{d(s)}, \tag{1}$$

where $d(s)$ is the common denominator of all the transfer functions and $n_{ab}(s)$ are the numerators whose nature depends on the particular input and output locations. The output at location 2 due to both the controlled input at 3 and the disturbance input at 1 is:

$$\hat{y}_2(s) = \frac{n_{12}(s)}{d(s)} W(s) + \frac{n_{32}(s)}{d(s)} K(s) W(s), \tag{2}$$

where $K(s)$ is the controller transfer function and $W(s)$ is the disturbance point force, as shown in Figure 1. Control design consists in selecting a suitable $K(s)$ for an acceptable response $\hat{y}_2(s)$. One choice is to choose the controller such that

$$\min_{\forall \text{ stable } K(s)} \quad \hat{y}_2(s) \times \hat{y}_2(-s).$$

For some systems it is possible to force the response $y_2(t) \equiv 0, \forall t$ [3] by choosing the controller $K(s) = -n_{12}(s)/n_{32}(s)$. But when $n_{32}(s)$ has roots in the rhp, this controller can only work for tonal disturbances.

In reference [5], a feedforward controller design methodology is developed which extends the above approach [3] to the broadband disturbances. The control

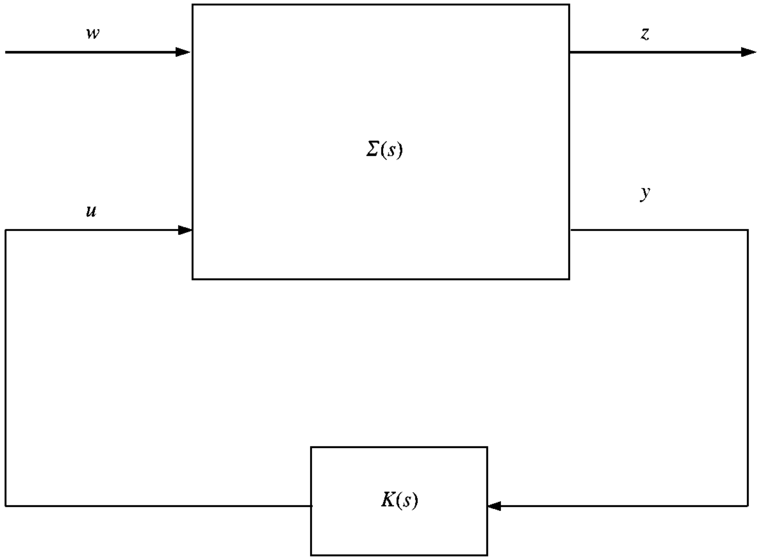

Figure 2. The general set up for $\mathscr{H}_\infty$ control problems.

algorithm allows for arbitrary placement of the poles of the transfer functions between the disturbance and the response at any point along the beam. Let the controller be given by $K(s) = a(s)/b(s)$, and the control algorithm selects these two polynomials $a(s)$ and $b(s)$ such that

$$\hat{y}_2(s) = \frac{n_{12}(s)b(s) + n_{32}(s)a(s)}{d(s)b(s)} W(s) = \frac{\tilde{n}(s)d(s)}{d(s)b(s)} W(s).$$

It can be seen that the controlled system characteristic polynomial is $b(s)$. In a way this controller design procedure ‘cancels’ the uncontrolled system poles using feedforward control.

Assuming that the disturbance is a wide-band noise with finite signal energy, the feed-forward control problem depicted in Figure 1 can be cast into a control framework and hence the $\mathscr{H}_\infty$ optimization methods can be used to design a controller. Indeed, the main objective is to minimize the $\mathscr{H}_\infty$ norm of the transfer function from the disturbance w to the displacements at x_2. Aside from allowing for wide-band disturbances, a major advantage of the $\mathscr{H}_\infty$ control is that it allows for controllers to be designed in a way which is robust to model uncertainties.

To define the so-called “standard $\mathscr{H}_\infty$ control problem”, consider the system of Figure 2. Suppose that the plant has two inputs and two outputs and that the transfer function of the plant is $\Sigma(s)$. The system can be described by

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}.$$

Here, u is the *control input*, w is the *disturbance input*, y is the *measured output* and z is the to-be-controlled *noise output*. The plant can be represented in a state-space form as

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t),$$
$$z(t) = C_1 x(t) + D_{12} u(t),$$
$$y(t) = C_2 x(t) + D_{21} w(t). \tag{1.3}$$

The controller K takes as an input y and produces the output u. The problem is to find a controller K which stabilizes the plant Σ such that the influence of the disturbance w on the to-be-controlled output z is minimized in an $\mathscr{H}_\infty$-norm sense. That is, if we denote the closed loop transfer function from w to z as $\Sigma_K(s)$, then $\Sigma_K(s)$ must be stable and the following condition holds:

$$\| \Sigma_K(s) \|_\infty = \sup_{w \in \mathbf{R}} \sigma_{\max} [\Sigma_K(\mathrm{j}\omega)] < \gamma,$$

where $\sigma_{\max}[\Psi]$ denotes the maximum singular value of the matrix Ψ. The parameter γ is called the disturbance attenuation factor.

The $\mathscr{H}_\infty$ controller design approach is indeed a worst-case design approach. For, if we assume that U is the set of all controllers K which stabilize the system Σ, then it can be shown that the real objective is to find a controller in U which achieves the following condition:

$$\inf_{K \in U} \sup_{w(\cdot) \in \mathscr{L}_2(0,\, \infty)} \frac{\| z(t) \|_2^2}{\| w(t) \|_2^2} < \gamma^2,$$

where $\mathscr{L}_2[0, \infty]$ denotes the Hilbert space of square integrable vector valued functions defined on $[0, \infty]$ and $\|z(t)\|_2^2 = \int_0^\infty z(t)' z(t)\, \mathrm{d}t$.

The $\mathscr{H}_\infty$ optimal control problem has been subject of extensive research ever since Zames published his famous paper in 1981 [6]. In this paper, Zames motivated the $\mathscr{H}_\infty$ control problem by considering the sensitivity of a feedback control system. Following his work many authors attempted to solve the problem using frequency-domain techniques (see reference [7]).

The frequency domain approach does give a solution to the $\mathscr{H}_\infty$ control problem. However, it was later found that a more straightforward solution could be found in a state-space setting. Indeed, [8] showed that the problem of $\mathscr{H}_\infty$ control via state feedback can be solved by solving a Riccati equation. Later, Doyle *et al.* [9] showed that the output feedback $\mathscr{H}_\infty$ optimal control problem can be solved via solving two algebraic Riccati equations. The connections with dynamic games have also been well understood [10].

The large number of research papers in this field following the work of Zames is due to the fact that many interesting problems can be formulated in an $\mathscr{H}_\infty$-design framework (see for example [11]). A complete treatment of the $\mathscr{H}_\infty$ optimal control problem may be found in references [12, 10].

An important class of systems for which $\mathscr{H}_\infty$ control problems can be considered are the so-called distributed parameter systems. These systems are important since many real-world systems can be modelled as distributed parameter systems. The

$\mathscr{H}_\infty$ control theory for such systems is currently under development. For example, reference [13] addresses such problem for a class of distributed parameter systems, the so-called Pritchard–Salamon systems.

However, although many systems can be modelled as distributed parameter systems, in most practical control problems, it is useful to approximate the system by another finite dimensional system. This is specially important if $\mathscr{H}_\infty$ control is to be used since it typically leads to a controller which has the same order as the state dimension of the model.

In this paper, we consider an approximate model of the pinned–pinned beam shown in Figure 1; see also reference [14]. The model is assumed to have a finite number of states. However, since the objective is to minimize the effect of the disturbance over the entire beam, we allow the system to have an output of infinite dimension. To be more precise, in this system, the disturbance and control inputs as well as the state vector and the measurements are finite-dimensional. However, the error output is an infinite-dimensional quantity. We solve this disturbance attenuation problem and apply our results to the pinned-pinned beam of Figure 1.

The rest of the paper is as follows. In Section 2, we present a dynamic model of the beam. The model consists of an infinite number of terms. However, for our purposes, we only consider the first six modes. Section 3 contains a standard $\mathscr{H}_\infty$ controller design for the beam. It is shown that a better disturbance attenuation may result by increasing the number of output points along the beam. In Section 4, we consider a problem of $\mathscr{H}_\infty$ control for a class of distributed parameter systems where only the output noise vector is allowed to be of infinite dimension. We give a solution to this problem in terms of two algebraic Riccati equations of the game type. We also show that this problem is equivalent to the $\mathscr{H}_\infty$ control problem for a finite-dimensional plant with a finite number of error outputs. In section 5, we apply our results to the beam problem assuming our infinite-dimensional model for the beam.

