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A NEW DERIVATION OF THE FREQUENCY RESPONSE FUNCTION MATRIX FOR VIBRATING NON-LINEAR SYSTEMS

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Frequency response function matrices relate the inputs and the outputs of structural dynamic systems. If a system is linear the frequency response function matrix is the same for any combination or types of inputs over the entire operating range. Furthermore, the frequency response matrix of a linear vibrating system is a simple combination of temporal and spatial characteristics, the modal frequencies, modal vectors and modal scale factors. When a system is non-linear, the inputs interact through an exchange of energy between the linear and non-linear elements in the system. No general combination of the temporal and spatial non-linear characteristics has to date been proposed to describe these linear–non-linear interactions. This article introduces a unifying perspective of non-linearities as internal feedback forces that act together with the external forces to generate the response of the non-linear system. This perspective of the non-linearities is spatial in nature and leads to two simple but conceptually powerful relationships between the frequency response function matrix of a non-linear multiple-degree-of-freedom system and its linear counterpart. Several single- and multiple-degree-of-freedom systems are used to demonstrate the use and interpretation of these relationships. The broad implication of the new input–output frequency response representation for both linear and non-linear systems are also addressed. In particular, the merits of the spatial perspective of non-linear systems and the new frequency response relationships are stated in the context of linear and non-linear system characterization and identification. One implication is that these relationships suggest there is an input–output-dependent temporal-spatial (modal) decomposition of the frequency response function matrix for non-linear systems.

1. INTRODUCTION

There are three essential pieces of information in every structural dynamics analysis or test: the input data, the output data, and the spatial arrangement of the analysis or measurement degrees of freedom (spatial data). The spatial data from the test system or model includes the absolute and relative locations of the degrees of freedom (d.o.f.s) as well as their measurement direction. For instance, if the model is one-dimensional then all of the d.o.f.s measure motion in the same direction.

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Traditional system identification techniques have focused on using input and output temporal measurements to estimate the parameters in an assumed model of some kind. These parametric models can be linear in the case of modal models and autoregressive moving-average (ARMA) models [1, 2], or non-linear in the case of direct parameter models [3], non-linear autoregressive moving average with exogenous input (NARMAX) models [4], or reverse path models [5]. A survey of these system identification techniques is given in reference [6]. All of these non-linear models are classified as parametric because they require a comprehensive knowledge of the system and its non-linear structure. Since characterization and model selection are the most challenging aspects of parametric non-linear multiple-degree-of-freedom (m.d.o.f.) system identification, one of the main goals of the research in this article was to produce a general m.d.o.f. method for characterizing non-linear systems.

In contrast to the popular use of input and output temporal measurements, the important uses of spatial information in linear systems analysis (e.g. modal analysis) has only recently been addressed and published by researchers in reference [7]. These researchers have taken a non-traditional perspective of the traditional modal parameter estimation problem. They have developed the unified matrix polynomial approach (UMPA), which emphasizes the efficient use of spatial information in the parameter estimation process. The benefits of using spatial data include better parameter estimates for heavily damped systems and more consistent estimates of frequency and damping for large d.o.f. test systems. Spatial concepts for linear vibrating systems are used in section 2 to draw analogies to the spatial nature of non-linear systems.

Methods of non-linear system characterization and identification have also relied heavily on input and output temporal measurements and have made only minimal use of spatial information. Reference [8] used linear modal vectors to achieve excellent results with the restoring force method, a non-parametric system identification technique. References [5,9] recognized the advantages of switching the roles of the inputs and outputs in their work with the reverse-path approach to non-linear system identification. It will be demonstrated in section 2.1 that the reverse-path technique is impedance-based; consequently, it is derived in part by viewing the non-linearity spatially as a response feedback force. A variation of the reverse-path method has previously been developed that is more intuitive and more direct [10].

The present research is focused on using spatial information to reveal a new and important perspective of non-linear systems. The spatial perspective views the non-linearity as an internal force, which acts together with the external forces on the underlying linear system to produce the responses. Two simple but powerful relationships between the measured frequency response function (FRF) matrix of a general non-linear system and the associated FRF matrix of the linearized system are derived using this perspective.

Section 2 introduces the spatial concepts that are key to understanding the derivation of the new FRF representations. The spatial nature, advantages, and physical basis for using impedance analysis for non-linear systems is discussed in section 2.1. Then the new FRF matrix representations for non-linear systems are derived in section 3. The implications of these relationships for linear and non-linear systems in general are outlined in section 4. Lastly, a summary is used to highlight the most imporant elements of the article and give a brief overview of future research that will expand on these topics.

2. SPATIAL CONCEPTS IN LINEAR AND NON-LINEAR SYSTEMS ANALYSIS

The measurement and analysis d.o.f.s are an important part of the linear system vibration analysis problem. D.o.f.s in an analytical context determine the possible motions that a model of a system can describe. Similarly, measurements d.o.f.s determine what types of motion an experimental model can replicate. These measurement d.o.f.s are chosen before a test. They represent the spatial data and determine how the sensors and actuators are configured relative to the system. The importance of spatial data is evident in the fundamental equation of modal analysis, which is presented shortly. First, the most often used formula in analytical and experimental structural dynamics is discussed.

When multiple inputs or forces act on a linear system, the multiple outputs or responses, $\{X(\omega)\}$, of a lumped parameter model of the system are given by

$$\{X(\omega)\}_{N_o \times 1} = [H_L(\omega)]_{N_o \times N_i}\{F(\omega)\}_{N_i \times 1}, \tag{1}$$

where the input and output vectors are written as frequency transforms of the corresponding time-domain quantities. There are N_o responses and N_i non-zero external forces. Although analytical models often have equal numbers of inputs and outputs, experimental models will usually have one much larger than the other. It is not only convenient but necessary in non-linear analysis to rewrite the model in equation (1) to accommodate those cases in which there are as many non-zero forces as there are responses. The rationale for doing this is given in section 2.1. Since the input d.o.f.s are usually a subset of the output d.o.f.s, the input–output model is written in terms of the number of outputs:

$$\{X(\omega)\}_{N_o \times 1} = [H_L(\omega)]_{N_o \times N_o}\{F(\omega)\}_{N_o \times 1}. \tag{2}$$

The FRF matrix of the linear system maps the external forces into the responses. $[H_L(\omega)]$ often is decomposed into three distinct parts. The decomposition of $[H_L(\omega)]$ is given below in the fundamental equation of modal analysis:

$$[H_L(\omega)]_{N_o \times N_o} = [\Phi]_{N_o \times 2N}[\Lambda]_{2N \times 2N}[L]^T_{2N \times N_o}. \tag{3}$$

This equation demonstrates that the path between the inputs and outputs passes through the spatial input elements first, the modal participation matrix $[L]$, then through the temporal elements, the eigenvalue matrix $[\Lambda]$, and finally through the spatial output elements, the matrix of modal vectors $[\Phi]$. There are N positive modal frequencies for a total of $2N$ modal frequencies and modal vectors. Figure 1 illustrates the transmission path through each of these three elements of the system. Since the path contains two pieces of spatial information, the analysis or test should always be designed to capture as much of the spatial data as possible.

Early methods of modal analysis used polynomial models with the measured inputs and outputs to estimate eigenvalues and eigenvectors. These methods are

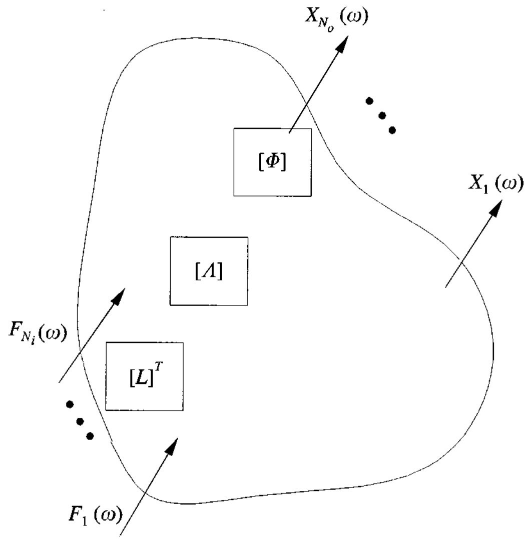

Figure 1. Illustration of the path between the inputs and outputs of a linear vibrating system.

well documented in the literature [11]. Two types of matrix coefficient polynomial models were later introduced that used spatial data to gain certain advantages over the temporal scalar polynomial models. The two fundamental time and frequency domain, multiple input, multiple output (MIMO) models are written as [7]

$$\sum_{k=0}^{m} \Big[[\alpha_k]\,(j\omega)^k\Big]\{X(\omega)\} = \sum_{k=0}^{n} \Big[[\beta_k]\,(j\omega)^k\Big]\{F(\omega)\}, \tag{4}$$

$$\sum_{k=0}^{m} [\alpha_k]\{x(t_{d+k})\} = \sum_{k=0}^{n} [\beta_k]\{f(t_{d+k})\}, \tag{5}$$

where d indicates the initial time shift in the time-domain model (equation (5)). These models can be thought of as general forms of the lumped parameter, MIMO mass-stiffness-damping models in the frequency and time domains.

The first goal of modal parameter estimation is to calculate the modal frequencies and modal vectors that fit into the matrices in equation (3). High order parameter estimation methods use small coefficient matrices in equations (4) and (5) to write high order characteristic equations to estimate the modal frequencies or poles of the system. In other words, high order techniques use primarily temporal data to estimate the modal frequencies.

In contrast, low order parameter estimation methods use large coefficient matrices to write lower order, matrix characteristic equations to estimate the poles. Note that a low order, high-dimensional matrix equation will produce the same number of modal frequency estimates as a high order scalar (one-dimensional) equation. The low order methods use multiple references in the experimental measurement to generate the matrix coefficients in equations (4) and (5). Because they use multiple references the low order methods require less temporal data and more spatial data than the single reference methods. The maximum number of modal frequencies that can be estimated with low and high order characteristic equations is the same:

$$\text{No. of } \lambda_r \leqslant (\text{Equation order}) \times (\text{No. of references}), \quad (6)$$

where λ_r are the modal frequencies of the linear system.

