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ON THE VIBRATION OF ONE-DIMENSIONAL PERIODIC STRUCTURES

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1. INTRODUCTION

A structure is said to be periodic, or repetitive, when its construction takes the form of a spatially repeated cell; a honeycomb sandwich panel is a good example of a two-dimensional (plate-like) periodic structure, whilst examples of one-dimensional (beam-like) periodic structures include rail track supported on equi-spaced sleepers, and trusses employed to provide a large span. A review of the various approaches to their analysis was given by Noor [1] in 1988; more recently, Mead [2] has provided an overview of the contributions made by researchers at the University of Southampton. Noor described four approaches to the analysis of large repetitive structures, these being

(a) *Direct* method in which the complete structure is analyzed as a system of discrete finite elements; computationally this is the least efficient, as the periodicity of the structure is not exploited.
(b) *Direct field* method in which the displacements on either side of the typical cell are related by finite difference equations; this approach has been developed extensively by Renton [3].
(c) *Periodic structure* approach, which typically employs a transfer matrix relating a state vector of displacement and force components on either side of the generic cell; application of Bloch's theorem leads to an eigenvalue problem for propagation constants or frequencies. This approach has been applied successfully to both one- and two-dimensional structures, and the theory is highly developed [4–9].
(d) *Substitute continuum* approach in which the original structure is replaced by a continuum whose properties are in some sense equivalent.

Periodic structures are analysed most efficiently when the periodicity is taken into account; in principle, this allows the behaviour of the complete structure to be determined through analysis of a single cell (and a knowledge of the boundary conditions if the structure is not of infinite extent). Of the above, approaches (b) and (c) do exploit the property of periodicity. On the other hand, the *substitute continuum* approach (d) is appealing for a variety of reasons, particularly if one is interested in the global (rather than local) behaviour of the structure, for example

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vibration and its feedback control, global buckling or thermal conductivity, these being areas where the engineer is accustomed to thinking in terms of continuum properties and theories.

The present paper describes an approach, applied to a one-dimensional structure, which seeks to combine the best features of methods (c) and (d); periodic structure theory is first employed to generate the equivalent continuum stiffness properties of a pin-jointed truss structure; inertia properties are determined by elementary means. These properties are then employed in a variety of continuum theories, for example the Euler–Bernoulli and Timoshenko theories for transverse vibration, in order to predict the natural frequencies of the structure; comparison is made with the predictions of a finite element analysis of the complete structure (Noor's *direct* method), which is taken as the benchmark for accuracy. A variety of truss lengths and end conditions are considered. Agreement is found to be very good.

2. THE PIN-JOINTED FRAMEWORK AND ITS CONTINUUM PROPERTIES

A planar one-dimensional (beam-like) pin-jointed framework is shown in Figure 1. Each member in the framework is of material having Young's modulus $E = 200 \times 10^9$ N/m^2, density $\rho = 8000$ kg/m^3. Horizontal and vertical members are of length 1 m, and have cross-sectional area 1 cm^2; diagonal members have length $\sqrt{2}$ m. and cross-sectional area 0·5 cm^2. These lengths, together with Young's modulus, are regarded as being equally applicable to the continuum beam.

The procedure, for finding the equivalent stiffness continuum properties is described fully in reference [10], and may be summarized as follows:

(a) Consider the single generic jth cell in Figure 1; ascribe half of the vertical pin-jointed members to the preceding and following cells, so that the vertical members are treated as having cross-sectional area 0·5 cm^2.

