Journal of Sound and Vibration (1999) **228**(1), 109–128
Article No. jsvi.1999.2398, available online at http://www.idealibrary.com on IDEAL®

SENSITIVITY OF PLANETARY GEAR NATURAL FREQUENCIES AND VIBRATION MODES TO MODEL PARAMETERS

J. LIN AND R. G. PARKER

Department of Mechanical Engineering, The Ohio State University Columbus, OH 43210, *U.S.A.*

(*Received* 8 *February* 1999, *and in final form* 25 *May* 1999)

The natural frequency and vibration mode sensitivities to system parameters are rigorously investigated for both tuned (cyclically symmetric) and mistuned planetary gears. Parameters under consideration include support and mesh stiffnesses, component masses, and moments of inertia. Using the well-defined vibration mode properties of tuned planetary gears, the eigensensitivities are calculated and expressed in simple, exact formulae. These formulae connect natural frequency sensitivity with the modal strain or kinetic energy and provide efficient means to determine the sensitivity to all stiffness and inertia parameters by inspection of the modal energy distribution. The natural frequency sensitivity to operating speed is calculated to estimate the impact of gyroscopic effects. While the terminology of planetary gears is used throughout, the results apply for general epicyclic gears.

1. INTRODUCTION

Dynamic analysis of planetary (epicyclic) gears is essential for the reduction of noise and vibration in helicopters, cars, turbo-prop/turbofan engines, and other power transmission systems. Sensitivity of the natural frequencies and vibration modes to system parameters provides important information for tuning resonances away from operating speeds, minimizing response, and optimizing the structural design. In industrial practice, planetary gear design balances many objectives in selecting component inertias and support/mesh stiffnesses. Load sharing among the planets, avoidance of resonance, and weight reduction are three important examples. The design process needs to balance how changes in key design parameters for these (or other) purposes alter the modal properties and impact dynamic response. This work identifies remarkably compact, closed-form eigensensitivity relations that can guide these decisions.

The influence of selected design parameters on planetary gear natural frequencies was touched on in a few papers. Botman [1] and Cunliffe *et al.* [2] both presented plots of natural frequencies versus planet bearing stiffness. Botman also studied the effect of carrier rotation through a numerical example. Kahraman [3] showed the effects of mesh/support stiffnesses on the natural frequencies in his torsional model

0022-460X/99/460109 + 20 $30.00/0

of planetary gears. Saada and Velex [4] discussed the influence of the ring support stiffness on free vibration. These analyses are based on parametric studies of example planetary gears. Systematic analysis of the eigensensitivities of general planetary gears has not been found in the published literature. In addition, most previous dynamic analyses assume the planetary gears to be cyclically symmetric (tuned) systems where all planets are identical and equally spaced. In practical applications, however, planetary gears may be mistuned by differing stiffnesses at the multiple tooth meshes, manufacturing variations, and assembling errors. Eigensensitivity analysis for a mistuned parameter can identify modes that are especially susceptible to irregularity. Frater *et al.* [5] studied the vibration modes with a single mistuned mesh stiffness, but general conclusions were not presented.

The objective of this paper is to analytically investigate the natural frequency and vibration mode sensitivity to the system design parameters, including mesh/support stiffnesses, component masses, and moments of inertia. Lin and Parker [6] rigorously characterized the highly structured natural frequency and vibration mode properties of planetary gears. Taking advantage of these properties, exact, simple formulae are obtained to calculate eigensensitivities when the perturbed system is tuned or mistuned. The natural frequency sensitivities are closely related to the modal strain and kinetic energy distributions such that eigensensitivity is calculable by inspection of these energy distributions. The lumped-parameter model of planetary gears is introduced first. The general eigensensitivity relations for distinct and degenerate natural frequencies are then formulated and applied to the three classes of planetary gear vibration modes. The derived results are applicable for general epicyclic gears with any number of planets. The application of these relations is illustrated through examples. Finally, the natural frequency sensitivity to the operating speed is analyzed and used to estimate how gyroscopic effects alter the natural frequency spectrum.