2. DYNAMIC MODEL OF THE BEAM

Consider a flexible beam as shown in Figure 1. Here, $y(x, t)$ denotes the elastic deformation of the beam as measured from the rest position. The elastic deflection $y(x, t)$ is governed by the following partial differential equation:

$$\frac{\partial^2}{\partial x^2}\left[EI\,\frac{\partial^2 y(x,t)}{\partial x^2}\right] + \rho A\,\frac{\partial^2 y(x,t)}{\partial t^2} = w(x,t), \tag{2.1}$$

where E, I, A, $w(x, t)$ and ρ represent respectively the Young's modulus, moment of inertia, cross-section area, external force per unit length, and the linear mass density of the beam. The differential equation (2.1) is the classical Bernoulli–Euler beam equation. Pinned–Pinned beam boundary conditions are

$$y(0,t) = 0, \quad EI\,\frac{\partial^2 y(0,t)}{\partial x^2} = 0, \quad y(l,t) = 0, \quad \text{and} \quad EI\,\frac{\partial^2 y(l,t)}{\partial x^2} = 0. \tag{2.2}$$

A dynamic model of the beam may be derived using the assumed modes modelling approach of Meirovitch [14]. First, the function $y(x, t)$ is expanded as an infinite series in the form [15]

$$y(x, t) = \sum_{i=1}^{\infty} q_i(t)\phi_i(x), \tag{2.3}$$

where $\phi_i(x)$ are the eigenfunctions satisfying the ordinary differential equations, resulting from the substitution of equation (2.3) into equations (2.1) and (2.2). The eigenfunctions also are chosen to be orthogonal according to the condition $\int_0^l \phi_i(x)\,\phi_j(x)\rho A\,\mathrm{d}x = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Now if the system input is a point force applied at position x_a and the output is the displacement measured at position x_b, then the transfer function [16] between applied force w and the ith modal response $q_i(t)$ is given by

$$\frac{\hat{q}_i(s)}{W(s)} = \frac{\phi_i(x_a)}{(s^2 + \omega_i^2)}. \tag{2.4}$$

Then after combination of equation (2.4) with equation (2.3) yields displacement at position b given by

$$\frac{\hat{y}(x_b, s)}{W(s)} = \sum_{i=1}^{\infty} \frac{\phi_i(x_a)\phi_i(x_b)}{(s^2 + \omega_i^2)}. \tag{2.5}$$

For the pinned–pinned beam system in Figure 1 the mode functions are given by [14]

$$\phi_n(x) = \sqrt{\frac{2}{\rho A l}}\sin\left(\frac{n\pi x}{l}\right), \tag{2.6}$$

and the corresponding natural frequencies are $\omega_n = (n\pi/l)^2\sqrt{EI/\rho A}$.

In this paper, we consider the beam described in reference [5]. The parameters are as follows: l = beam length = 0·38 m; $x_1 = 0{\cdot}038$ m; $x_2 = 0{\cdot}171$ m; $x_3 = 0{\cdot}247$ m; $\rho A = 0{\cdot}6265$ kg/m; $EI = 5{\cdot}329$ Nm^2.

We also assume a damping ratio of $\xi = 0{\cdot}01$. Figure 3 shows the three dimensional plot of the disturbance response of the uncontrolled beam. Notice that in Figure 3, λ represents the distance, i.e., $\lambda = x$. This notation is adopted so that in the following sections, the state vector is not mixed with the distance.

3. THE STANDARD $\mathscr{H}_\infty$ CONTROLLER DESIGN

In this section, we design an $\mathscr{H}_\infty$ controller for the pinned–pinned beam of Figure 1. The block diagram corresponding to this $\mathscr{H}_\infty$ control problem is shown in Figure 4. In this figure, w and u correspond to w and u in Figure 1, while z is the displacement in position 2 along the beam. The controller $K(s)$ is to be designed in a way to minimize the $\mathscr{H}_\infty$ norm of the transfer function from the disturbance w to the displacement at position 2, z.

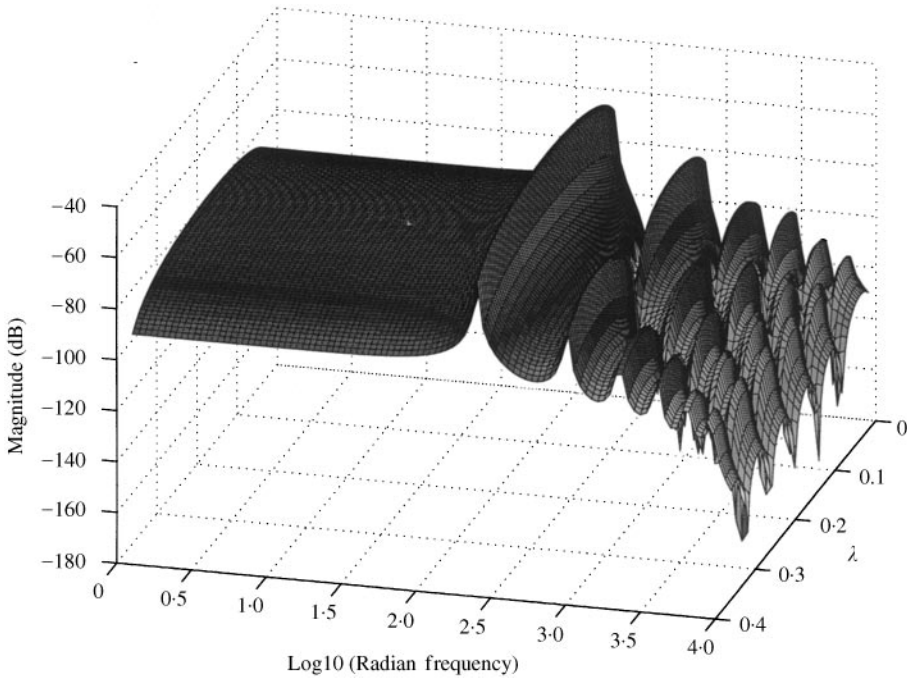

Figure 3. Three-dimensional disturbance response of the uncontrolled beam.

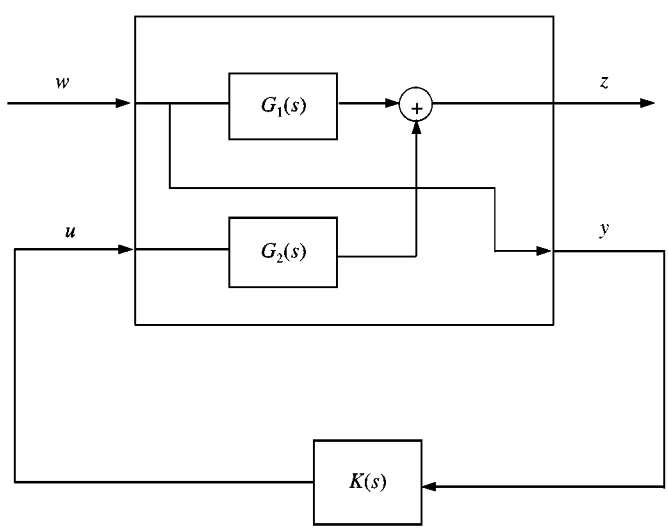

Figure 4. A block diagram representation of the beam and controller.

A state-space realization of this system is as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= Cx(t), \\ y(t) &= w(t) \end{aligned} \tag{3.1}$$

where

$$A = \begin{bmatrix} 0 & 1 & & 0 & 0 \\ -\omega_1^2 & -2\xi\omega_1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & & -\omega_n^2 & -2\xi\omega_n \end{bmatrix},$$

$$B_1 = [0 \;\; \phi_1(x_1)\phi_1(x_2) \;\ldots\; 0 \;\; \phi_n(x_1)\phi_n(x_2)]'$$

$$B_2 = [0 \;\; \phi_1(x_3)\phi_1(x_2) \;\ldots\; 0 \;\; \phi_n(x_3)\phi_n(x_2)]'$$

$$C = [1 \;\; 0 \;\ldots\; 1 \;\; 0]$$

$$x = [q_1 \;\; \dot{q}_1 \;\ldots\; q_n \;\; \dot{q}_n]'.$$

In the above model, only the first n modes of the beam are taken into account.

A major problem in designing a $\mathscr{H}_\infty$ controller for system (3.1) is that such a design will result in a controller with an infinitely large gain. Such a problem is referred to as a singular $\mathscr{H}_\infty$ *control problem* in the literature and is due to the fact that there is no direct feed through term from the control signal to the noise output (i.e., $D_{12} = 0$). Therefore optimization of the cost function $\|z\|_2/\|w\|_2$ may result in a controller with an infinitely large gain since there is no weighting on u in the cost function. To overcome this difficulty, we add a fictitious noise output to the system such that the new system is defined by

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t),$$

$$z(t) = \begin{bmatrix} C \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} u(t),$$

$$y(t) = w(t).$$

Here, ε is a parameter chosen by the designer. In this new system, the matrix D_{12} is no longer zero, and hence a practical $\mathscr{H}_\infty$ controller may be designed for this system. Moreover, the smaller the ε, the better the new system approximates the original system. However, care must be taken in choosing ε. For, if a very small ε is chosen, a very large gain controller will be obtained which may cause implementation difficulties. This is due to the nature of the singular $\mathscr{H}_\infty$ control problem. Indeed, if $\varepsilon \to 0$, the controller gain will approach infinity. Here, we choose $\varepsilon = 10^{-5}$.