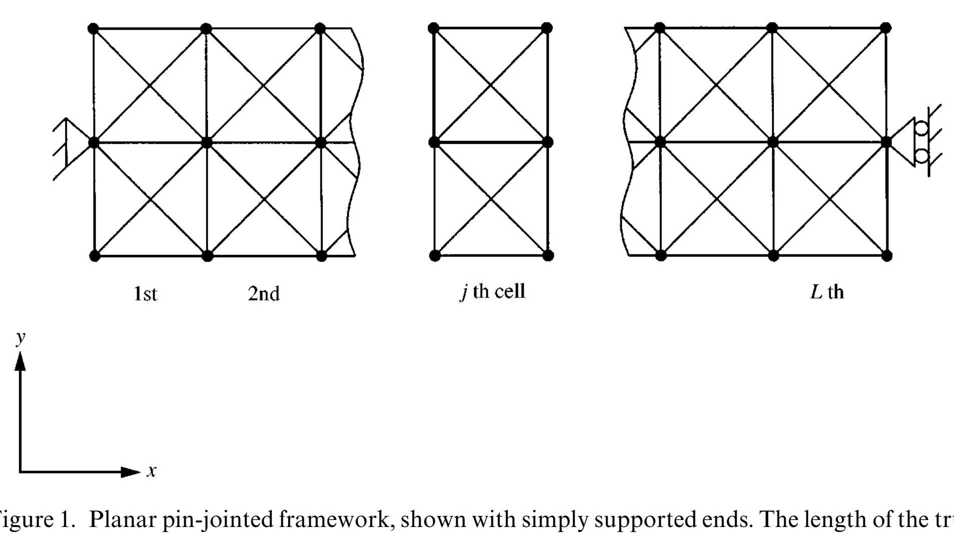

Figure 1. Planar pin-jointed framework, shown with simply supported ends. The length of the truss is equal to the number of cells, L.

(b) Construct the stiffness matrix **K** of the cell, and then partition and manipulate to find the transfer matrix **G**, which has the properties of being symplectic, defective and derogatory.

(c) Rigid-body displacements and the transmission modes of tension, bending moment and shear are associated with the multiple eigenvalue $\lambda = 1$. The eigenvectors pertain to rigid body displacements in the axial (x) and transverse (y) directions. Coupled to these are principal vectors (or generalised eigenvectors) which describe rigid-body rotation, and the force and moment transmission modes; these are found using the reduced row echelon form, which relates nodal force and displacement components in their simplest form.

The equivalent continuum properties are found to be: cross-sectional area $A = 3{\cdot}5224\ \mathrm{cm}^2$, second moment of area $I = 2{\cdot}13061 \times 10^{-4}\ \mathrm{m}^4$, Poisson's ratio $v = 0{\cdot}2612$, shear modulus $G = E/2(1+v) = 79{\cdot}29 \times 10^9\ \mathrm{N/m}^2$ and shear coefficient $\kappa = 0{\cdot}4956$.

The inertia properties are found as follows:

(a) The mass per unit length m is calculated simply as the sum of the individual masses of the members which constitute the generic cell, which has length 1 m. Thus, the cell consists of three horizontal members having length 1 m and cross-sectional 1 cm^2, four diagonal members having length $\sqrt{2}$ m and cross-sectional area $0{\cdot}5\ \mathrm{cm}^2$, and four vertical members having length 1 m and cross-sectional area $0{\cdot}5\ \mathrm{cm}^2$. The resulting mass per unit length is calculated as 6·2627 kg/m; note that this property is exact (as opposed to equivalent) in the sense that it could equally be obtained by weighting the generic cell.

(b) The rotary moment of inertia of the cell is calculated as the sum of the individual moments of inertia of the members about the centre of gravity of the cell; this involves only the simple formula $J = mL^2/12$ for the moment of inertia of the individual member about its own centre of gravity, and application of the parallel axis theorem where appropriate. The rotary moment of inertia per unit length is thus calculated as $J = 3{\cdot}6761$ kg m, and again this property is exact.

3. CONTINUUM THEORIES

3.1. FLEXURAL VIBRATION

The two coupled equations of Timoshenko theory for a continuum beam are normally written as

$$-\kappa AG\left(\frac{\partial\phi}{\partial x} - \frac{\partial^2 v}{\partial x^2}\right) = \rho A\frac{\partial^2 v}{\partial t^2}, \qquad EI\frac{\partial^2\phi}{\partial x^2} - \kappa AG\left(\phi - \frac{\partial v}{\partial x}\right) = \rho I\frac{\partial^2\phi}{\partial t^2}. \quad \text{(1a,b)}$$

For the continuum model of the periodic truss, the right-hand side inertia parameters are modified to read

$$-\kappa AG\left(\frac{\partial\phi}{\partial x} - \frac{\partial^2 v}{\partial x^2}\right) = m\frac{\partial^2 v}{\partial t^2}, \qquad EI\frac{\partial^2\phi}{\partial x^2} - \kappa AG\left(\phi - \frac{\partial v}{\partial x}\right) = J\frac{\partial^2\phi}{\partial t^2}. \quad \text{(2a,b)}$$

where m and J are mass and moment of inertia, respectively, per unit length. These changes are necessary as the cross-sectional area, A, is the equivalent property pertaining to axial stiffness (EA/L); similarly the second moment of area, I, is the equivalent property pertaining to the flexural stiffness (EI).