The advantages of the low order or polyreference parameter estimation methods are that they use spatial data to identify heavily damped systems, estimate modal vectors directly, lower the order of the frequency domain model in equation (5) to prevent numerical errors at higher frequencies or detect and estimate closely spaced (repeated) modal frequencies. Another merit of the spatial perspective is that it enables UMPA to describe all of the seemingly complex parameter estimation algorithms (e.g. complex exponential, least-squares complex exponential, Ibrahim time domain, eigensystem realization algorithm, polyreference time domain, polyreference frequency domain) with a single model form. In summary modal parameter estimation methods all start with the same basic matrix polynomial models; it is the order of the polynomials and the extent to which spatial data is used that makes the methods different [7].

Spatial information is also important for analyzing non-linear system vibrations. In fact, it is even more important than for linear systems because it provides the missing piece of data that is needed to diagnose the non-linearities in the system. The temporal data does not contain enough information to do this alone when there is more than one d.o.f. in the system. Indeed, there are two distinct elements at the same location in space that interact to produce the temporal response: the linear elements and the non-linear elements. The spatial information helps to distinguish between these two types of elements.

The authors propose that the best way to recognize non-linearities in many situations is to treat them as hidden inputs to the corresponding linear system. Thus, the non-linearities becomes unmeasured, internal, feedback forces that are non-linear functions of the outputs. The notion of a non-linearity as a feedback force is illustrated symbolically in Figure 2. The input energy enters the system at d.o.f.s q, flows through the system, exits at output d.o.f.s p, and then returns as an internal forces to the d.o.f.s at which non-linear elements exist. The system in the illustration contains a single non-linearity N around which the internal forces feedback. The figure is symbolic because the feedback process is actually more a type of generation phenomenon than it is a flow across the system boundary.

Feedback is not a new concept in dynamics theory, where it has been used to solve analytical non-linear ordinary differential equations with sinusoidal inputs [12]. There are many more uses for the feedback perspective in non-linear dynamics. For example, feedback helps to explain drops in the ordinary coherence function between the measured inputs and outputs of a non-linear system. Since the non-linearities are unmeasured correlated forces, there is energy in the output spectrum that is not accounted for by the measured inputs. Reference [8] used modal vectors for the purpose of identifying the internal, feedback forces due to the non-linearities that are generated by the outputs; however, this spatial information is in a sense borrowed from the linear system and cannot represent the absolute

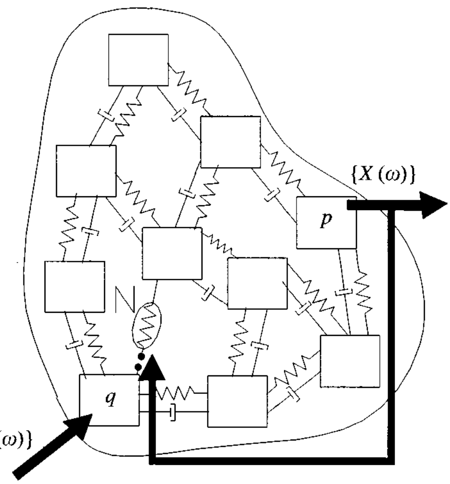

Figure 2. Illustration of the path between the inputs and outputs of a non-linear vibrating system showing the feedback of internal forces due to the non-linearities.

behavior of the non-linear system. The estimated feedback forces are used to characterize and identify the non-linearities. The authors have also used spatial information in the form of input–output reciprocity checks to characterize nonlinearities in vibrating systems [13].

2.1. IMPEDANCE AS A SPATIAL PERSPECTIVE

There is another important use for the spatial or feedback perspective in non-linear structural dynamics: it helps to determine how to best select an input–output model for the purpose of characterizing and identifying non-linear systems. The feedback perspective suggests that the model should explictly represent the internal feedback forces as functions of the outputs. To better illustrate this idea, consider the model for linear vibrating systems in equation (2). It is a natural way to relate the inputs and outputs of linear structural dynamic systems because once $\{x(t)\}$ and $\{f(t)\}$ are measured, $\{X(\omega)\}$ and $\{F(\omega)\}$ can be calculated to provide all of the important input–output information. When the system is non-linear, the input and output spectra do not provide enough information to develop a model of the system.

There are two equivalent ways of ruling out the linear FRF model in equation (2) as a model for non-linear systems (refer to Figure 3). The system on the left in Figure 3 contains a non-linearity in the hatched region, so the linear model is automatically incomplete and cannot be used to model the system. The system on the right is linear; however, the non-linear element has been replaced with an internal force that has the same effect as the non-linearity. Note that this internal force has not been measured. Since the model in equation (2) only accounts for the path between the measured external inputs and the outputs, the model is again

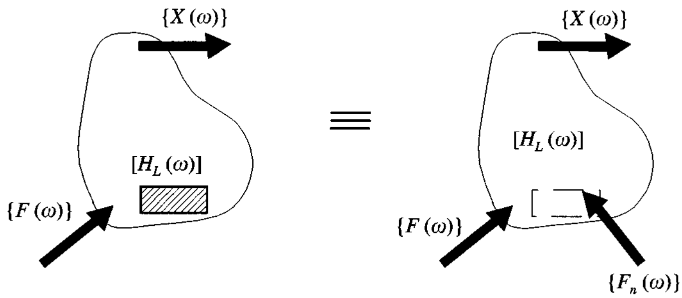

Figure 3. Two equivalent ways of ruling out the linear frequency response function model (equation (2)) for non-linear systems: non-linearity in the linear system (left), internal feedback into the linear system (right).

concluded to be inappropriate for the non-linear system. In both of these cases, the linear and non-linear dynamics combine to create an FRF matrix of the non-linear system that is different from the FRF matrix of the linear system.

A review of the literature shows that the use of FRF matrix model has declined as researchers have made important advances using the impedance model instead [5]. The motivation for using these impedance models is now given in terms of the feedback perspective. To that end if a m.d.o.f non-linear system has a single non-linearity that can be modelled as a lumped element, then the fundamental frequency domain equation of motion for that system can be written as

$$[B_L(\omega)]_{N_o \times N_o}\{X(\omega)\}_{N_o \times 1} + \mu_1(\omega)\{B_{n1}\}_{N_o \times 1}X_{n1}(\omega) = \{F(\omega)\}_{N_o \times 1}. \quad (7)$$

The linear parts of this equation are contained in the linear impedance matrix $[B_L(\omega)]$, whereas the non-linear contributions are contained in the impedance vector $\mu_1(\omega)\{B_{n1}\}X_{n1}(\omega)$.

There are three parts to the non-linear term. $X_{n1}(\omega)$ is a scalar non-linear function of the outputs; it determines the class of the non-linearity. Each element of $\{B_{n1}\}$ is either a 1 or a -1; these elements determine the location of the non-linearity. Lastly, $\mu_1(\omega)$ is the coefficient of the non-linearity; it determines the strength of the internal feedback force. Note that a different $[B_{ni}\}$ and $X_{ni}(\omega)$ pair are used to model each non-linear element in the system. This kind of impedance model is used to derive the FRF matrix in section 3. Also note that since the internal feedback forces are derived from non-linear functions of the responses, the impedance model must always have a greater number of responses than non-zero external forces ($N_i \leqslant N_o$). Finally, note that $\mu_1(\omega)$ is in general a frequency-dependent non-linear parameter.

As a specific example of the model in equation (7), consider the three d.o.f. system in Figure 4 with a non-linear stiffness $\mu_1 x_1^3(t)$ to ground at d.o.f. 1 and a coupling non-linear stiffness $\mu_2(x_2(t) - x_3(t))^3$ between d.o.f.s 2 and 3. Note that both of the non-linear stiffness characteristics are functions of the responses across the ends of the non-linear springs. This system has the following $\{B_{ni}\}$ vectors and $X_{ni}(\omega)$ scalar functions:

$$X_{n1}(\omega) = \mathbf{F}[x_1^3(t)],\ \mu_1\{B_{n1}\} = \mu_1\begin{pmatrix}1\\0\\0\end{pmatrix}, \tag{8}$$

$$X_{n2}(\omega) = \mathbf{F}[(x_2(t) - x_3(t))^3],\ \mu_2\{B_{n2}\} = \mu_2\begin{pmatrix}0\\1\\-1\end{pmatrix}, \tag{9}$$

in which $F[\cdot]$ is the Fourier transform operator. Additional impedance vectors and non-linear scalar functions of the outputs must be added to the left-hand side of equation (7) for each additional non-linear element.

When $[H_L(\omega)]$ in equation (2) is derived for linear systems, it is defined as the inverse of the impedance matrix $[B_L(\omega)]$. FRF matrices of non-linear systems couple together the linear and non-linear dynamics. The advantage of the impedance relationship in equation (7) is that it describes the motion of the system without as much coupling between the linear and non-linear dynamics. Hence, the reverse-path approach to non-linear system identification has become so attractive. In fact, the reverse-path spectral method has been used successfully to estimate non-linear system parameters (e.g., $\mu_1(\omega)$ in equation (7)) in a wide variety of systems [9, 14]. The general idea of the reverse-path method is to switch the roles of the inputs and outputs in an effort to use the measured data more efficiently in the system identification process. This technique uses the terminology and procedures of conditioned spectral analysis to decouple the linear and non-linear dynamics in non-linear systems. For a better understanding of the reverse-path technique, it is helpful to view it as a generalized dynamic impedance approach. It uses an impedance model like the one in equation (7) to express the internal feedback forces as functions of the outputs. This interpretation of the reverse-path approach is discussed further in reference [10].

In summary, the use of spatial information in vibrating linear systems reduces the order of the characteristic equation and the amount of temporal data needed to estimate the modal parameters. The spatial information also makes it possible to detect closely spaced (repeated) modal frequencies. The spatial nature of non-linear systems involves the feedback of internal forces that are made up of non-linear functions of the outputs (responses). Feedback in non-linear systems can be modelled with impedance functions, which are useful to help distinguish between the linear and non-linear dynamics. Although impedance is useful, it cannot be measured easily and accurately. The frequency response matrix can be measured directly and is therefore a more practical experimental way of characterizing a vibrating system. Unfortunately, the measured frequency response functions of non-linear systems have not been quantified in a consistent way for m.d.o.f. systems. The next section combines the model in equation (7) with the spatial notion of a non-linearity as an internal feedback force to derive the true FRF matrix of a general non-linear system.