Our $\mathscr{H}_\infty$ design results in a controller with disturbance attenuation level of 8×10^{-5}. Figure 5 shows the disturbance response of the controlled beam as a function of the beam length, Figure 6 shows the $\mathscr{H}_\infty$ norm of the closed-loop system as a function of the beam length, and finally, Figure 7 shows the Bode plot of the controller.

So far, we have designed a controller which minimizes the $\mathscr{H}_\infty$ norm of the transfer function from the disturbance input to a particular point along the beam.

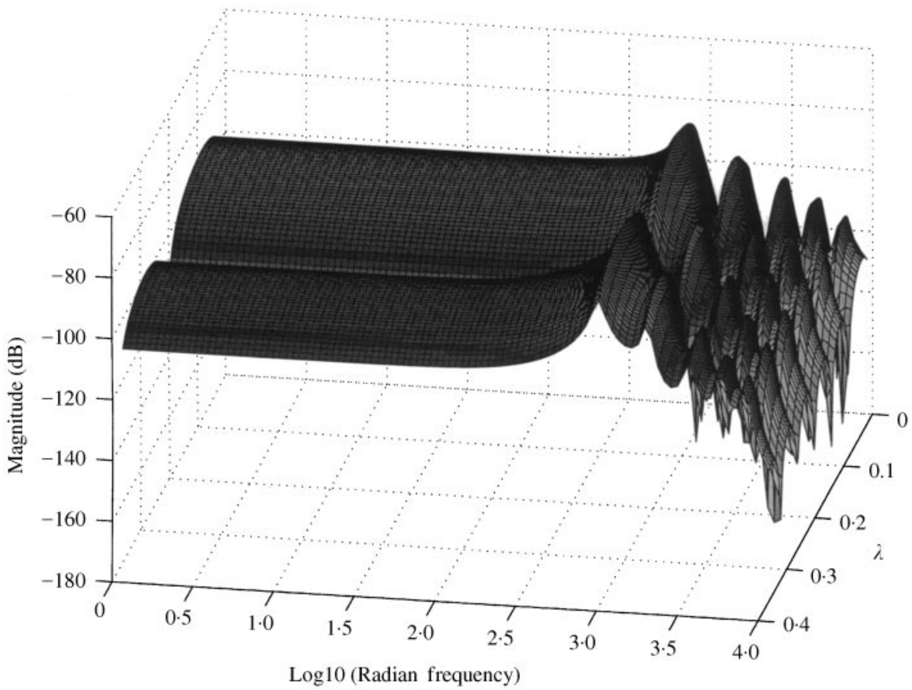

Figure 5. The three-dimensional frequency response of the controlled beam.

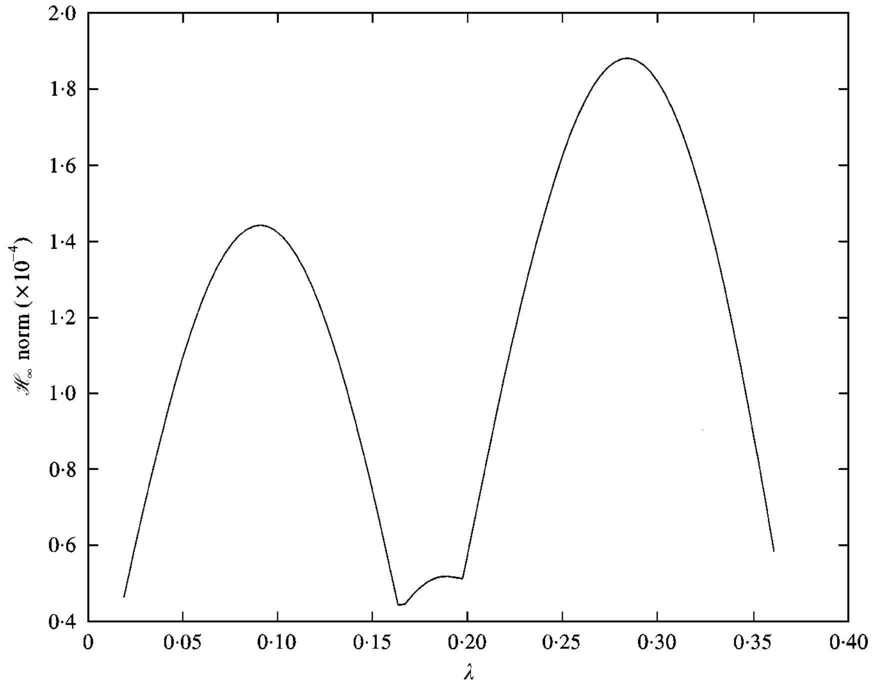

Figure 6. The $\mathscr{H}_\infty$ norm of the disturbance to various locations along the beam.

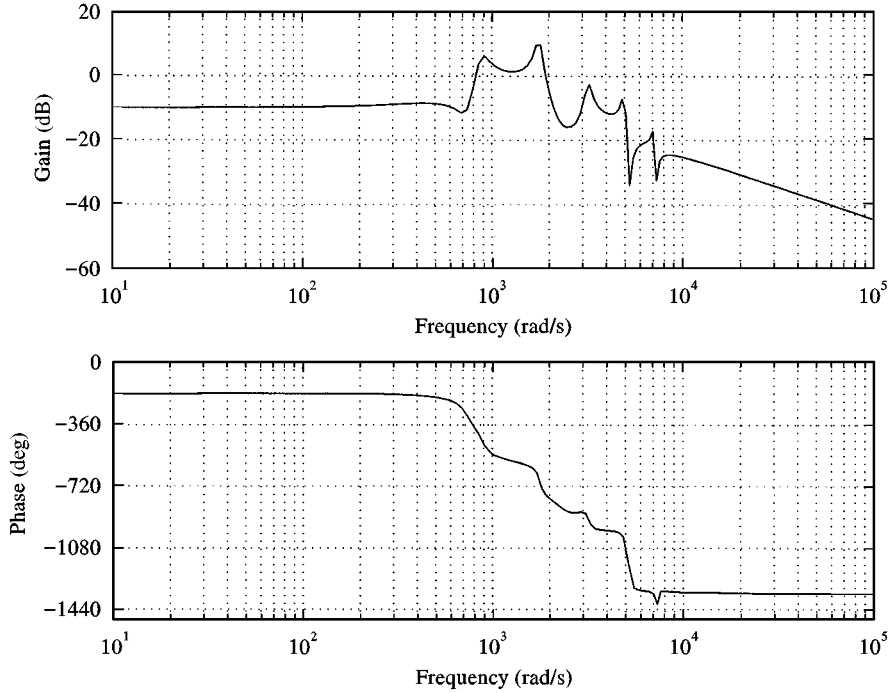

Figure 7. Bode plot of the $\mathscr{H}_\infty$ controller.

This controller does not guarantee noise reduction at other locations along the beam. Indeed, as Figure 6 shows, the $\mathscr{H}_\infty$ norm increases considerably as we move from x_2 in any direction. One approach to attenuating the disturbance more evenly along the beam is to increase the number of error outputs. Therefore, we will have to design a controller for a single input, multiple output system. To illustrate this approach, we assume that

$$x_2 = [0{\cdot}095 \quad 0{\cdot}171 \quad 0{\cdot}285]'.$$

The system can be described in state-space form as

$$\dot{x}(t) = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} x(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t),$$

$$z(t) = [\, C_1 \;\; C_2 \,]\, x(t),$$

$$y(t) = w(t),$$

where A is as above and

$$C_1 = \begin{bmatrix} \phi_1(x_1)\phi_1(x_2(1)) & 0 & \dots & \phi_n(x_1)\phi_n(x_2(1)) & 0 \\ \phi_1(x_1)\,\phi_1(x_2(2)) & 0 & \dots & \phi_n(x_1)\phi_n(x_2(2)) & 0 \\ \phi_1(x_1)\phi_1(x_2(3)) & 0 & \dots & \phi_n(x_1)\,\phi_n(x_2(3)) & 0 \end{bmatrix},$$

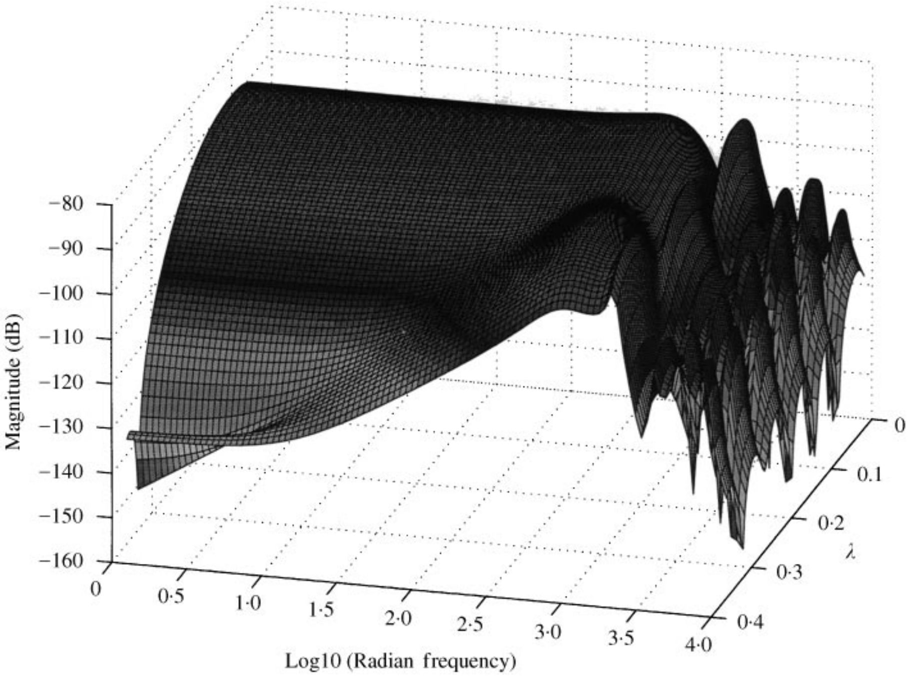

Figure 8. The three-dimensional disturbance response of the beam.