The single fourth order differential equation becomes

$$EI\frac{\partial^4 v}{\partial x^4} + m\frac{\partial^2 v}{\partial t^2} - \left(J + \frac{EIm}{\kappa AG}\right)\frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{mJ}{\kappa AG}\frac{\partial^4 v}{\partial t^4} = 0. \tag{3}$$

For a simply supported beam, the centre line deflection curve is

$$v(x, t) = \sin\frac{n\pi x}{L}\sin\omega t, \quad n = 1, 2, 3, \ldots \text{ etc.} \tag{4}$$

when the frequency predictions become:

1 *Timoshenko beam*: the lower root of

$$\frac{mJ}{\kappa AG}\omega^4 - \left(m + \left(\frac{n\pi}{L}\right)^2\left(J + \frac{EIm}{\kappa AG}\right)\right)\omega^2 + EI\left(\frac{n\pi}{L}\right)^4 = 0. \tag{5}$$

2 *Rayleigh beam*: set the inverse of the shear coefficient κ equal to zero in equation (5), to give

$$\omega = \left(\frac{n\pi}{L}\right)^2\sqrt{\frac{EI}{m + (n\pi/L)^2 J}}. \tag{6}$$

3 *Shear beam*: set the moment of inertia J equal to zero in equation (5), to give

$$\omega = \left(\frac{n\pi}{L}\right)^2\sqrt{\frac{EI}{m(1 + (n\pi/L)^2 EI/\kappa AG)}}. \tag{7}$$

4 *Euler–Bernoulli beam*: apply both of the above simplifications, to give

$$\omega = \left(\frac{n\pi}{L}\right)^2\sqrt{\frac{EI}{m}}. \tag{8}$$

3.2. LONGITUDINAL VIBRATION

The Raleigh differential equation governing longitudinal vibration of a continuum rod, including the transverse inertia correction for Poisson's ratio contraction is (see Love [11])

$$EA\frac{\partial^2 u}{\partial x^2} = \rho A\left(\frac{\partial^2 u}{\partial t^2} - v^2 r_g^2\frac{\partial^4 u}{\partial x^2 \partial t^2}\right) = \rho A\frac{\partial^2 u}{\partial t^2} - v^2 I\rho\frac{\partial^4 u}{\partial x^2 \partial t^2}, \tag{9}$$

where r_g is the radius of gyration. For the continuum model of the periodic structure, the inertia terms are modified, to read

$$EA\frac{\partial^2 u}{\partial x^2} = m\frac{\partial^2 u}{\partial t^2} - v^2 I\frac{m}{A}\frac{\partial^4 u}{\partial x^2 \partial t^2}. \tag{10}$$

For a fixed–fixed rod, the mode shape is

$$u(x, t) = \sin\frac{n\pi x}{L}\sin\omega t, \quad n = 1, 2, 3, \dots \text{ etc.} \tag{11}$$

when the frequency predictions becomes:

1. *Rayleigh*:

$$\omega = \frac{n\pi}{L}\sqrt{\frac{EA}{m + (n\pi/L)^2 v^2 Im/A)}}. \tag{12}$$

2. *Bernoulli*: set the second moment of area I equal to zero in equation (12), to give

$$\omega = \frac{n\pi}{L}\sqrt{\frac{EA}{m}}. \tag{13}$$

The end conditions employed in all of the above continuum theories lead to a (co)sinusoidal mode shape and, consequently, to relatively simple expressions for the natural frequencies. As will become apparent in the discussion which follows, best agreement with the finite element predictions is provided by the shear beam, and the Rayleigh rod theories, for bending and extensional vibration, respectively. In addition to the above end conditions, a comparison has been made for a truss having free–free end conditions, using these "best" theories. For extensional vibration, it is noted that equation (12) is equally applicable to free–free end conditions; the frequency equation of the shear beam theory, with free–free ends, is recorded as

$$(r^2 - s^2)\sinh rL \sin sL - 2rs\cosh rL \cos sL + 2rs = 0, \tag{14}$$

where

$$r = \frac{k}{2}\sqrt{-2k^2 q + 2\sqrt{k^4 q^2 + 4}},$$

$$s = \frac{k}{2}\sqrt{2k^2 q + 2\sqrt{k^4 q^2 + 4}},$$

$$k = \sqrt[4]{\frac{m\omega^2}{EI}}, \qquad q = \frac{EI}{\kappa AG},$$

4. DISCUSSION OF RESULTS

Consider first the bending frequency predictions in Table 1 for a slender simply supported truss having length $L = 30$ m, and regard the ANSYS predictions as the benchmark. The Euler–Bernoulli (EB) prediction is fairly accurate for the first two frequencies, and quite inaccurate for the higher modes, which are not listed. The third mode, for which the EB prediction is an unacceptable overestimate of 14·6%, has a wavelength $\Lambda = 20$ m, which is ten-fold the beam depth. For a continuum beam of the same depth, the radius of gyration is $r_g = (3)^{-1/2} = 0{\cdot}5774$ and