2. SYSTEM MODEL

The lumped-parameter model of a planetary gear derived in reference [6] is the basis for this study. Figure 1 shows a planetary gear with N planets. Each of the carrier, ring, sun, and planets has two translational and one rotational degree of freedom, so the system has $L = 3(N + 3)$ degrees of freedom. The system equation for free vibration is

$$\mathbf{M}\ddot{\mathbf{q}} + \Omega_c \mathbf{G}\dot{\mathbf{q}} + [\mathbf{K}_b + \mathbf{K}_m - \Omega_c^2 \mathbf{K}_\Omega]\mathbf{q} = \mathbf{0},$$

$$\mathbf{q} = (\underbrace{x_c, y_c, u_c}_{carrier}, \underbrace{x_r, y_r, u_r}_{ring}, \underbrace{x_s, y_s, u_s}_{sun}, \underbrace{\zeta_1, \eta_1, u_1}_{planet\ 1}, \ldots, \underbrace{\zeta_N, \eta_N, u_N}_{planet\ N})^{\mathrm{T}}, \tag{1}$$

where $\mathbf{M}$ is the inertia matrix, $\mathbf{K}_b$ is the support (bearing) stiffness matrix, and $\mathbf{K}_m$ is the mesh stiffness matrix, which is taken to be time invariant. The gyroscopic matrix $\mathbf{G}$ and centripetal stiffness matrix $\mathbf{K}_\Omega$ result from carrier rotation. $\mathbf{M}$, $\mathbf{K}_b$, $\mathbf{K}_m$ and $\mathbf{K}_\Omega$ are symmetric and $\mathbf{G}$ is skew-symmetric. $x_h, y_h, h = c, r, s$ denote the translations of the carrier, ring, and sun, and $\zeta_n, \eta_n, n = 1, \ldots, N$ are the radial and tangential translations of planet n. $u_h = r_h\theta_h, h = c, r, s, 1, \ldots, N$ are rotational co-ordinates, where θ_h is the rotation angle of component h; r_h is the base circle radius for the sun, ring and planets, and the radius of the circle passing through the planet centers for the carrier. All co-ordinates are with respect to a basis fixed to the carrier and rotating with the constant carrier rotation speed Ω_c. Appendix A lists the nomenclature used throughout.

The associated eigenvalue problem of (1) is obtained from the separable solution $\mathbf{q} = \phi_i e^{j\omega_i t}$:

$$[-\omega_i^2\mathbf{M} + j\omega_i\Omega_c\mathbf{G} + (\mathbf{K} - \Omega_c^2\mathbf{K}_\Omega)]\phi_i = \mathbf{0}, \tag{2}$$

where $j = \sqrt{-1}$ and $\mathbf{K} = \mathbf{K}_b + \mathbf{K}_m$. The eigensensitivity analysis calculates natural frequency and vibration mode derivatives with respect to stiffnesses, masses, moments of inertia, and operating speed. Eigensensitivity to stiffness and inertia parameters are examined in the absence of gyroscopic effects ($\Omega_c = 0$), where equation (2) reduces to

$$(\mathbf{K} - \lambda_i\mathbf{M})\phi_i = \mathbf{0} \tag{3}$$

and $\lambda_i = \omega_i^2$. Gyroscopic effects are important in high-speed applications, and eigensensitivity with respect to Ω_c is studied later in this paper. The eigensensitivity for problems in form (3) has been investigated in references [7–12] and the necessary results are introduced below. The unique modal properties of planetary gears are then invoked to reduce these general results to simple expressions specific to planetary gears.

3. CALCULATION OF EIGENSENSITIVITY

Let $(\ )'$ and $(\ )''$ denote the first and second derivatives with respect to a model parameter (i.e., mesh/support stiffness, component mass, or moment of inertia). For simplicity, we calculate the eigenvalue derivatives λ_i' and λ_i''; the relations $\omega_i' = \lambda_i'/(2\omega_i)$ and $\omega_i'' = (2\lambda\lambda_i'' - \lambda_i'^2)/(4\omega_i^3)$ yield the natural frequency sensitivities. For a distinct eigenvalue, the eigensensitivities are [7, 12]

$$\lambda_i' = \phi_i^{\mathrm{T}}(\mathbf{K}' - \lambda_i\mathbf{M}')\phi_i, \tag{4}$$

$$\phi_i' = -\frac{1}{2}(\phi_i^{\mathrm{T}}\mathbf{M}'\phi_i)\phi_i + \sum_{k=1, k\neq i}^{L} \frac{\phi_k^{\mathrm{T}}(\mathbf{K}' - \lambda_i\mathbf{M}')\phi_i}{\lambda_i - \lambda_k}\phi_k \tag{5}$$

$$\lambda_i'' = 2\phi_i^{\mathrm{T}}(\mathbf{K}' - \lambda_i\mathbf{M}')\phi_i + \phi_i^{\mathrm{T}}(\mathbf{K}'' - \lambda_i\mathbf{M}'' - 2\lambda_i'\mathbf{M}')\phi_i. \tag{6}$$