$$
\mathrm{C}_2 = \begin{bmatrix} \phi_1(x_3)\phi_1(x_2(1)) & 0 & \dots & \phi_n(x_3)\phi_n(x_2(1)) & 0 \\ \phi_1(x_3)\phi_1(x_2(2)) & 0 & \dots & \phi_n(x_3)\phi_n(x_2(2)) & 0 \\ \phi_1(x_3)\phi_1(x_2(3)) & 0 & \dots & \phi_n(x_3)\phi_n(x_2(3)) & 0 \end{bmatrix},
$$

$$
B = [0 \quad 1 \ \dots \ 0 \quad 1]'.
$$

Note that the state of this new system is twice as large as that of equation (3.1). The reason is that here, the C matrix has more than one row, which results in a minimal system. However, such a state-space model for system (3.1) would result in a non-minimal system which would be reducible to equation (3.1).

As before, we face a singular $\mathscr{H}_\infty$ control problem. To solve this problem, we add an extra noise output as follows:

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} x(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t), \\
z(t) &= \begin{bmatrix} C_1 & C_2 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} u(t), \\
y(t) &= w(t).
\end{aligned} \tag{3.2}
$$

We assume that $\varepsilon = 10^{-5}$ and design a standard $\mathscr{H}_\infty$ controller with $\gamma = 8 \times 10^{-5}$. Figure 8 shows the three-dimensional disturbance response of the beam as a function of the beam position and Figure 9 shows the corresponding $\mathscr{H}_\infty$ norm as

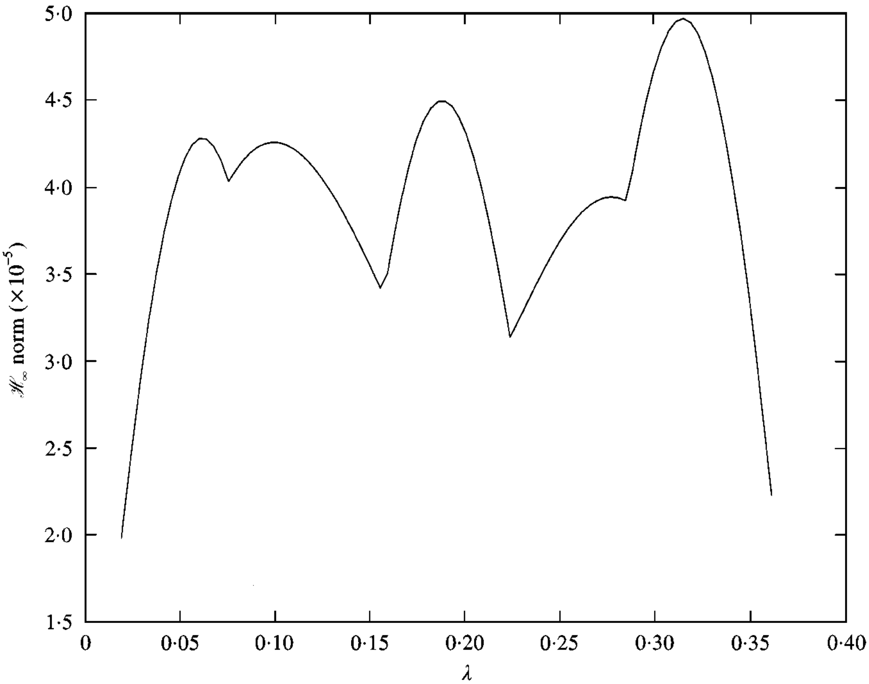

Figure 9. The $\mathscr{H}_\infty$ norm of the disturbance to various locations along the beam.

a function of the beam position. It can be observed that the disturbance attenuation is distributed more evenly along the beam this time. Figure 10 shows the Bode plot of the designed $\mathscr{H}_\infty$ controller.

A drawback of the above $\mathscr{H}_\infty$ design is that there is no systematic method of choosing the error output points along the beam and no indication as to how many points will be needed. Of special interest is the problem of guaranteeing a level of disturbance attenuation over the entire beam. However, this requires the solution to an infinite-dimensional $\mathscr{H}_\infty$ control problem. In the next section, we set up this problem and give a corresponding solution to it.

4. $\mathscr{H}_\infty$ CONTROL OF A CLASS OF DISTRIBUTED PARAMETER SYSTEMS

Consider a system described by the following dynamics:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t, \lambda) &= C_1(\lambda) x(t) + D_{12}(\lambda) u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t), \end{aligned} \tag{4.1}$$

where $\lambda \in \Lambda$. In this system, $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^p$ is the disturbance input, $u \in \mathbf{R}^m$ is the control input, $z \in \mathbf{R}^q$ is the error output and $y \in \mathbf{R}^l$ is the measured output. The state, disturbance inputs, measurements and control inputs are finite-dimensional. However, the error output is allowed to be infinite-dimensional

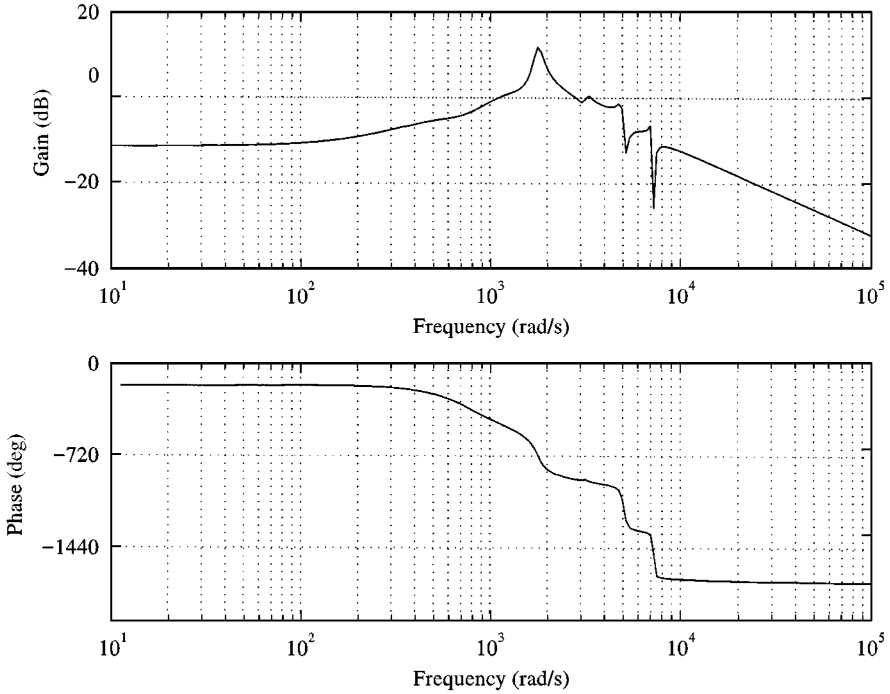

Figure 10. Bode plot of the $\mathscr{H}_\infty$ controller.

since it is a function of parameter λ. We also assume that $\Lambda = [0, l]$. The $\mathscr{H}_\infty$ control problem for this system is defined as follows.

The $\mathscr{H}_\infty$ control problem for distributed parameter systems. Design a controller

$$\dot{x}(t) = A_k x(t) + B_k y(t),$$

$$u(t) = C_k x(t) + D_k y(t),$$

such that the closed-loop system satisfies

$$\inf_{K(\cdot)\in U} \sup_{w(\cdot)\in \mathscr{L}_2[0,\infty)} \frac{\int_0^\infty \int_\Lambda z(t,\lambda)' z(t,\lambda)\, \mathrm{d}\lambda \mathrm{d}t}{\int_0^\infty w(t)' w(t)\, \mathrm{d}t} < \gamma^2, \tag{4.2}$$

where U is the set of all stabilizing controllers.

Condition (4.2) is an extension of the condition arising in the standard $\mathscr{H}_\infty$ control problems to the case of distributed parameter systems. To understand the motivations behind this definition, assume that in the beam problem considered in the previous section, we increase the number of error points to be controlled along the beam. As this number approaches infinity, the $\mathscr{H}_\infty$ norm objective approaches the objective function (4.2). Condition (4.2) guarantees that a level of disturbance attenuation less than γ will be achieved over the entire set of Λ in an average sense.

However, it does not guarantee that the $\mathscr{H}_\infty$ norm of the transfer function from the disturbance to each particular point inside the set Λ will be lower than γ.