3. DERIVATION OF A FREQUENCY RESPONSE FUNCTION MATRIX FOR NON-LINEAR SYSTEMS

The reverse-path technique and all other parametric models assume prior knowledge of the discrete distribution of non-linearities throughout a system. Since m.d.o.f. non-linear systems are complicated by nature, this *a priori* assumption of direct access to the model structure is usually not realistic. Moreover, characterization and model selection are usually the most challenging tasks in non-linear structural dynamics analysis. There are many techniques for characterizing non-linear systems. A survey of many of these techniques is provided in reference [6]. Since FRFs are among the most popular and convenient measurements to make in experimental dynamics, a complete understanding of the measured FRF matrix of a non-linear system would be very helpful for characterizing non-linearities in the system and selecting parametric models. Although researchers have studied the distortions that appear in the measured FRFs of non-linear single-degree-of-freedom (s.d.o.f.) systems [15, 16], a consistent and unified explanation of FRF distortion for m.d.o.f. systems is not available. The spatial perspective of non-linearities as internal feedback forces will be used in this section to derive two distinct FRF matrix representations for non-linear systems that are decomposed into pseudo-linear and non-linear parts. This result will be consistent for all s.d.o.f. and m.d.o.f. non-linear systems that can be modelled using lumped elements.

Before deriving an expression for the FRF matrices of non-linear systems, it is prudent to consider how these FRFs will behave in the presence of non-linearities. First, they should clearly depend on the response amplitudes at the d.o.f.s at which non-linearities exist. This is a defining feature of non-linearities in general. Second, the strength or impedance coefficient of the non-linearities should determine the difference between the measured $[H(\omega)]$ and the linear $[H_L(\omega)]$. For instance, the $\mu_1(\omega)$ in equation (8) will certainly help to determine the effects of the non-linear spring, $\mu_1 x_1^3(t)$, on the system dynamics (refer to Figure 4). In fact, if $\mu_1(\omega)$ is zero, the system behaves linearly. Third, since non-linear systems are path- and direction-dependent, $[H(\omega)]$ should not exhibit reciprocity, that is $H_{pq}(\omega)$ should not be equal to $H_{qp}(\omega)$, where $H_{pq}(\omega)$ denotes the FRF between the input q and the output p with all other inputs set to zero. These three expectations will help to interpret the meaning of the derived FRF matrix.

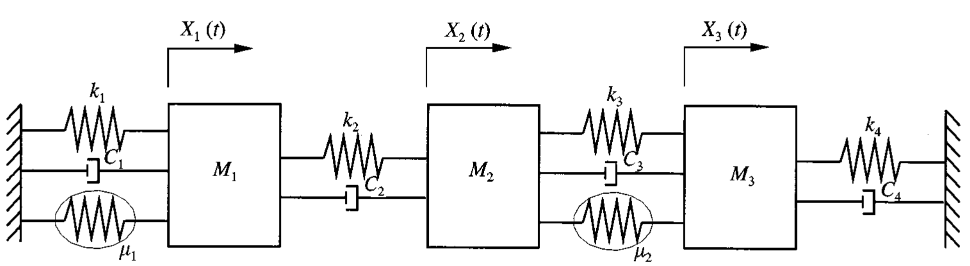

Figure 4. A three-d.o.f.s system with a non-linear spring to ground, $\mu_1 x_1^3(t)$, and a coupling non-linear spring, $\mu_2(x_2(t) - x_3(t))^3$, between degrees d.o.f.s 2 and 3.

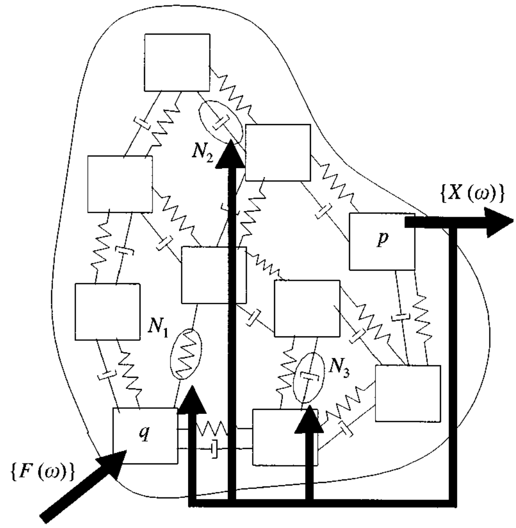

Figure 5. A general non-linear vibrating system with different types of non-linearities at different d.o.f.s

The MIMO non-linear system model being considered is shown in Figure 5. This lumped element model is completely general and contains many different types of non-linearities at many locations. The impedance relation that governs the dynamics of this model is

$$[B_L(\omega)]_{N_o \times N_o}\{X(\omega)\}_{N_o \times 1} + \sum_{i=1}^{N_n} \mu(\omega)\{B_{ni}\}_{N_o \times 1} X_{ni}(\omega) = \{F(\omega)\}_{N_o \times 1}, \tag{10}$$

or after moving the internal forces due to the non-linearities to the right-hand side of the equation,

$$[B_L(\omega)]\{X(\omega)\} = \{F(\omega)\} - \sum_{i=1}^{N_n} \mu_i(\omega)\{B_{ni}\} X_{ni}(\omega). \tag{11}$$

N_n in these equations denotes the number of non-linear elements in the system. The relative strength of each non-linearity is determined by the size of its coefficient, $\mu_i(\omega)$. Note that these coefficients can be functions of frequency. Also notice that equation (11) is simply a statement of the linear superposition principal for linear systems: the sum of the responses of the linear system, $[B_L(\omega)]$, due to the sum of two forces is the sum of the individual responses to each of the forces when they act alone. If the second term on the right-hand side of equation (11) is mistakenly or intentionally ignored, as it often is in linear systems analysis, the non-linearities are transformed into unmeasured sources of correlated noise, which will generate bias errors in the linear model parameter estimates.

The first step in the derivation is to write part of the internal feedback forces, $\{B_{ni}\}X_{ni}(\omega)$, as a linear combination at every frequency of the measured response vector, $\{X(\omega)\}$. This creates the following MIMO, spectral, total least-squares set of equations:

$$\{B_{ni}\}_{N_o \times 1} X_{ni}(\omega) = [{}_xB_{ni}(\omega)]_{N_o \times N_o}\{X(\omega)\}_{N_o \times 1}, \tag{12}$$

in which $\{_xB_{ni}(\omega)\}$ is the projection matrix of the non-linear response feedback onto the measured response vector, $\{X(\omega)\}$. In other words, the projection matrix is the FRF matrix between the measured responses and the non-linear functions of the responses that travel through the feedback path.

There is an alternative projection to the one in equation (12) that projects onto the external force spectra instead of the response spectra. Moreover, when the internal feedback forces are projected onto the external forces, the following set of equations is formed:

$$\{B_{ni}\}_{N_o\times 1}X_{ni}(\omega) = [_fB_{ni}(\omega)]_{N_o\times N_o}\{F(\omega)\}_{N_o\times 1}. \tag{13}$$

Both of these projections accomplish the same goal: they eliminate the unmeasured internal forces in favor of the measured external forces and the measured responses. This is desirable because only the measured responses and the external forces determine the measured FRF matrix. Furthermore, most useful relationships in experimental structural dynamics involve measured quantities only.

The second step in the derivation is to recognize the equations (10) and (12) can be combined into the following equation:

$$[B_L(\omega)]_{N_o\times N_o}\{X(\omega)\}_{N_o\times 1} + \sum_{i=1}^{N_n}\mu_i(\omega)[_xB_{ni}(\omega)]_{N_o\times N_o}\{X(\omega)\}_{N_o\times 1} = \{F(\omega)\}_{N_o\times 1}. \tag{14}$$

This equation is illustrated in block diagram form in Figure 6. Note that the entire summation in equation (14) is replaced with a single impedance matrix, $[B_n(\omega)]$, for convenience. In the language of classical control theory, the linear system in the figure is the open-loop system and the non-linear system is the closed-loop system. The input–output relationship for the closed-loop system is defined in the following set of equations:

$$\{X(\omega)\}_{N_o\times 1} = \left[[I]_{N_o\times N_o} + [H_L(\omega)]_{N_o\times N_o}\sum_{i=1}^{N_n}\mu_i(\omega)[_xB_{ni}(\omega)]_{N_o\times N_o}\right]^{-1} [H_L(\omega)]_{N_o\times N_o}\{F(\omega)\}_{N_o\times 1}, \tag{15}$$

$$\{X(\omega)\} = [_xH_M(\omega)]\,[H_L(\omega)]\,\{F(\omega)\}, \quad \{X(\omega)\} = [_xH(\omega)]\,\{F(\omega)\}. \tag{16, 17}$$

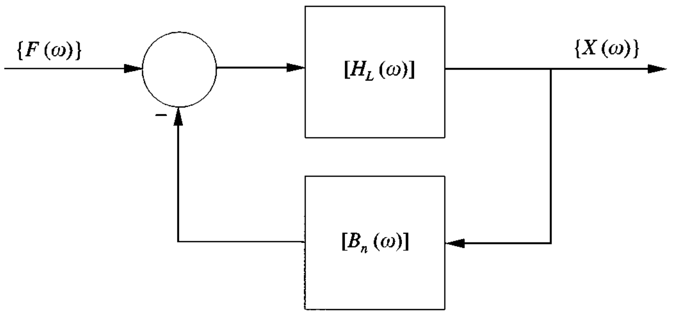

Figure 6. Equivalent closed-loop representation of the non-linear system dynamics between the measured inputs and outputs.

Equation (16) states that the measured FRF matrix of the non-linear system is a left-hand modulation of the linear FRF matrix. $[{}_xH_M(\omega)]$ will be defined as the non-linear modulation matrix on the outputs (NMMO). Since the internal non-linear feedback forces are written as explicit functions of the output or response vector, the modulation matrix is literally associated with the outputs. In symbolic form, equation (15) is a very familiar formula from classical control theory (Mason's formula):

$$\{X(\omega)\} = \frac{[H_L(\omega)]}{[I] + [H_L(\omega)]\,[B_n(\omega)]}\,\{F(\omega)\}, \tag{18}$$

where all the terms in the summation of equation (15) are grouped into one matrix, $[B_n(\omega)]$.