TABLE 1

Comparison of natural frequencies (Hz) *in bending*; *simply supported*, $L = 30$ m

n	ANSYS	Euler–Bernoulli beam	Rayleigh beam	Shear beam	Timoshenko beam
1	4·4730	4·5526 (+1·78%)	4·5380 (+1·45%)	4·4777 (+0·09%)	4·4642 (−0·20%)
2	17·0425	18·2105 (+6·85%)	17·9805 (+5·5%)	17·0929 (+0·30%)	16·9242 (−0·69%)
3	35·7534	40·9737 (+14·6%)	39·8360 (+11·42%)	35·8833 (+0·36%)	35·2812 (−1·32%)
4	58·5776	72·8420 (+24·35%)	69·3579 (+18·40%)	58·6946 (+0·20%)	57·4330 (−1·95%)
5	83·9620	113·8157 (+35·56%)	105·6331 (+25·81%)	83·8148 (−0·18%)	81·8261 (−2·54%)
6	110·8810	—	—	110·1137 (−0·69%)	107·4627 (−3·08%)
7	138·7053	—	—	136·9263 (−1·28%)	133·7426 (−3·58%)
8	167·0633	—	—	163·8902 (−1·90%)	160·3163 (−4·04%)
9	195·7394	—	—	190·8208 (−2·51%)	186·9847 (−4·47%)
10	224·6075	—	—	217·6315 (−3·11%)	213·6373 (−4·88%)

$r_g/\Lambda = 0{\cdot}0289$; for such a ratio one would expect EB theory to provide acceptable predictions. For the pseudo-continuum truss model, however, the equivalent radius of gyration has the value $r_g = 0{\cdot}7777$ when calculated as the ratio of equivalent second moment of area to equivalent area, or $r_g = 0{\cdot}7662$ when calculated as the ratio of moment of inertia to mass (both per unit length); thus the truss is not as *slender* as its overall dimensions first suggest. Moreover, the parameter $E/\kappa G$ is approximately 3 for a continuum rectangular cross-section, while the equivalent parameter for the truss model is approximately 5, indicating a cross-section which is much more flexible in shear. Both these factors contribute to the failure of EB theory to provide accurate frequency predictions at relatively low mode numbers. For all modes the shear beam model provides best agreement (error less than 0·4%), overestimating at the lower and underestimating at the higher modes. The Timoshenko model consistently underestimates, with a maximum error of −2·54%. Comparing the effect of rotary inertia alone (Rayleigh), with the effect of shear, one concludes that the latter is 3 to 4 times more important in terms of depressing the EB frequency.

Next consider the predictions for the short beam in Table 2: as might be expected from the above discussion, the EB prediction is inaccurate even for the lowest

TABLE 2

Comparison of natural frequencies (Hz) *in bending*; *simply supported*, $L = 10$ m

n	ANSYS	Euler–Bernoulli beam	Rayleigh beam	Shear beam	Timoshenko beam
1	35·6391	40·9737 (+14·97%)	39·8360 (+11·78%)	35·8833 (+0·7%)	35·2812 (−1·00%)
2	110·0462	163·8946 (+48·93%)	147·6748 (+34·19%)	110·1137 (+0·06%)	107·4627 (−2·35%)
3	193·6421	368·7629	298·9686	190·8208 (−1·46%)	186·9847 (−3·44%)
4	278·7818	655·5784	472·2713	270·7825 (−2·87%)	266·6919 (−4·34%)
5	362·9798	1024·3	654·6507	349·3771 (−3·75%)	345·4376 (−4·83%)

mode, and the inclusion of shear is essential for accuracy. The effect of shear is now 2 to 3 times more important than that of rotary inertia. Again the shear beam provides the best accuracy (+0·7 to −3·75%), while the Timoshenko model consistently underestimates (−1 to −4·83%).