For the case of degenerate eigenvalues, consider a system having a group of eigenvalues $\lambda_1 = \cdots = \lambda_m$ with multiplicity m. We select an arbitrary set of independent eigenvectors $\Gamma = [\gamma_1, \ldots, \gamma_m]$ associated with this degenerate eigenvalue and normalize them such that $\Gamma^{\mathrm{T}}\mathbf{M}\Gamma = \mathbf{I}_{m\times m}$. The transformation $\Phi = \Gamma\mathbf{A}$ produces another set of eigenvectors $\Phi = [\phi_1, \ldots, \phi_m]$, where the matrix $\mathbf{A}$ is to be determined. Note that $\phi_i = \Gamma\mathbf{a}_i$ where $\mathbf{a}_i$ is the ith column of $\mathbf{A}$. Differentiation of equation (3) gives

$$(\mathbf{K} - \lambda_i\mathbf{M})\phi_i' = (\lambda_i'\mathbf{M} + \lambda_i\mathbf{M}' - \mathbf{K}')\Gamma\mathbf{a}_i = \mathbf{f}. \tag{7}$$

The m solvability conditions $\gamma_1^T\mathbf{f} = \cdots = \gamma_m^T\mathbf{f} = 0$ of equation (7) yield the symmetric $m \times m$ eigenvalue problem

$$\mathbf{D}\mathbf{a}_i = \lambda_i'\mathbf{a}_i, \qquad \mathbf{D} = \Gamma^T(\mathbf{K}' - \lambda_i\mathbf{M}')\Gamma. \tag{8}$$

The first order eigenvalue derivatives λ_i' are the eigenvalues of equation (8). For the case when all λ_i' obtained from equation (8) are distinct, the $\mathbf{a}_i$ are uniquely obtained with the normalization $\mathbf{a}_i^T\mathbf{a}_i = 1$. This procedure determines the set of independent eigenvectors $\phi_i = \Gamma\mathbf{a}_i$ that admit continuous change of the eigenvectors as the degenerate eigenvalues split into distinct ones when a parameter is varied. The eigenvector derivatives for distinct λ_i' are expressed as

$$\phi_i' = \mathbf{v}_i + \Phi\mathbf{c}_i, \quad i = 1, \ldots, m. \tag{9}$$

The term $\Phi\mathbf{c}_i$ is associated with the solution of the homogeneous form of equation (7) with the $\mathbf{c}_i$ to be determined. $\mathbf{v}_i$ is a particular solution of equation (7) so that

$$(\mathbf{K} - \lambda_i\mathbf{M})\mathbf{v}_i = \mathbf{f}. \tag{10}$$

The matrix $(\mathbf{K} - \lambda_i\mathbf{M})$ is singular with rank $L - m$. Many papers [9–11] apply Nelson's method [8] to calculate $\mathbf{v}_i$ by assigning m arbitrary elements of $\mathbf{v}_i$ to be zero. In order to get a closed-form expression for $\mathbf{v}_i$, however, we use an alternate method where $\mathbf{v}_i$ is expressed as a linear combination of the eigenvectors $\phi_{m+1}, \ldots, \phi_L$ (the contributions of $\phi_1, \ldots, \phi_m$ to ϕ_i' are included in the term $\Phi\mathbf{c}_i$):

$$\mathbf{v}_i = \sum_{k=m+1}^{L} d_{ki}\phi_k. \tag{11}$$

Substitution of equation (11) into equation (10) and premultiplication by ϕ_k^T yield the solution for $\mathbf{v}_i$,

$$d_{ki} = \frac{\phi_k^T(\lambda_i\mathbf{M}' - \mathbf{K}')\phi_i}{\lambda_k - \lambda_i}, \quad k = m+1, \ldots, L. \tag{12}$$