Recall that equations (16) and (18) were derived after projecting the internal feedback forces onto the output spectra. If the projection in equation (13) is used instead, the following input–output relationship is obtained:

$$\{X(\omega)\}_{N_o\times 1} = [H_L(\omega)]_{N_o\times N_o}\left[[I]_{N_o\times N_o} - \sum_{i=1}^{N_n}\mu_i(\omega)[{}_fB_{ni}(\omega)]_{N_o\times N_o}\right]\{F(\omega)\}_{N_o\times 1}, \tag{19}$$

$$\{X(\omega)\} = [H_L(\omega)]\,[{}_fH_M(\omega)]\,\{F(\omega)\}, \quad \{X(\omega)\} = [{}_fH(\omega)]\,\{F(\omega)\}. \tag{20, 21}$$

Equations (19)–(21) are the mirror images of equations (15)–(17) .

Equation (19) indicates that when the internal feedback forces due to the non-linearities are written as implicit functions of the external inputs, a right-hand modulation of the linear FRF matrix produces the FRF matrix of the non-linear system. The modulation matrix in this case is the non-linear modulation matrix on the inputs (NMMI). Although N_i, the number of non-zero external inputs, is usually much smaller than N_o, the NMMI has certain computational advantages over the NMMO. These include: (1) a matrix inversion is not required to compute the NMMI, whereas the NMMO does require an inversion, and (2) unlike the NMMO, the NMMI is not numerically difficult to compute near-modal frequencies of vibration, around which the responses become highly correlated. These advantages are illustrated in sections 3.1 and 3.2. There is also one disadvantage to using the NMMI formulation: only the columns of the FRF matrix that correspond to non-zero forces can be calculated using equation (21).

The remainder of this section is devoted to examples of the set-up and use of equations (16) and (20). The broader implications of these new FRF relationships are subsequently addressed in section 4.

3.1. S.D.O.F. EXAMPLE

Three s.d.o.f. systems are shown in Figure 7. System 1 has a hardening cubic stiffness to ground, system 2 has a clearance non-linearity to ground, and system 3 has both a cubic stiffness and a clearance non-linearity to ground. The FRFs of these three systems will be constructed using equations (16) and (20).

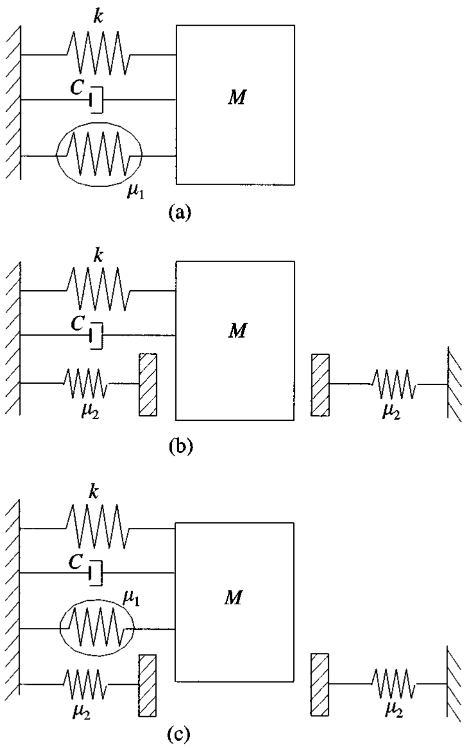

Figure 7. Three s.d.o.f.s non-linear systems: (a) system 1—hard spring to ground (b) system 2—clearance element to ground; (c) system 3—hard spring and clearance element to ground.

The form of equation (15) for each of these three systems is simple to compute because the matrices shrink into scalars. When the NMMO is used to calculate the FRF matrices of the three non-linear systems, the results are as follows:

$$X(\omega)|_{\text{system 1}} = \frac{H_L(\omega)}{1 + \mu_1 H_L(\omega) \cdot {}_xB_{n1}(\omega)} F(\omega), \quad {}_xB_{n1}(\omega) \equiv H_{x^3,x}(\omega); \quad (22, 23)$$

$$X(\omega)|_{\text{system 2}} = \frac{H_L(\omega)}{1 + \mu_2 H_L(\omega) \cdot {}_xB_{n2}(\omega)} F(\omega), \quad {}_xB_{n2}(\omega) \equiv H_{x_{dz},x}(\omega); \quad (24, 25)$$

$$X(\omega)|_{\text{system 3}} = \frac{H_L(\omega)}{1 + \mu_1 H_L(\omega) \cdot {}_xB_{n1}(\omega) + \mu_2 H_L(\omega) \cdot {}_xB_{n2}(\omega)} F(\omega), \quad (26)$$

$${}_xB_{n1}(\omega) \equiv H_{x^3,x}(\omega), {}_xB_{n2}(\omega) \equiv H_{x_{dz},x}(\omega). \quad (27)$$

x_{dz} in these equations denotes the deadzone or clearance non-linear characteristic. Equations (22), (24), and (26) are exact formulas for the FRFs of the three non-linear systems. The relations for the projection matrices are also given in equations (23), (25), and (27). Note that the projections are simply FRFs between the measured response and the non-linear functions of the response that generate the internal feedback forces. Bode plots of equations (22), (24), and (26) are shown in Figures 8–10. The internal forces due to the non-linearities in all three cases are 15% of the

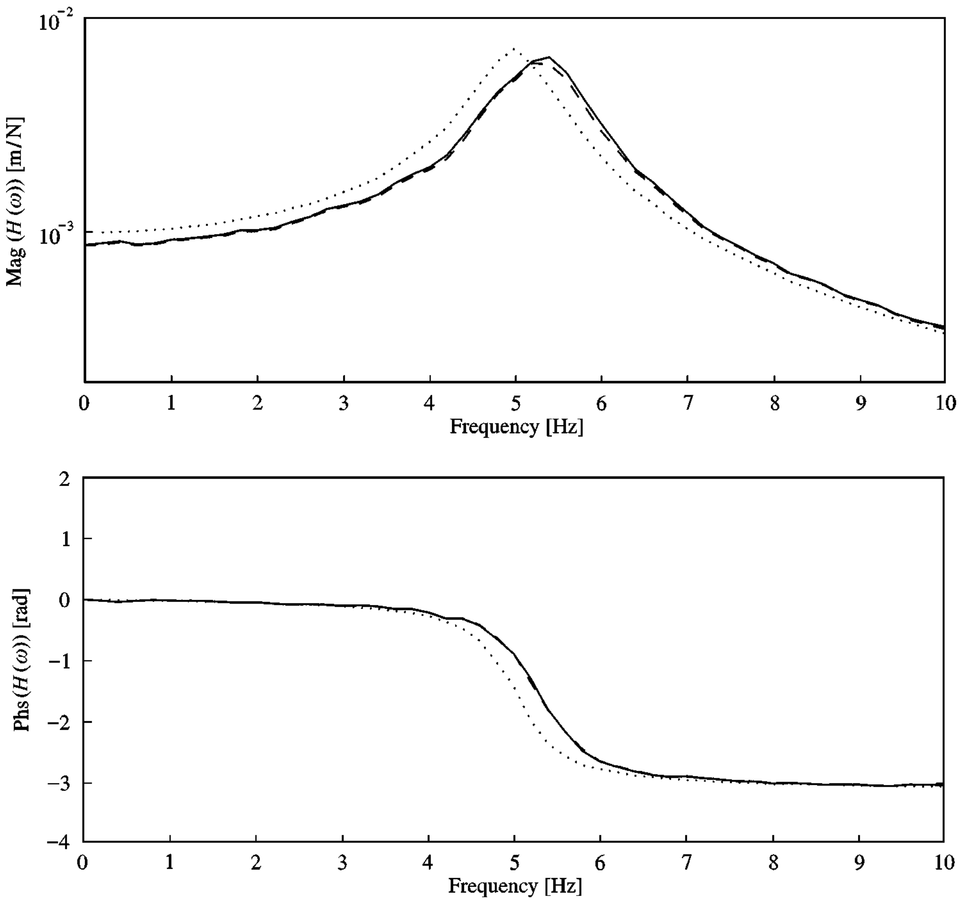

Figure 8. Comparison of the linear FRF, measured FRF, and FRF computed from equation (22) for system 1—hard spring to ground; …., linear FRF; - - - -, measured FRF; —computed FRF.

linear forces to ground. The dashed curves in these figures are the true FRFs of the non-linear systems, the dotted curves are the corresponding linear FRFs, and the solid curves are the calculated values of the FRFs from equations (22), (24), and (26). Note that the magnitude and phase of the measured and calculated FRFs in all three systems are in excellent agreement; however, the FRF magnitude for system 3 is slightly in error near the modal frequency (peak). This error is due to the inversion in equation (26) in the sensitive region of the FRF near the peak. Recall that errors near the modes of vibration are one of the disadvantages of the HMMO calculation.