The solution to this problem will be given in terms of the following algebraic Riccati equations:

$$(A - B_2R^{-1}P')'X + X(A - B_2R^{-1}P') - X(B_2R^{-1}B_2' - \gamma^{-2}B_1B_1')X$$
$$+ Q - PR^{-1}P' = 0, \tag{4.3}$$
$$(A - LN^{-1}C_2)\, Y + Y(A - LN^{-1}C_2)' - Y(C_2{}'N^{-1}C_2 - \gamma^{-2}Q)\,Y$$
$$+ B_1B_1' - LN^{-1}L' = 0, \tag{4.4}$$

where

$$R = \int_0^l D_{12}(\lambda)' D_{12}(\lambda)\, d\lambda,$$
$$Q = \int_0^l C_1(\lambda)' C_1(\lambda)\, \mathrm{d}\lambda,$$
$$P = \int_0^l C_1(\lambda)' D_{12}(\lambda)\, \mathrm{d}\lambda,$$
$$L = B_1 D_{21}'$$
$$N = D_{21}D_{21}'.$$

In the sequel we will need the following assumption.

Assumption. The following matrix is positive semi-definite:

$$\begin{bmatrix} Q & P \\ P' & R \end{bmatrix} \geqslant 0.$$

Theorem 4.1. *Consider the distributed parameter system* (4.1) *and the related $\mathscr{H}_\infty$ control problem and assume that* $(A - B_2R^{-1}P', (Q - PR^{-1}P')^{1/2})$ *is detectable and* $(A - LN^{-1}C_2, (B_1B_1{}' - LN^{-1}L)^{1}/^{2})$ *is stabilizable. This problem has a solution if and only if the Riccati equations* (4.3) *and* (4.4) *admit* (*minimal*) *non-negative-definite solutions X and Y such that*

$$\rho(YX) < \gamma^2. \tag{4.5}$$

In this case, a suitable controller is defined by

$$A_k = A - B_2R^{-1}(B_2'X + P') + \gamma^{-2}B_1B_1'X$$
$$- (I - \gamma^{-2}YX)^{-1}(YC_2' + L)N^{-1}(C_2 + \gamma^{-2}D_{21}B_1{}'X), \tag{4.6}$$
$$B_k = (I - \gamma^{-2}YX)^{-1}(YC_2' + L)N^{-1}, \tag{4.7}$$
$$C_k = -R^{-1}(B_2'X + P'), \tag{4.8}$$
$$D_k = 0. \tag{4.9}$$

Proof. Consider system (4.1) and the corresponding $\mathscr{H}_\infty$ objective (4.2). Condition (4.2) can be written as

$$\inf_{K(\cdot)\in U} \sup_{w(\cdot)\in \mathscr{L}_2[0,\infty)} \frac{\int_0^\infty \int_0^l z(t,\lambda)' z(t,\lambda) \mathrm{d}\lambda \mathrm{d}t}{\int_0^\infty w(t)'w(t)\,\mathrm{d}t} < \gamma.$$

This disturbance attenuation problem is equivalent to the differential game defined by the system

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t),$$

$$y(t) = C_2 x(t) + D_{21} w(t),$$

and the cost function (see reference [10])

$$J_\gamma = \int_0^\infty \left\{ \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \left(\int_0^l \begin{bmatrix} C_1(\lambda)' C_1(\lambda) & C_1(\lambda)' D_{12}(\lambda) \\ D_{12}(\lambda)' C_1(\lambda) & D_{12}(\lambda)' D_{12}(\lambda) \end{bmatrix} \mathrm{d}\lambda \right) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} - \gamma^2 \|w(t)\|^2 \right\} \mathrm{d}t. \tag{4.10}$$

The necessary and sufficient condition for existence of a solution to this game problem is existence of (minimal) solutions to Riccati equations (4.3) and (4.4) that satisfy condition (4.5). Moreover, a $\mathscr{H}_\infty$ controller is given by equations (4.6)–(4.8) (see section 5.5 and Theorem 5.6 of reference [10]). □

Remark. The above problem is equivalent to a finite-dimensional $\mathscr{H}_\infty$ control problem, where the underlying system is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t) &= \Sigma x(t) + \Pi u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t) \end{aligned} \tag{4.11}$$

Here, Σ and Π are determined from

$$[\Sigma \quad \Pi] = \begin{bmatrix} Q & P \\ P' & R \end{bmatrix}^{1/2}.$$

This can be verified by comparing the Riccati equations (4.3) and (4.4) with the Riccati equations arising in the $\mathscr{H}_\infty$ control for the finite-dimensional system (4.11) (see, e.g., Chapter 5 of reference [10]).

Now, let us give a frequency-domain interpretation of the induced norm defined above. Consider a system T defined by

$$\dot{x}(t) = Ax(t) + Bw(t),$$

$$z(t,\lambda) = C(\lambda)x(t),$$

and define

$$\ll \mathscr{T} \gg^2 \triangleq \sup_{w(\cdot)\in\mathscr{L}_2[0,\infty]} \frac{\int_0^\infty \int_\Lambda z(t,\lambda)' z(t,\lambda)\,\mathrm{d}\lambda \mathrm{d}t}{\int_0^\infty w(t)'w(t)\,\mathrm{d}t}.$$

The following theorem gives a frequency domain interpretation of this norm in terms of $\mathscr{H}_\infty$ norm of a finite-dimensional system.

Theorem 4.2. *Suppose a stable linear system has a transfer matrix $T(s,\lambda)$ for $\lambda \in [0,l]$. And let $\mathscr{T}$ denote the linear map it induces from the finite-dimensional $\mathscr{L}_2$ spaces of its inputs to its infinite-dimensional outputs. Its induced linear operator norm $\ll \mathscr{T} \gg$ satisfies*

$$\ll \mathscr{T} \gg = \|\tilde{T}\|_\infty$$

where $\tilde{T}(s)$ is a finite-dimensional system defined by

$$\tilde{T}(s) = M(sI - A)^{-1}B$$

and

$$M = \left(\int_0^l C(\lambda)'C(\lambda)\,\mathrm{d}\lambda\right)^{1/2}.$$

Proof. The linear relationship between the disturbance and the infinite-dimensional output can be written as $z = \mathscr{T}w$. The Fourier transform of w and z will be denoted by $W(\mathrm{j}\omega)$ and $Z(\mathrm{j}\omega,\lambda)$. By Parseval's identity we have

$$\begin{aligned}\int_0^\infty \int_0^l z(t,\lambda)'z(t,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}t &= \frac{1}{2\pi}\int_{-\infty}^\infty \int_0^l W(\mathrm{j}\omega)^* T(\mathrm{j}\omega,\lambda)^*\, T(\mathrm{j}\omega,\lambda)\, W(\mathrm{j}\omega)\,\mathrm{d}\lambda \mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty W(\mathrm{j}\omega)^*\left(\int_0^l S(\mathrm{j}\omega)^* C(\lambda)'C(\lambda)\, S(\mathrm{j}\omega)\mathrm{d}\lambda\right) W(\mathrm{j}\omega)\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty W(\mathrm{j}\omega)^* \tilde{T}(\mathrm{j}\omega)^* \tilde{T}(\mathrm{j}\omega)\, W(\mathrm{j}\omega)\,\mathrm{d}\omega \\ &\leqslant \frac{1}{2\pi}\sigma_{\max}(\tilde{T}(\mathrm{j}\omega))^2 \int_{-\infty}^\infty W(\mathrm{j}\omega)^* W(\mathrm{j}\omega)\,\mathrm{d}\omega\end{aligned}$$

where $S(\mathrm{j}\omega) = (\mathrm{j}\omega I - A)^{-1}B$.

Since $\|\tilde{T}\|_\infty = \sup_\omega \sigma_{\max}(\tilde{T}(\mathrm{j}\omega))$, it follows that

$$\int_0^\infty \int_0^l z(t,\lambda)'z(t,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}t \leqq \|\tilde{T}\|_\infty^2 \int_0^\infty w(t)'w(t)\,\mathrm{d}t.$$

Therefore,

$$\ll z \gg \leqslant \|\tilde{T}\|_\infty \|w\|_2.$$

Conversely, suppose $\gamma < \|\tilde{T}\|_\infty$. This means that for some ω_0, we have $\sigma_{\max}(\tilde{T}(\mathrm{j}\omega_0)) > \gamma$. Therefore, by continuity, there exists $\eta > 0$ such that

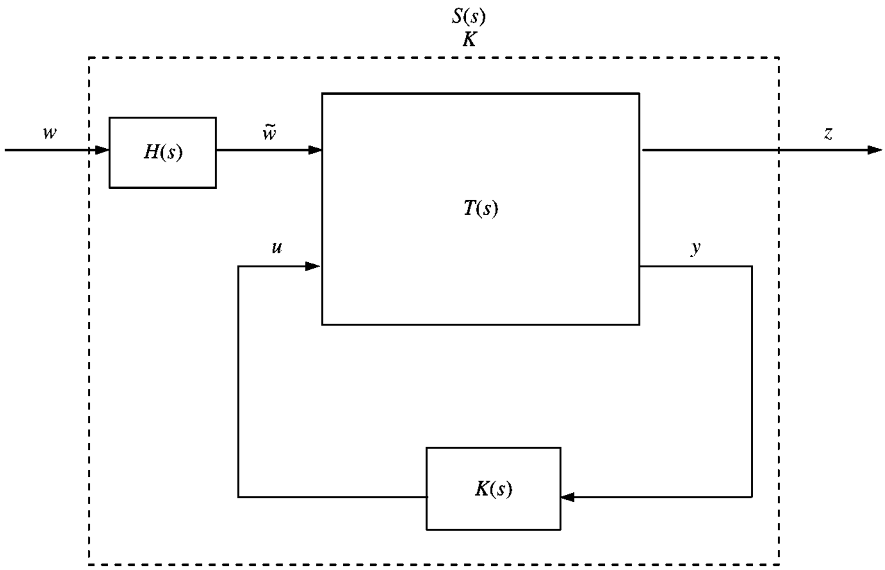

Figure 11. The closed-loop system with a pre-filter.