The predictions for extensional vibration are shown in Tables 3 and 4; for the slender rod, Table 3, the elementary Bernoulli theory provides excellent agreement (−0·2 to +0·5%), and inclusion of the effects of Poisson's ratio contraction improves accuracy still further (−0·04 to −0·08%). However, it is doubtful whether this improvement is worthwhile, if one bears in mind that the 1st, 2nd, 3rd, 4th and 5th extensional modes are, respectively, the 4th, 7th, 10th, 13th and 16th vibration modes of the truss, the remainder in this range being bending modes not predictable to such a high degree of accuracy. For the short rod, Table 4, agreement is again very good; the inclusion of Poisson's ratio contraction does, in the main, improve accuracy, but again this is probably not worthwhile given that the 1st, 2 and 3rd extensional modes are respectively, the 3rd, 6th and 10th vibration modes of the truss; the remainder are bending modes, with the exception of the 8th modes which is characterized by axial shear. Extensional modes higher than the 3rd are not listed, as the ANSYS mode shapes indicate that one of the essential characteristics of extensional vibration—that of plane sections remaining plane—becomes unrealistic at higher modes.

Last, consider the first 20 natural frequencies of a free–free truss having length $L = 30$ m, Table 5: only the Rayleigh rod and shear beam predictions are presented, as these generally provided best agreement with the ANSYS predictions which are again regarded as the benchmark. Of these 20 modes, 6 are extensional in nature and can be predicted using the Rayleigh rod theory to an accuracy of approximately ±0·5%. The remaining 14 modes are flexural in nature and the shear beam model provides agreement within approximately ±2·5%. At frequencies

TABLE 3

Comparison of natural frequencies (Hz) *in extension*; *fixed-fixed*, $L = 30$ m

n	ANSYS	Bernoulli rod	Rayleigh rod
1	55·9094	55·8985 (−0·02%)	55·8858 (−0·04%)
2	111·7594	111·7970 (+0·03%)	111·6959 (−0·06%)
3	167·4796	167·6955 (+0·13%)	167·3550 (−0·07%)
4	222·9749	223·5939 (+0·28%)	222·7888 (−0·08%)
5	278·1058	279·4924 (+0·5%)	277·9246 (−0·07%)

TABLE 4

Comparison of natural frequencies (Hz) *in extension*; *fixed-fixed*, $L = 10$ m

n	ANSYS	Bernoulli rod	Rayleigh rod
1	167·6177	167·6955 (+0·05%)	167·3550 (−0·16%)
2	333·0602	335·3909 (+0·70%)	332·6918 (−0·11%)
3	489·5158	503·0864 (+2·77%)	494·1116 (+0·94%)

higher than those listed in Table 5, there is an increased density of modes, most of which cannot be categorized simply as flexural or extensional. This occurs as the semi-wavelength of bending vibration, which is approximately 2 m for the 14th bending mode, approaches the depth of the truss, so allowing the possibility of depth-wise modes of vibration; this defines the extent to which the present one-dimensional approach is useful for the prediction of natural frequencies.

5. CONCLUSIONS

Comparisons with finite element predictions indicate that the combined approach employing both periodic structure theory and well-known one-dimensional continuum theories suitably modified, can provide very good accuracy, at least for the pin-jointed trusses considered here. For bending vibration, where the inclusion of shear deformation is essential, comparison suggests that the shear beam model, which neglects rotary inertia, provides the best accuracy. However, it should be borne in mind that the consistent mass matrix formulation

TABLE 5

Comparison of first 20 *natural frequencies* (Hz) *of free–free truss*, $L = 30$ m; *E denotes extensional modes, as indicated by the ANSYS mode shape*

Mode	ANSYS	Shear beam	Rayleigh rod
1	9·924	10·1096	—
2	25·822	26·4373	—
3	47·032	48·2015	—
4	55·651 (E)	—	55·8858
5	71·584	73·1775	—
6	98·164	99·8432	—
7	111·239 (E)	—	111·6959
8	125·912	127·2510	—
9	154·302	154·8795	—
10	166·69 (E)	—	167·3550
11	183·01	182·4645	—
12	211·826	209·8849	—
13	221·903 (E)	—	222·7888
14	240·587	237·0946	—
15	269·121	264·0853	—
16	276·736 (E)	—	277·9246
17	297·14	290·8663	—
18	324·073	317·4548	—
19	330·972 (E)	—	332·6918
20	347·176	343·8698	—

employed in the finite element analysis leads to an overestimate of the natural frequencies and it is quite possible that the Timoshenko predictions, which are consistently lower than the ANSYS values, are the more accurate. Future work will extend the approach to rigid-jointed frameworks, for which micro-polar continuum descriptions may be necessary, and to provide experimental comparisons which are, of course, the ultimate benchmark.

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