It remains to calculate $\mathbf{c}_i$ of equation (9). Denoting $\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_m]$, the diagonal terms are obtained by differentiation of the mass normalization equation $\Phi^T\mathbf{M}\Phi = \mathbf{I}_{m\times m}$ and use of equation (9):

$$C_{ii} = -\tfrac{1}{2}\,\phi_i^T\mathbf{M}'\phi_i, \quad i = 1, \ldots, m, \tag{13}$$

where $\mathbf{C}_{ij}$ is the (i, j)th element of $\mathbf{C}$. From the second derivative of equation (3), the off-diagonal terms of $\mathbf{C}$ and the second order eigenvalue derivatives are [9–11]

$$C_{ji} = \frac{2\phi_j^T(\mathbf{K}' - \lambda_i\mathbf{M}')\mathbf{v}_i - \phi_j^T(2\lambda_i'\mathbf{M}' - \mathbf{K}'' + \lambda_i\mathbf{M}'')\phi_i}{2(\lambda_i' - \lambda_j')}, \quad i \neq j, \tag{14}$$

$$\lambda_i'' = 2\phi_i^T(\mathbf{K}' - \lambda_i\mathbf{M}')\mathbf{v}_i - \phi_i^T(2\lambda_i'\mathbf{M}' - \mathbf{K}'' + \lambda_i\mathbf{M}'')\phi_i, \quad i = 1, \ldots, m. \tag{15}$$

For the case when all λ_i' obtained from equation (8) are degenerate (i.e., the degenerate λ_i do not separate), the $\mathbf{a}_i$ are not unique and hence ϕ_i, $i = 1, \ldots, m$ are arbitrary in the eigenspace. The eigenvector derivatives ϕ_i' cannot be determined. However, λ_i'' can be obtained from the eigenvalues of [11]

$$\mathbf{E} = 2\Phi^T(\mathbf{K}' - \lambda_i\mathbf{M}')\mathbf{V} + \Phi^T(\mathbf{K}'' - \lambda_i\mathbf{M}'' - 2\lambda_i'\mathbf{M}')\Phi, \tag{16}$$

where $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_m]$ is determined by equations (11) and (12). These λ_i'' are not affected by the selection of Φ.

The foregoing development is used subsequently to derive general, closed-form eigensensitivity relations for λ_i', ϕ_i' and λ_i'' for planetary gears. These expressions yield eigensolution approximations according to

$$\tilde{\lambda} = \lambda + \sum_{\rho} \left.\frac{\partial \lambda}{\partial \rho}\right|_{\rho=\rho_0} (\rho - \rho_0), \quad \tilde{\phi} = \phi + \sum_{\rho} \left.\frac{\partial \phi}{\partial \rho}\right|_{\rho=\rho_0} (\rho - \rho_0), \tag{17}$$

where ρ represents any system parameter with nominal value ρ_0 and multiple parameter perturbations are permitted. Eigensolutions λ, ϕ are for a nominal set of model parameters referred to as the *unperturbed* system, and the derivatives are evaluated for this unperturbed system. Eigensolutions $\tilde{\lambda}$, $\tilde{\phi}$ are for the *perturbed* system with varied parameters. Two types of systems are considered. *Tuned* systems are those where all planets are identical and equally spaced, all sun–planet mesh stiffnesses are equal, and all ring-planet mesh stiffnesses are equal. *Mistuned* systems are those where one or more planet parameters violate this cyclic symmetry. The unperturbed system is taken to be tuned in this study. Note that this does not meaningfully restrict the results because parameter variations leading to both tuned and mistuned perturbed systems are examined. The remarkable simplicity of the derived eigensensitivity expressions results because of the unique, highly structured natural frequency and vibration mode properties of tuned planetary gears [6]. These properties result from the cyclic symmetry and are outlined below.

Let vibration mode $\phi_i = [\mathbf{p}_c, \mathbf{p}_r, \mathbf{p}_s, \mathbf{p}_1, \ldots, \mathbf{p}_N]^{\mathrm{T}}$ with $\mathbf{p}_h = [x_h, y_h, u_h]^{\mathrm{T}}$, $h = c, r, s$ for deflections of the carrier, ring, and sun, and $\mathbf{p}_n = [\zeta_1, \eta_n, u_n]^{\mathrm{T}}$, $n = 1, \ldots, N$ for deflections of the planets. For a tuned planetary gear with N planets, all eigensolutions fall into one of the following three classes.

Rotational modes: Their natural frequencies are distinct and the carrier, ring, and sun execute pure rotation (no translation). A rotational mode has the form

$$\phi_i = [\mathbf{p}_c, \mathbf{p}_r, \mathbf{p}_s, \mathbf{p}_1, \ldots, \mathbf{p}_1]^{\mathrm{T}}, \tag{18}$$

where $\mathbf{p}_h = [0, 0, u_h]^{\mathrm{T}}$, $h = c, r, s$ and $\mathbf{p}_1 = [\zeta_1, \eta_1, u_1]^{\mathrm{T}}$. All planets have the same motion. There are exactly six rotational modes.