The HMMI does not require an inversion, so it should provide better accuracy near the peak in the FRF magnitude plot. This will now be demonstrated. When the FRFs for the three systems are calculated using the HMMI, the following equations are obtained:

$$X(\omega)|_{\text{system 1}} = H_L(\omega)\,[1 + \mu_1 \cdot {}_fB_{n1}(\omega)]\; F(\omega), \quad {}_fB_{n1}(\omega) \equiv H_{x^3,f}(\omega); \quad (28, 29)$$

$$X(\omega)|_{\text{system 2}} = H_L(\omega)\,[1 + \mu_2 \cdot {}_fB_{n2}(\omega)]\; F(\omega), \quad {}_fB_{n2}(\omega) \equiv H_{x_{dz},f}(\omega); \quad (30, 31)$$

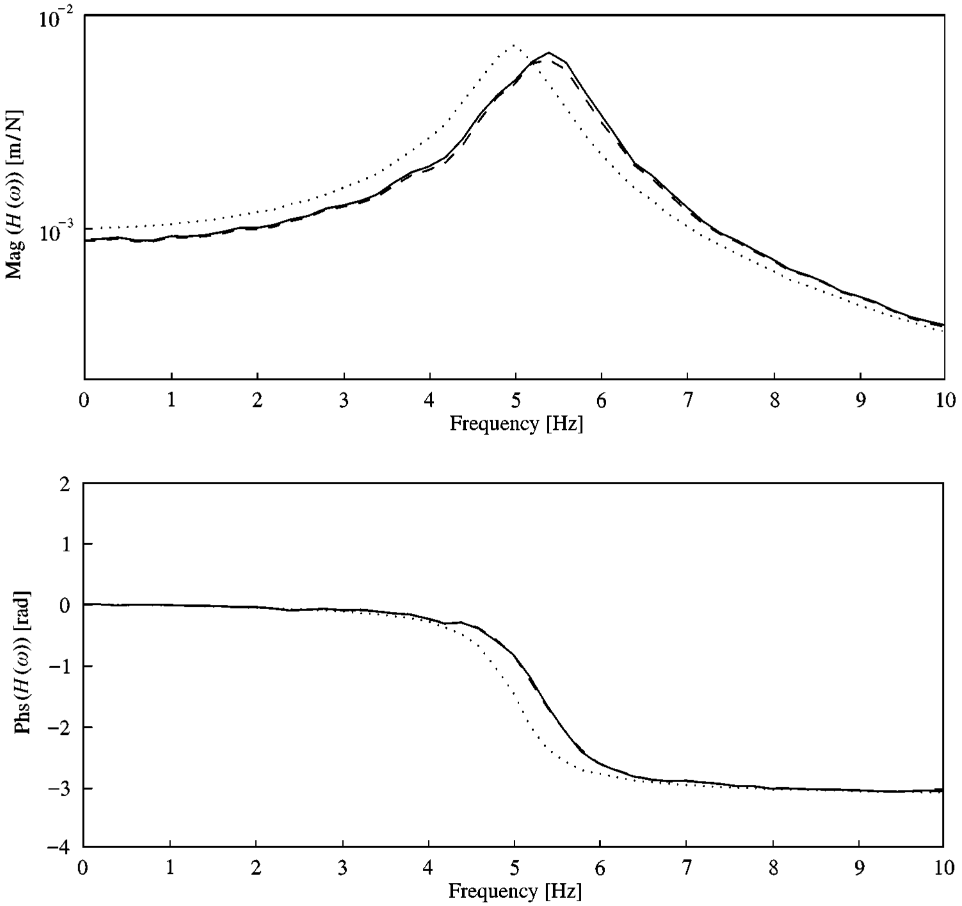

Figure 9. Comparison of the linear FRF, measured FRF, and FRF computed form equation (24) for system 2 — clearance element to ground; ...., linear FRF; ----, measured FRF; — computed FRF.

$$X(\omega)|_{\text{system } 3} = H_L(\omega)\,[1 + \mu_1 \cdot {}_fB_{n1}(\omega) + \mu_2 \cdot {}_fB_{n2}(\omega)]\; F(\omega), \tag{32}$$

$${}_fB_{n1}(\omega) \equiv H_{x^3,f}(\omega),\; {}_fB_{n2}(\omega) \equiv H_{x_{dz},f}(\omega). \tag{33}$$

The plots using the NMMI that correspond to the plots for the NMMO are shown in Figures 11–13. Note that the magnitude and phase of the measured and calculated FRFs in Figure 13 for the system with multiple non-linearities are in better agreement than in the corresponding results for the NMMO calculation.

3.2. M.D.O.F. EXAMPLE

A two-d.o.f. system is shown in Figure 14 with a single non-zero input at d.o.f. 1 and a single hard spring to ground at d.o.f. 1. The procedure for calculating the FRFs of this non-linear system are the same as for the s.d.o.f. systems in the

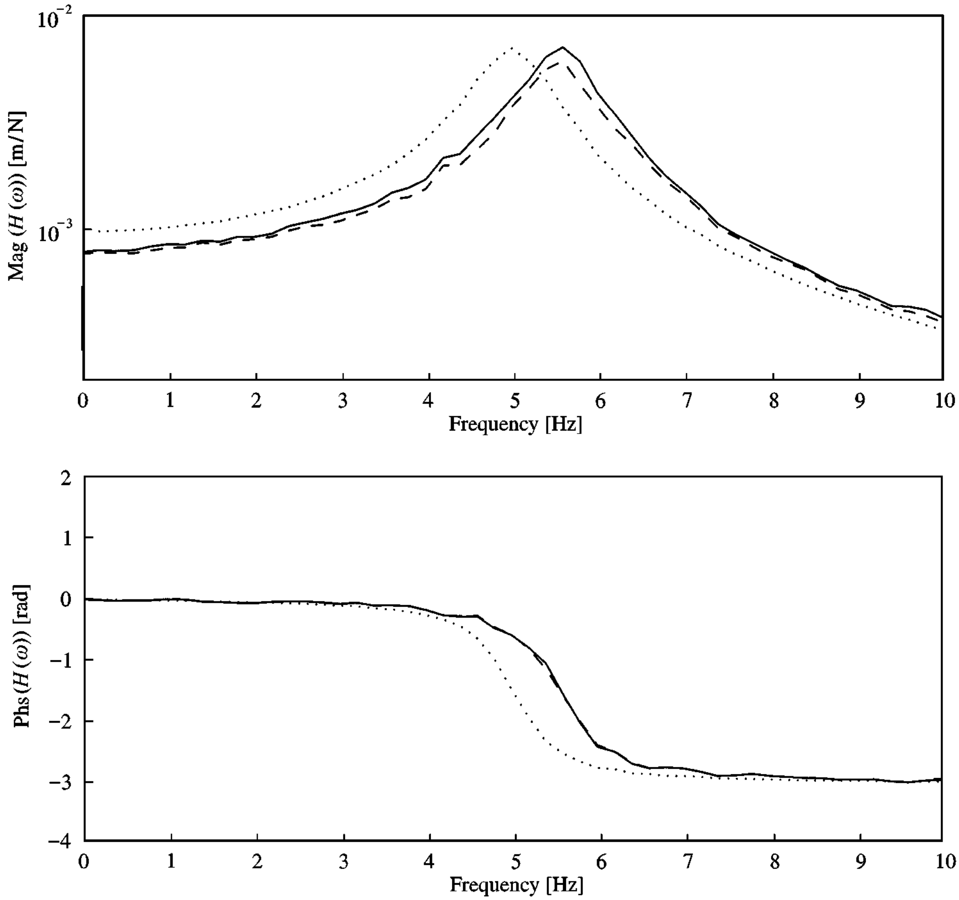

Figure 10. Comparison of the linear FRF, measured FRF, and FRF computed from equation (26) for system 3 — hard spring and clearnace element to ground; ...., linear FRF; ---- measured FRF; — computed FRF.

previous section. The results are shown in Figures 15–18. Note that although the NMMI produces better results in the $H_{11}(\omega)$ and $H_{21}(\omega)$ magnitude plots, the NMMI calculation cannot produce the FRFs $H_{12}(\omega)$ and $H_{22}(\omega)$ because there is no input at d.o.f. 2.

Note that a subset of the responses can be used for the projection instead of all the responses. This offers moderate improvements in the accuracy of the NMMO FRF calculation near the damped natural frequencies.

4. IMPLICATIONS OF THE NEW FREQUENCY RESPONSE RELATIONSHIPS

There are many implications of the FRF matrix relationships in equations (16) and (20). After the fundamental properties of these two relationships are discussed, their implications for the analysis of vibrating linear and non-linear systems are

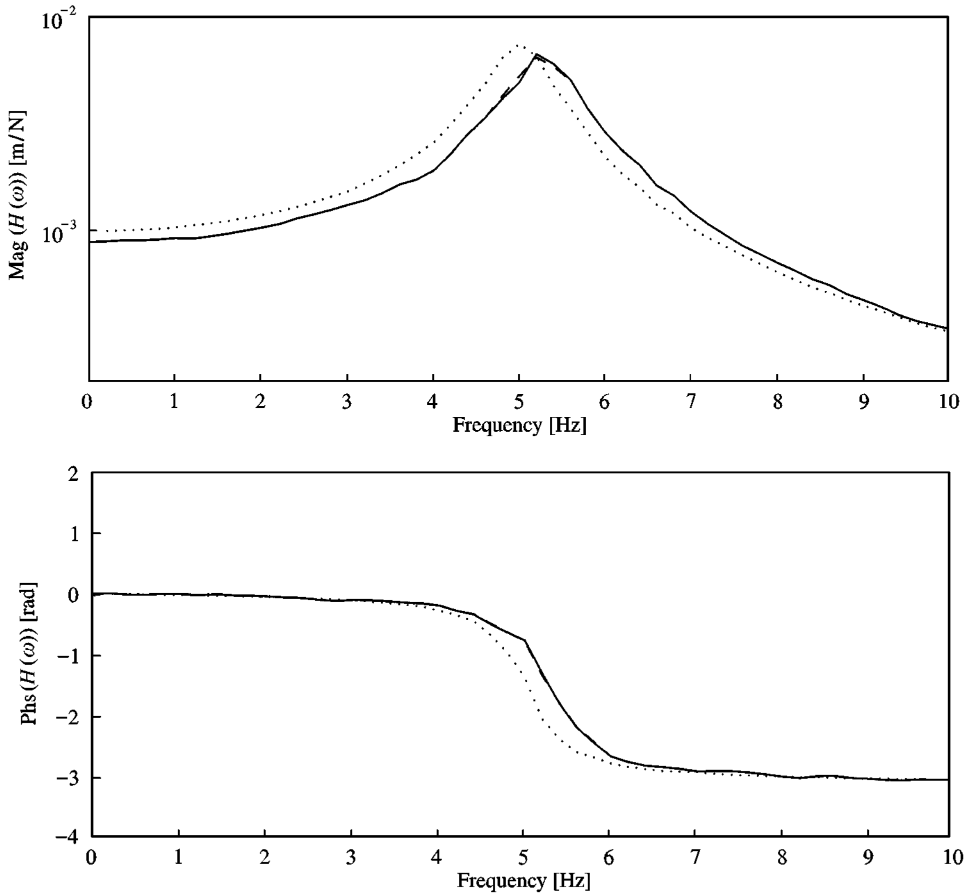

Figure 11. Comparison of the linear FRF, measured FRF, and FRF computed from equation (28) for system 1 — hard spring to ground; ...., linear FRF; ----, measured FRF; — computed FRF.