$\sigma_{\max}(\tilde{T}(\mathrm{j}\omega)) \geqslant \gamma$ for all ω in $[\omega_0 - \eta, \omega_0 + \eta]$ and $[-\omega_0 - \eta, -\omega_0 + \eta]$. Now, consider a $W(\mathrm{j}\omega)$ which is zero outside of these ranges of frequencies, and which coincides with an eigenvalue corresponding to the largest eigenvalue of $\tilde{T}(\mathrm{j}\omega)^*\tilde{T}(\mathrm{j}\omega)$ in these ranges. For the corresponding output z we then have

$$\begin{aligned}
\ll z \gg^2 &= \int_0^\infty \int_0^l z(t,\lambda)' z(t,\lambda)\,\mathrm{d}\lambda\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\omega_0-\eta}^{-\omega_0+\eta}\int_0^l W(\mathrm{j}\omega)^* T(\mathrm{j}\omega,\lambda)^* T(\mathrm{j}\omega,\lambda)\, W(\mathrm{j}\omega)\,\mathrm{d}\lambda\mathrm{d}\omega \\
&\quad + \frac{1}{2\pi}\int_{\omega_0-\eta}^{\omega_0+\eta}\int_0^l W(\mathrm{j}\omega)^* T(\mathrm{j}\omega,\lambda)^* T(\mathrm{j}\omega,\lambda)\, W(\mathrm{j}\omega)\,\mathrm{d}\lambda\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\omega_0-\eta}^{-\omega_0+\eta} W(\mathrm{j}\omega)^* \tilde{T}(\mathrm{j}\omega)^* \tilde{T}(\mathrm{j}\omega)\, W(\mathrm{j}\omega)\,\mathrm{d}\omega \\
&\quad + \frac{1}{2\pi}\int_{\omega_0-\eta}^{\omega_0+\eta} W(\mathrm{j}\omega)^* \tilde{T}(\mathrm{j}\omega)^* \tilde{T}(\mathrm{j}\omega)\, W(\mathrm{j}\omega)\,\mathrm{d}\omega \\
&\geqq \frac{1}{2\pi}\int_{-\omega_0-\eta}^{-\omega_0+\eta} \gamma^2 W(\mathrm{j}\omega)^* W(\mathrm{j}\omega)\,\mathrm{d}\omega + \frac{1}{2\pi}\int_{\omega_0-\eta}^{\omega_0+\eta} \gamma^2 W(\mathrm{j}\omega)^* W(\mathrm{j}\omega)\,\mathrm{d}\omega \\
&= \gamma^2 \|w\|_2^2.
\end{aligned}$$

Hence, $\ll \mathscr{T} \gg \geqslant \gamma$, which proves the theorem. □

Now, let us consider the system in Figure 11 and let $\mathscr{S}_{\mathrm{K}}$ denote the linear operator mapping w to z. That is, $\mathscr{S}_{\mathrm{K}}$ corresponds to a transfer function $S_K(s, \lambda)$.

Also, assume that T_{K} has a state-space realization

$$\dot{x}(t) = Ax(t) + B\tilde{w}(t),$$
$$z(t) = C(\lambda)\,x(t).$$

Also consider $H(s)$ to be a given weighting function with state-space realization

$$\dot{x}_H(t) = A_H x_H(t) + B_H w(t),$$
$$\tilde{w}(t) = C_H x_H(t)$$

Theorem 4.3. *Consider the system of Figure* 11. *The following statements are equivalent*:

(i) $\ll \mathscr{S}_{\mathrm{K}} \gg\; < \gamma$.

(ii) $\ll T_{\mathrm{K}}(\mathrm{j}\omega, \lambda) \gg\; < \gamma / \| H(\mathrm{j}\omega) \|\ \forall \omega \in \mathbf{R}$.

Here $\ll T_K(\mathrm{j}\omega, \lambda) \gg\; = (\int_0^l T_K(j\omega, \lambda)^* T_K(\mathrm{j}\omega, \lambda)\,\mathrm{d}\lambda)^1/^2$.

Proof. Consider Figure 11; we can write

$$S_K(s, \lambda) = T_K(s, \lambda)\,H(s).$$

Therefore, $S_K(s, \lambda)$ will have the following state-space realization

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}w(t),$$
$$z(t) = \hat{C}(\lambda)\,\hat{x}(t),$$

where

$$\hat{A} \triangleq \begin{bmatrix} A & BC_{\mathrm{H}} \\ 0 & A_{\mathrm{H}} \end{bmatrix}, \quad \hat{B} \triangleq \begin{bmatrix} 0 \\ B_{\mathrm{H}} \end{bmatrix}, \quad \hat{C}(\lambda) \triangleq [C(\lambda) \quad 0], \quad \hat{x} \triangleq \begin{bmatrix} x \\ x_H \end{bmatrix}.$$

Corresponding to this system is the finite-dimensional system $\tilde{S}_K(s)$ which has the following state-space realization:

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}w(t),$$
$$\tilde{z}(t) = M\hat{x}(t),$$

where

$$M = \left(\int_0^l C(\lambda)' C(\lambda)\,\mathrm{d}\lambda \right)^{1/2}.$$

It is easy to show that

$$\tilde{S}_K(s) = \tilde{T}(s) \times H(s),$$

where $\tilde{T}(s)$ has the following state-space realization:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bw(t),$$
$$\tilde{z}(t) = M\tilde{x}(t).$$

Now, if we assume that $\ll \mathscr{S}_{\mathrm{K}} \gg\ < \gamma$ we have

$$\begin{gathered}
\ll \mathscr{S}_{\mathrm{K}} \gg\ < \gamma \\
\Updownarrow \\
\| \tilde{S}_K(\mathrm{j}\omega) \|_\infty < \gamma \\
\Updownarrow \\
\| \tilde{T}_K(\mathrm{j}\omega) H(\mathrm{j}\omega) \|_\infty < \gamma \\
\Updownarrow \\
\| \tilde{T}_K(\mathrm{j}\omega) \| < \frac{\gamma}{\| H(\mathrm{j}\omega) \|} \quad \forall \omega \in \mathbf{R} \\
\Updownarrow \\
\ll T_{\mathrm{K}}(j\omega, \lambda) \gg\ < \frac{\gamma}{\| H(\mathrm{j}\omega) \|} \quad \forall \omega \in \mathbf{R}.
\end{gathered}$$

which proves the statement of the theorem. $\square$

5. APPLICATIONS TO THE BEAM PROBLEM

The pinned–pinned beam system can be represented by the following state-space system:

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} x(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t), \\
z(t, \lambda) &= [C_1(\lambda) \quad C_2(\lambda)]\, x(t), \\
y(t) &= w(t).
\end{aligned} \tag{5.1}$$

However, as in the finite-dimensional case, a $\mathscr{H}_\infty$ controller design for this system results in a controller with an infinitely large gain since $D_{12} = 0$. To overcome this difficulty, we add an extra noise output as follows:

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} x(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t) \\
z(t, \lambda) &= \begin{bmatrix} C_1(\lambda) & C_2(\lambda) \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} u(t), \\
y(t) &= w(t).
\end{aligned} \tag{5.2}$$

Here, ε is a small design parameter chosen by the designer. Again, whilst choosing ε one has to be careful not to choose a very small value for ε since the smaller the value of ε the higher the gain of the controller will be.