addressed. The three forms of each of the main equations are repeated and renumbered here without matrix dimensions for more convenient reference. For the projection onto the responses,

$$\{X(\omega)\} = \left[[I] + [H_L(\omega)] \sum_{i=1}^{N_n} \mu_i(\omega) [{}_xB_{ni}(\omega)] \right]^{-1} [H_L(\omega)] \{F(\omega)\}, \tag{34}$$

$$\{X(\omega)\} = [{}_xH_M(\omega)] [H_L(\omega)] \{F(\omega)\}, \quad \{X(\omega)\} = [{}_xH(\omega)] \{F(\omega)\} \quad (35, 36)$$

and for the projection onto the forces,

$$\{X(\omega)\} = [H_L(\omega)] \left[[I] - \sum_{i=1}^{N_n} \mu_i(\omega) [{}_fB_{ni}(\omega)] \right] \{F(\omega)\}, \tag{37}$$

$$\{X(\omega)\} = [H_L(\omega)] [{}_fH_M(\omega)] \{F(\omega)\}, \quad \{X(\omega)\} = [{}_fH(\omega)] \{F(\omega)\}. \quad (38, 39)$$

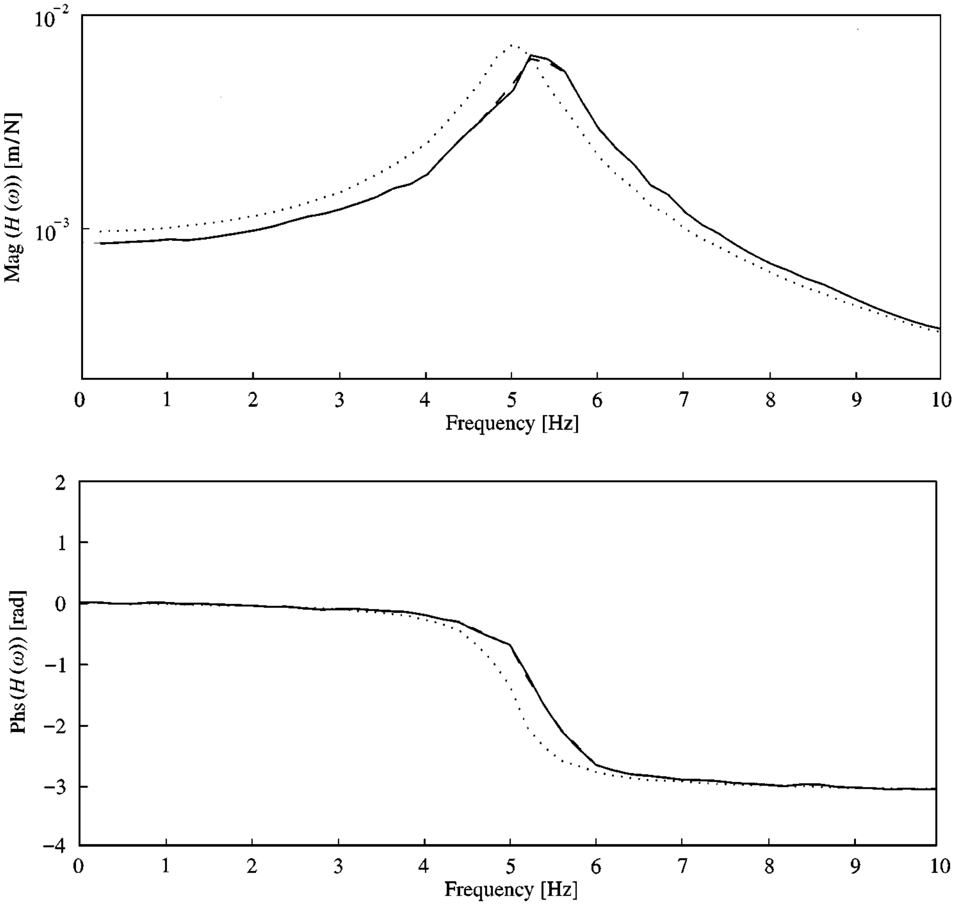

Figure 12. Comparison of the linear FRF, measured FRF, and FRF computed from equation (30) for system 2—clearance element to ground; ...., linear FRF; - - - -, measured FRF; — computed FRF.

Section 3 began by proposing what the properties of the FRF matrix of the non-linear system should be. It will now be demonstrated that the matrices in equations (36) and (39) possess these properties. All of these important properties are dictated by the modulation matrices, NMMO and NMMI.

The first property of equations (35) and (38) is that the degree to which the linear FRFs are modulated or distorted is determined by the amplitudes of the internal feedback forces. This was clearly the case in equations (22)–(32) in which the FRFs between the measured response/force and the non-linear function of the response ($H_{x^3,x}(\omega)$, $H_{x^3,f}(\omega)$, $H_{x_{dz},x}(\omega)$, and $H_{x_{dz},f}(\omega)$) in each system determined the amount of modulation. As the amplitude of the internal feedback force grew relative to the linear internal forces, so did the amount of modulation.

The second property concerns the non-linear coefficients $\mu_i(\omega)$ in equations (34) and (37); they also help to determine how pervasive the non-linearity is in the response. In fact, as $\mu_i(\omega)$ goes to zero, both of the modulations become identities

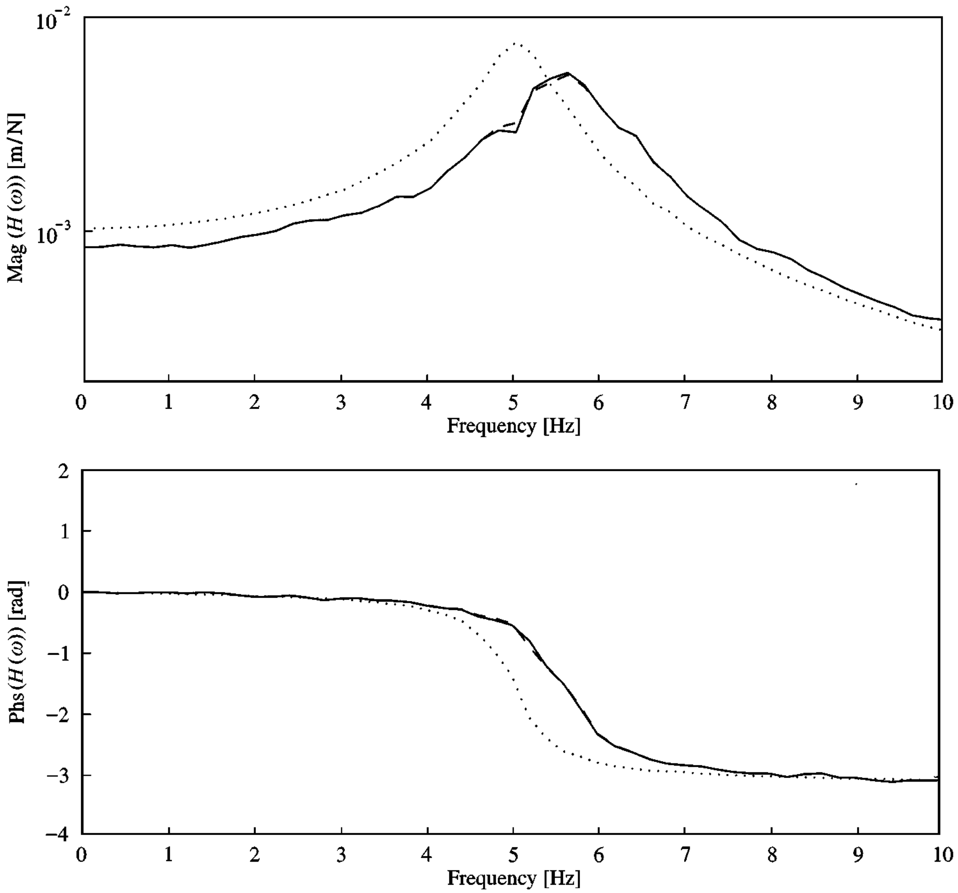

Figure 13. Comparison of the linear FRF, measured FRF, and FRF computed from equation (32) for system 3—hard spring and clearance element to ground; ...., linear FRF; ----, measured FRF; — computed FRF.

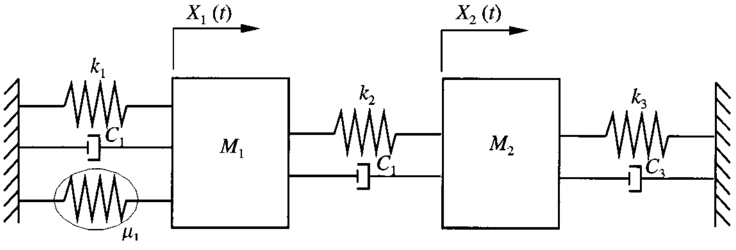

Figure 14. A two-d.o.f. non-linear system with a hard spring to ground.

and the measured FRF matrices become the linear FRF matrices:

$$\lim_{\mu_i(\omega) \to 0} [{}_x H_M(\omega)] = [I] \Rightarrow [{}_x H(\omega)] = [H_L(\omega)]. \quad (40)$$

$$\lim_{\mu_i(\omega) \to 0} [{}_f H_M(\omega)] = [I] \Rightarrow [{}_f H(\omega)] = [H_L(\omega)]. \quad (41)$$

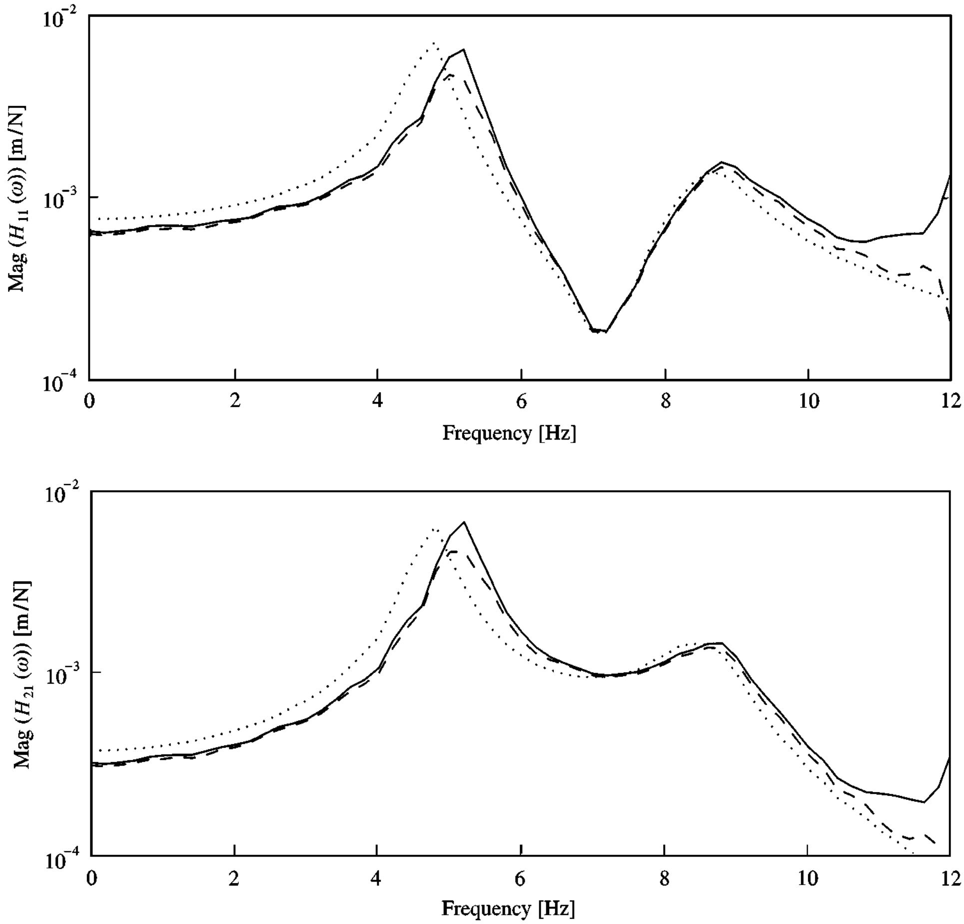

Figure 15. Comparison of the linear FRF, measured FRF, and calculated FRF ($H_{11}(\omega)$ and $H_{21}(\omega)$) of a two-d.o.f. system with a hard spring to ground using a projection onto the outputs (magnitude); ...., linear FRF; ----, measured FRF; — computed FRF.

The third property of the FRF matrices in equations (36) and (39) is that they are non-symmetric for non-linear systems. This property has already been used by the authors to characterize non-linear systems [13].

The three previous properties combine to produce some interesting and useful features of equations (35) and (38). For example, a kind of pseudo-input can be calculated in both of these equations. These pseudo-inputs reduce the equations to the following form:

$$\{X(\omega)\} = [{}_xH_M(\omega)]\,[H_L(\omega)]\,\{F(\omega)\} = [{}_xH_M(\omega)]\,\{X_L(\omega)\}, \tag{42}$$

in which

$$\{X_L(\omega)\} \equiv [H_L(\omega)]\,\{F(\omega)\}, \tag{43}$$

and for the NMMI FRF,

$$\{X(\omega)\} = [H_L(\omega)]\,[{}_fH_M(\omega)]\,\{F(\omega)\}\{X(\omega)\} = [H_L(\omega)]\,\{F_f(\omega)\}, \tag{44}$$

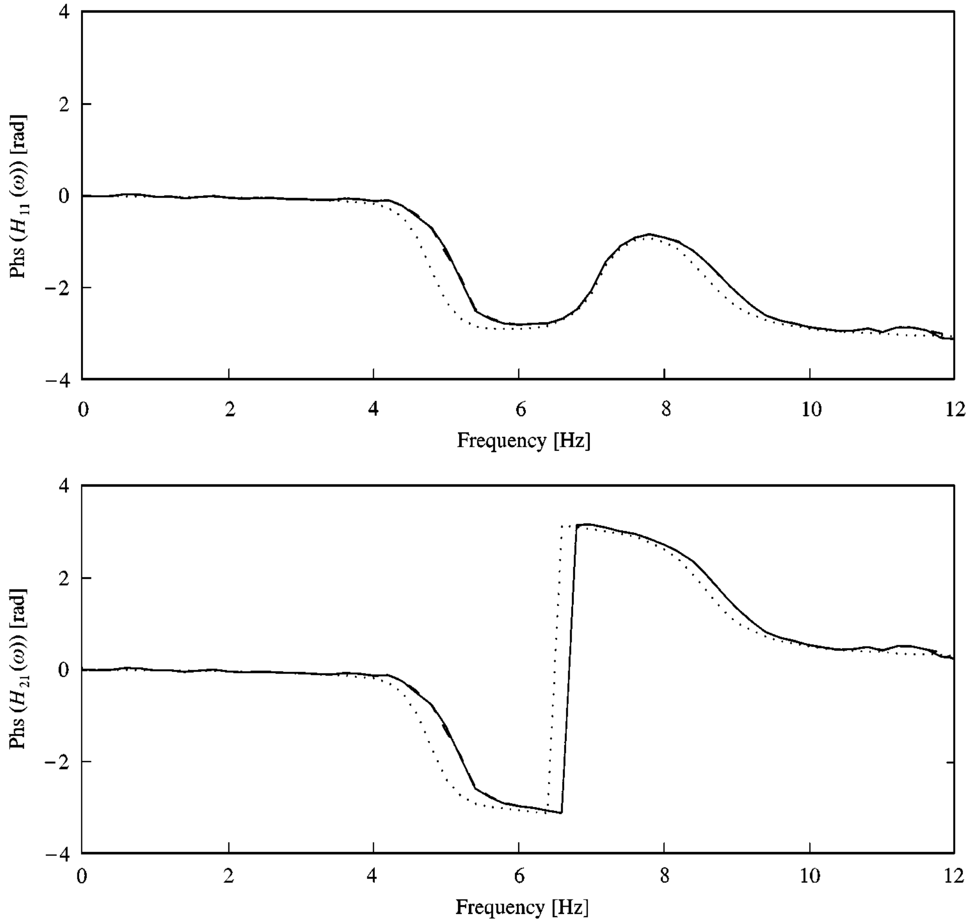

Figure 16. Comparison of the linear FRF, measured FRF, and calculated FRF ($H_{11}(\omega)$ and $H_{21}(\omega)$) of a two-d.o.f. system with a hard spring to ground using a projection onto the outputs (phase); ....., linear FRF; ----, measured FRF; — computed FRF.

in which

$$\{F_f(\omega)\} \equiv [{}_fH_M(\omega)]\,\{F(\omega)\}. \tag{45}$$

These equations have interesting interpretations. For instance, from equation (42) the actual response of the non-linear system can be written as a pure modulation of the response of the linearized system. In other words, for a given measurement time history, the response of the linear system becomes the input to NMMO, which acts like an FRF matrix to produce the measured output. Likewise, the response in equation (44) is found by forcing the system at a modulated version of the original force $\{F_f(\omega)\}$. Equations (35) and (38) can also be viewed as two different rotations and dilations of the FRF matrix of the linear system. This is the basis for the "modulation" terminology. These different interpretations are currently being expanded and investigated.

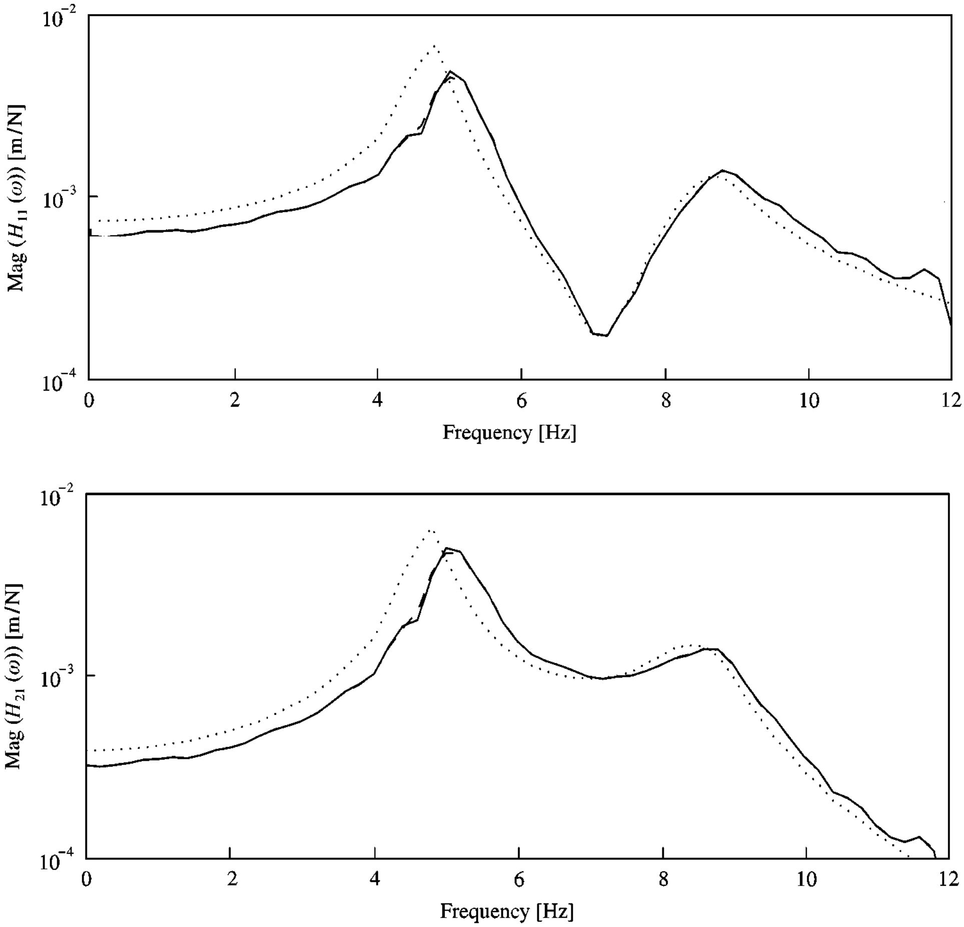

Figure 17. Comparison of the linear FRF, measured FRF, and calculated FRF ($H_{11}(\omega)$ and $H_{21}(\omega)$) of a two-d.o.f. system with a hard spring to ground using a projection onto the input (magnitude); ...., linear FRF; ----, measured FRF; — computed FRF.