In this state-space realization, $C_1(\lambda)$ and $C_2(\lambda)$ are defined by

$$C_1(\lambda) = [\phi_1(x_1)\phi_1(\lambda) \;\; 0 \;\; \dots \;\; \phi_n(x_1)\phi_n(\lambda) \;\; 0]$$
$$C_2(\lambda) = [\phi_1(x_3)\phi_1(\lambda) \;\; 0 \;\; \dots \;\; \phi_n(x_3)\phi_n(\lambda) \;\; 0].$$

Let us define

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 \\ B \end{bmatrix}.$$

Then, it can also be shown that $R = \varepsilon^2 l$, $P = 0$, $N = 1$, $L = \tilde{B}_1$ and

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix},$$

where

$$Q_{11} = \begin{bmatrix} \frac{\phi_1(x_1)^2}{\rho A} & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & \frac{\phi_n(x_1)^2}{\rho A} & 0 \\ 0 & 0 & & 0 & 0 \end{bmatrix},$$

$$Q_{12} = \begin{bmatrix} \frac{\phi_1(x_1)\ \phi_1(x_3)}{\rho A} & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & \frac{\phi_n(x_1)\phi_n(x_3)}{\rho A} & 0 \\ 0 & 0 & & 0 & 0 \end{bmatrix},$$

$$Q_{22} = \begin{bmatrix} \frac{\phi_1(x_3)^2}{\rho A} & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & \frac{\phi_n(x_3)^2}{\rho A} & 0 \\ 0 & 0 & & 0 & 0 \end{bmatrix}.$$

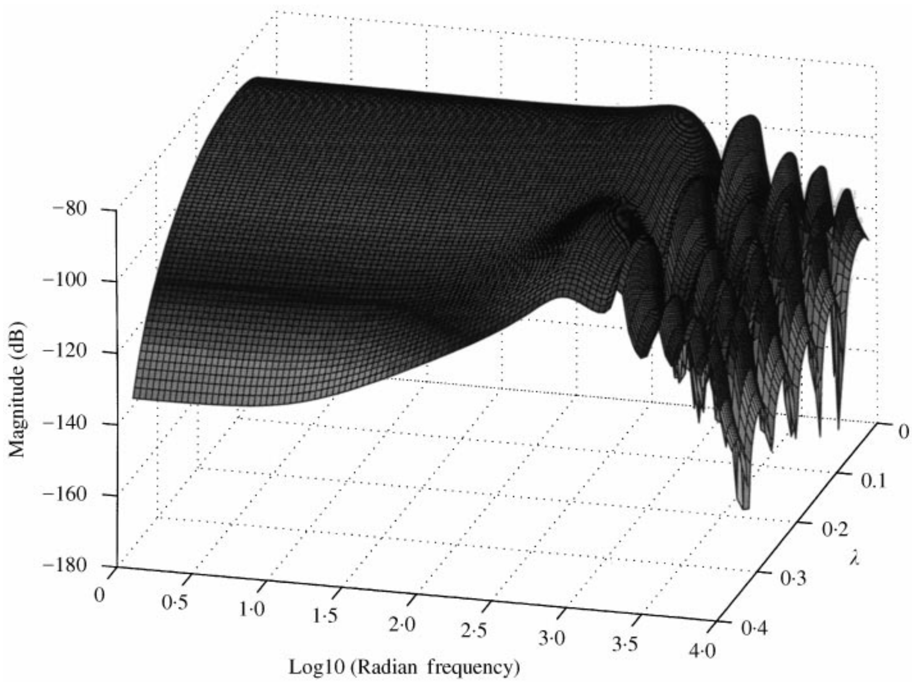

Figure 12. Disturbance responses of the controlled beam.

With these definitions it is possible to show that the Riccati equation (4.3) reduces to

$$\tilde{A}'X + X\tilde{A} - X(\tilde{B}_2R^{-1}\tilde{B}_2' - \gamma^{-2}\tilde{B}_1\tilde{B}_1')X + Q = 0, \tag{5.3}$$

and the Riccati equation (4.4) reduces to

$$\tilde{A}Y + Y\tilde{A}' - \gamma^{-2}YQY = 0.$$

Since $\tilde{A}$ is a stability matrix, the minimal solution to this Riccati equation is $Y = 0$. Hence, the controller can be written as

$$\dot{x}_k(t) = (\tilde{A} - \tilde{B}_2R^{-1}\tilde{B}_2'X)x_k(t) + \tilde{B}_1y(t),$$
$$u(t) = -R^{-1}\tilde{B}_2{}'Xx_k(t).$$

As explained above, we choose $\varepsilon = 10^{-5}$ and we design the $\mathscr{H}_\infty$ controller for $\gamma = 4{\cdot}0052 \times 10^{-5}$. Figure 12 shows the three-dimensional Bode magnitude plot for disturbance response of the beam at various locations along the beam. Also, Figure 13 shows the $\mathscr{H}_\infty$ norm of the closed-loop system with the $\mathscr{H}_\infty$ controller designed for the distributed parameter system. Figure 14 shows the Bode plot of the corresponding $\mathscr{H}_\infty$ controller.

In practical noise and vibration control problems, the high-frequency noise or vibrations can be reduced by passive methods such as damping. Therefore, the main emphasis in active control techniques is on reducing the low-frequency components of noise or vibrations.

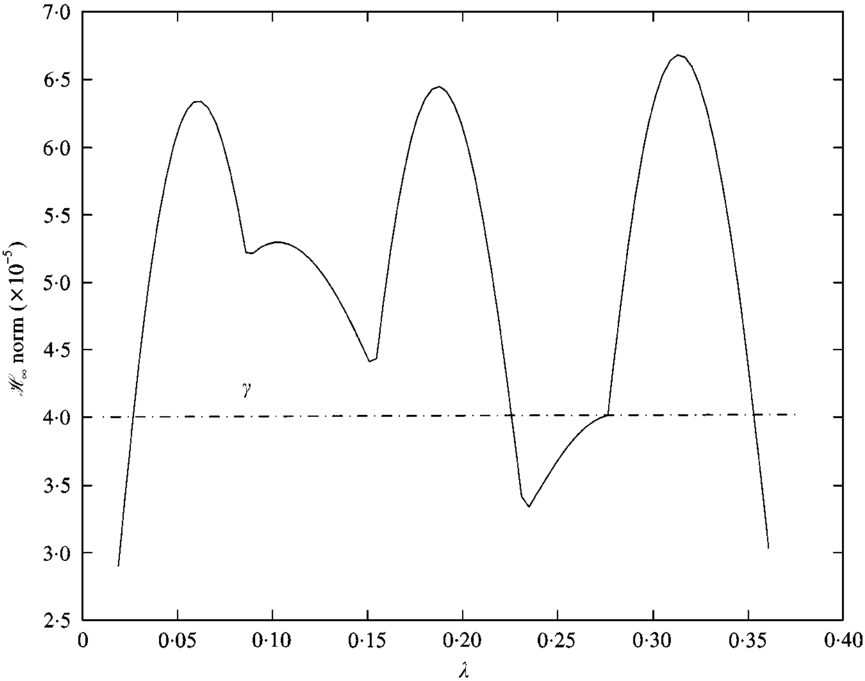

Figure 13. The $\mathscr{H}_\infty$ norm of the disturbance to various locations along the beam with the controller designed for the distributed parameter system.

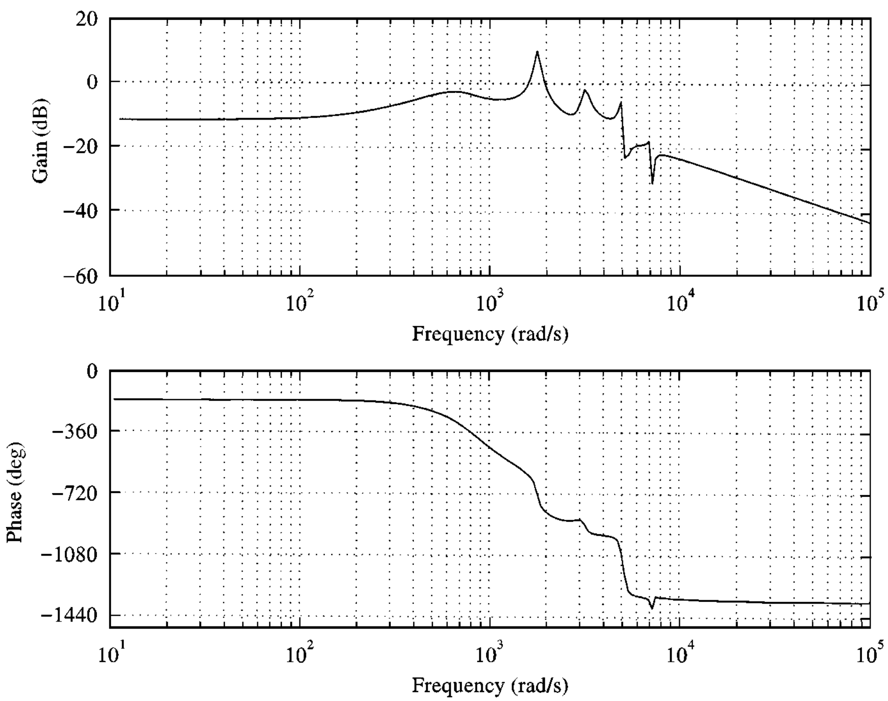

Figure 14. Bode plot of The $\mathscr{H}_\infty$ controller designed for the distributed parameter system.

The procedure discussed in the previous section allows us to design a controller with a new low-frequency closed-loop behaviour by means of including a weighting function in our design. Indeed, if in Figure 11 the weighting function $H(s)$ is assumed to be a low-pass filter with a suitable cutoff frequency, then Theorem 4.3 guarantees that the closed-loop frequency response of the beam is shaped by $H(s)^{-1}$.

If the transfer function $T(s)$ in Figure 11 is as in equation (5.1), then $S(s)$ may be written as follows:

$$\dot{x}_s(t) = \begin{bmatrix} A & 0 & BC_{\mathrm{H}} \\ 0 & A & 0 \\ 0 & 0 & A_{\mathrm{H}} \end{bmatrix} x_s(t) + \begin{bmatrix} 0 \\ 0 \\ B_{\mathrm{H}} \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ B \\ 0 \end{bmatrix} u(t),$$

$$z(t) = [C_1(\lambda) \;\; C_2(\lambda) \;\; 0] \; x_s(t),$$

$$y(t) = [0 \;\; 0 \;\; C_{\mathrm{H}}] \; x_s(t).$$

Note that in the above system, the matrices D_{12} and D_{21} are zero. Therefore, to avoid consequent problems which will arise while designing a $\mathscr{H}_\infty$ controller for such a system, we consider the following fictitious system which approximates our original system:

$$\dot{x}_s(t) = \begin{bmatrix} A & 0 & BC_{\mathrm{H}} \\ 0 & A & 0 \\ 0 & 0 & A_{\mathrm{H}} \end{bmatrix} x_s(t) + \begin{bmatrix} 0 \\ 0 \\ B_{\mathrm{H}} \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ B \\ 0 \end{bmatrix} u(t),$$

$$z(t, \lambda) = \begin{bmatrix} C_1(\lambda) & C_2(\lambda) & 0 \\ 0 & 0 & 0 \end{bmatrix} x_s(t) + \begin{bmatrix} 0 \\ \varepsilon_1 \end{bmatrix} u(t),$$

$$y(t, \lambda) = [0 \;\; 0 \;\; C_{\mathrm{H}}] \; x_s(t) + \varepsilon_2 w(t).$$

We designed a controller for this system with the assumption that $\varepsilon_1 = 10^{-5}$, $\varepsilon_2 = 10^{-3}$ and

$$H(s) = \frac{10^6}{s^2 + 10^3 s + 10^6}.$$

The $\mathscr{H}_\infty$ controller is designed for a disturbance attenuation level of $\gamma = 4.95 \times 10^{-5}$. The 3D disturbance response and a plot of the $\mathscr{H}_\infty$ norm versus length of the beam can be observed in Figures 15 and 16. Figure 17 shows the Bode plot of the designed controller.

6 CONCLUSIONS

In this paper, we presented a method of reducing vibrations of a flexible structure. We allowed the disturbance to have a wide-band nature and showed that the problem could be reduced to a spatial $\mathscr{H}_\infty$ control problem. Moreover, we

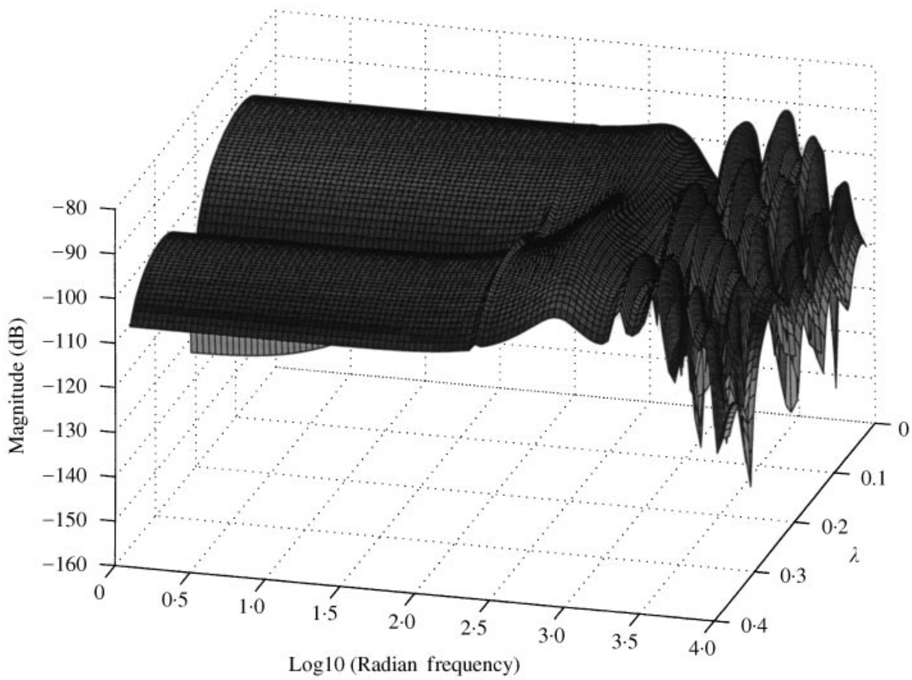

Figure 15. Disturbance responses of the controlled beam with a shaping filter included in the design.

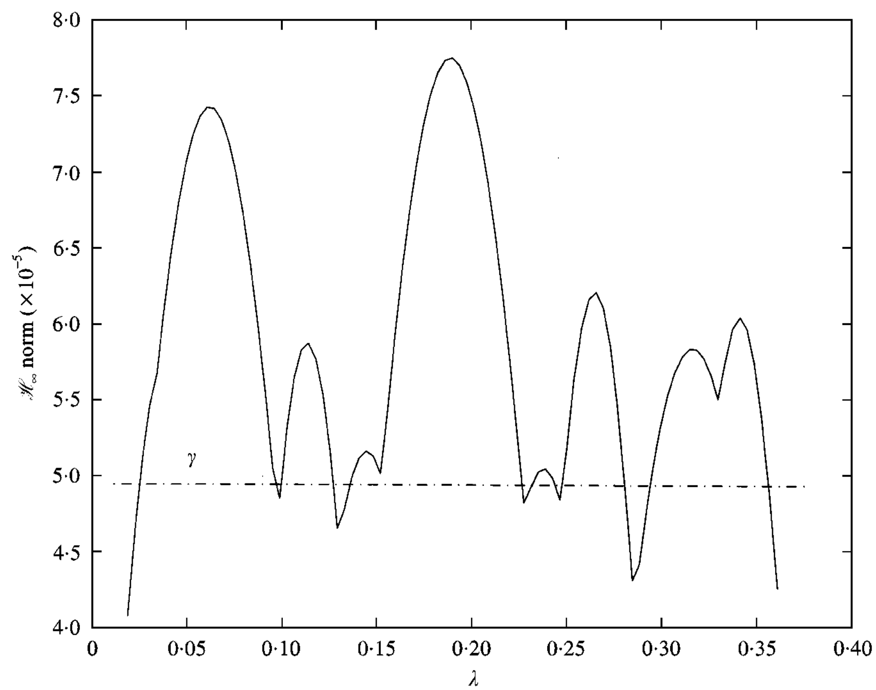

Figure 16. The $\mathscr{H}_\infty$ norm of the disturbance to various locations along the beam with the controller designed for the distributed parameter system with a shaping filter.

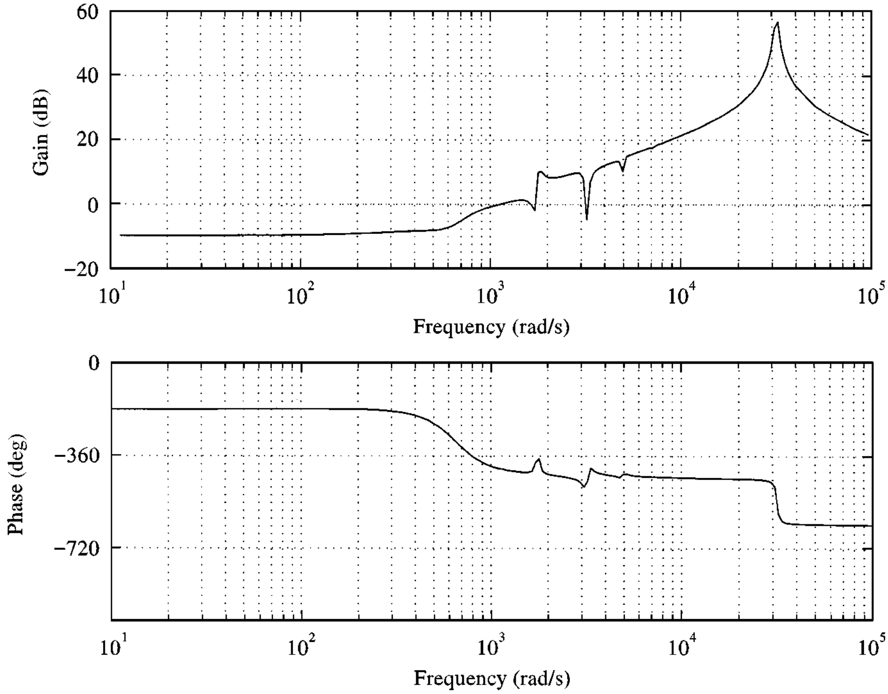

Figure 17. Bode plot of the $\mathscr{H}_\infty$ controller designed for the distributed parameter system with a shaping filter.

showed that this problem is reducible to an standard $\mathscr{H}_\infty$ control problem that can be solved using standard techniques. It was shown that a spatial $\mathscr{H}_\infty$ controller design could result in a better vibration reduction in a global sense.

ACKNOWLEDGMENT

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