One way in which these FRF expressions can be used is to extend the useful operating range of linear input–output FRF models. By modulating the FRF matrix of the linear system with the proposed non-linear model, the parameters of the non-linear model can be found from a least-squares solution in either equation (36) or (39). Then the effects of the non-linearities can be removed to estimate the modal parameters of the underlying linear system. This modulation of the modal parameters can be viewed from the same feedback perspective as before except in this instance the feedback path is in the modal domain. From equation (35) the modal path, from right to left, can be expressed as

$$[{}_xH(\omega)] = [{}_xH_M(\omega)]\,[H_L(\omega)] = [\Phi]_n[\Lambda]_n[L]_n^{\mathrm{T}}[\Phi]\,[\Lambda]\,[L]^{\mathrm{T}}. \tag{46}$$

Figure 19 illustrates the conceptual idea behind this approach. The input travels through the linear system first and then the response from the linear system

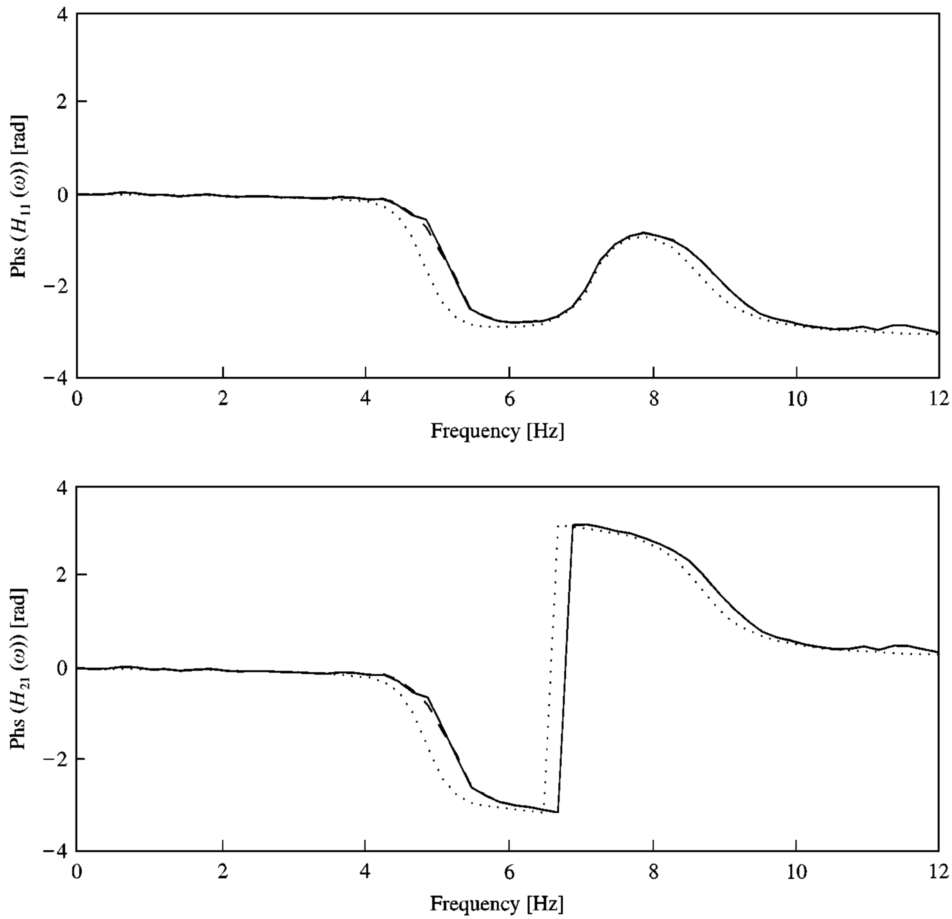

Figure 18. Comparison of the linear FRF, measured FRF, and calculated FRF ($H_{11}(\omega)$ and $H_{21}(\omega)$) of a two-d.o.f. system with a hard spring to ground using a projection onto the input (phase); ...., linear FRF; ----, measured FRF; — computed FRF.

(equation (42)) feeds back via the "modal parameters" of the modulation matrix. Note that the modal parameters of the modulation matrix depend on the amplitudes of the responses and the character of the non-linearities.

A second use for the FRF relationships is to characterize and identify non-linear systems without prior knowledge of the linear FRF matrix. Characterization through modulation is tantamount to a frequency-domain version of the restoring force method. The shape of the modulation matrices determines the presence, type, and location of non-linearities throughout the system. After estimating the linear system FRFs, the parameters of the non-linear model can be estimated directly. Although conditioned spectral analysis can be used to estimate the non-linear parameter of a non-linear system and the FRF matrix of the underlying linear system, a more direct method using equation (11) is being developed for this purpose.

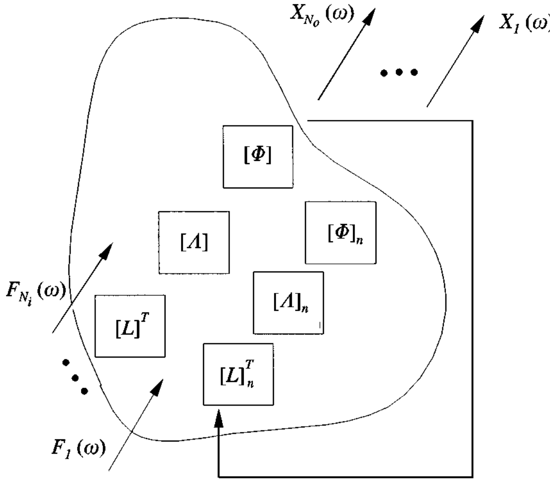

Figure 19. Feedback in the modal domain of the internal forces due to the non-linearities-illustration of the modulation of the modal parameters of a linear model.

5. SUMMARY AND FUTURE WORK

The spatial features of non-linear structural dynamic systems have been used to view non-linearities as internal feedback forces in the framework of a general impedance model. This perspective provided a means to derive two new relationships between the measured FRF matrix of a non-linear system and the FRF matrix of the corresponding linear system. These relationships were used to predict the effects of non-linearities on the FRFs of the linear system for several s.d.o.f. systems and a m.d.o.f. system. The NMMO and NMMI FRF relationships are currently being evaluated for their use in characterizing non-linearities and building non-linear models. The feedback perspective of non-linearities in this research has also been used to simplify certain methods of identification including the reverse-path approach. Further research is underway to use the new relationships in this article to characterize and identify non-linear systems, to extend the linear models of vibration in the presence of non-linearities, and to quantify the effects of non-linearities on estimated modal parameters.

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APPENDIX A: NOMENCLATURE

D.o.f.(s)	degree(s) of freedom
S.d.o.f	single degree of freedom
M.d.o.f	multiple degrees of freedom
SIMO	single input multiple output testing configuration
MIMO	multiple input multiple output testing configuration
N_o	number of output (response) degrees of freedom
N_i	number of input (forced) degrees of freedom at which the input is nonzero; there are N_o total input degrees of freedom
UMPA	unified matrix polynomial approach
ARMA	autoregressive moving average
NARMAX	non-linear autoregressive moving average with exogenous inputs
FRF(s)	frequency response function(s)
$\{x(t)\}_{N_o \times 1}$	measured output time history vector of length N_o
$\{f(\mathrm{t})\}_{N_o \times 1}$	measured input time history vector of length N_o
$\{X(\omega)\}_{N_o \times 1}$	linear Fourier spectrum of the output vector
$\{X_L(\omega)\}_{N_o \times 1}$	linear Fourier spectrum of the output vector of a linear or linearized system
$\{F(\omega)\}_{N_o \times 1}$	linear Fourier spectrum of the input vector
$\{F_f(\omega)\}_{N_o \times 1}$	modulated version of the linear Fourier spectrum of the input vector with N_i non-zero components
$[H_L(\omega)]_{N_o \times N_o}$	frequency response function matrix of a linear or linearized system

$[H(\omega)]_{N_o \times N_o}$	frequency response function matrix of a linear or possibly non-linear system
$[{}_xH(\omega)]_{N_o \times N_o}$	frequency response function matrix estimate using the projection onto the outputs
$[{}_fH(\omega)]_{N_o \times N_o}$	frequency response function matrix extimate using the projection onto the external inputs
$H_{pq}(\omega)$	frequency response function between input degree of freedom q and output d.o.f.p
λ_r	modal frequencies of the linear system, $\sigma_r + j\omega_r$
σ_r	damping factor
ω_r	damped natural frequency
$[\Phi]_{N_o \times 2N}$	matrix of modal vectors of a linear or linearized system
$[\Lambda]_{2N \times 2N}$	matrix of modal frequencies of a linear or linearized system $1/(j\omega - \lambda_r)$
$[L]^T_{2N \times N_o}$	matrix of modal participation factors of a linear or linearized system
$[B_L(\omega)]_{N_o \times N_o}$	impedance matrix of a linear or linearized system
$\mu_i(\omega)$	scalar non-linear parameter for non-linear element i
$X_{ni}(\omega)$	scalar non-linear function of the outputs for non-linear element i
$\{B_{ni}\}_{N_o \times 1}$	Vector of impedance with non-linear coefficient and non-linear spectral function factored out to yield entries of 1 and -1 only; associated with non-linear element i
$[{}_xB_{ni}(\omega)]_{N_o \times 1}$	frequency response (projection) matrix between the outputs and $X_{ni}(\omega)$ associated with non-linear element i
$[{}_fB_{ni}(\omega)]_{N_o \times 1}$	frequency response (projection) matrix between the external inputs and $X_{ni}(\omega)$ associated with the non-linear element i
$H_{g(x),x}(\omega)$	frequency response function between the ouptut of a single-degree-of-freedom system and the non-linear function $g(x)$
$H_{g(x),f}(\omega)$	frequency response function between the input to a single-degree-of-freedom system and the non-linear function $g(x)$
$[{}_xH_M(\omega)]_{N_o \times N_o}$	non-linear modulation matrix on the outputs (NMMO)
$[{}_fH_M(\omega)]_{N_o \times N_o}$	non-linear modulation matrix on the inputs (NMMI)