Translational modes: Their natural frequencies have multiplicity $m = 2$ and the carrier, ring, and sun execute pure translation (no rotation). A pair of translational modes (selected such that $\phi_i^{\mathrm{T}} \mathbf{M} \hat{\phi}_i = 0$) have the form

$$\begin{aligned} \phi_i &= [\mathbf{p}_c, \mathbf{p}_r, \mathbf{p}_s, (\cos\psi_1 \mathbf{p}_1 + \sin\psi_1 \hat{\mathbf{p}}_1), \ldots, (\cos\psi_N \mathbf{p}_1 + \sin\psi_N \hat{\mathbf{p}}_1)]^{\mathrm{T}}, \\ \hat{\phi}_i &= [\hat{\mathbf{p}}_c, \hat{\mathbf{p}}_r, \hat{\mathbf{p}}_s, (\cos\psi_1 \hat{\mathbf{p}}_1 - \sin\psi_1 \mathbf{p}_1), \ldots, (\cos\psi_N \hat{\mathbf{p}}_1 + \sin\psi_N \mathbf{p}_1)]^{\mathrm{T}}, \end{aligned} \tag{19}$$

where the deflections of the carrier, ring, and sun are $\mathbf{p}_h = [x_h, y_h, 0]^{\mathrm{T}}$ and $\hat{\mathbf{p}}_h = [-y_h, x_h, 0]^{\mathrm{T}}$, $h = c, r, s$ and the first planet deflections are $\mathbf{p}_1$ and $\hat{\mathbf{p}}_1$. The nth planet has position angle $\psi_n = (n-1)2\pi/N$ measured from the first planet (Figure 1). There are exactly six pairs of translational modes.

Planet modes: Their natural frequencies have multiplicity $m = N - 3$ and the carrier, ring, and sun have no motion; only the planets move. A planet mode has the

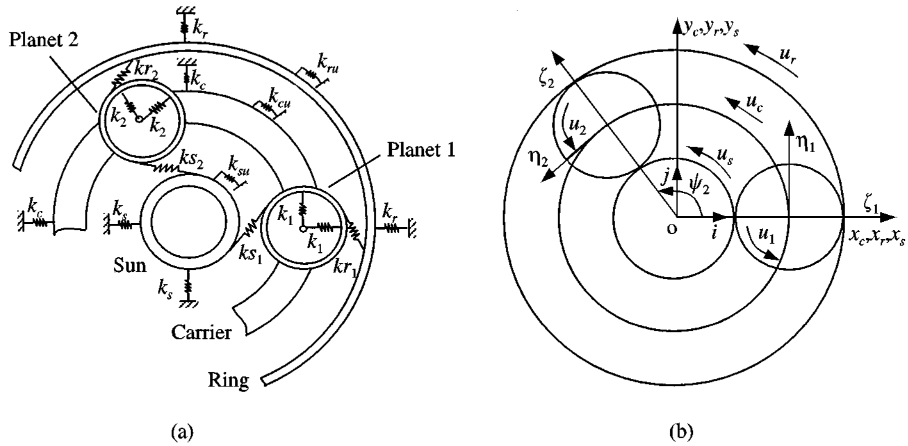

Figure 1. Lumped parameter model of planetary gears and system co-ordinates. (b) All translational co-ordinates $x_j, y_j, j = c, r, s$ and $\zeta_n, \eta_n, n = 1, \dots, N$ are with respect to the frame $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ rotating at constant carrier speed Ω_c.

form

$$\phi_i = [\mathbf{0}, \mathbf{0}, \mathbf{0}, w_1\mathbf{p}_1, \dots, w_N\mathbf{p}_1]^{\mathrm{T}}, \tag{20}$$

where w_n are a set of scalars satisfying $\sum w_n \sin\psi_n = 0, \sum w_n \cos\psi_n = 0, \sum w_n = 0$ with summation over $n = 1, \dots, N$. There are exactly three groups of planet modes, each having multiplicity $m = N - 3$. The m degenerate vibration modes in a group have the same $\mathbf{p}_1$ but m different sets of w_n.

4. EIGENSENSITIVITY TO MESH AND SUPPORT STIFFNESSES

The stiffnesses under consideration include mesh stiffnesses (k_{rn}, k_{sn}), transverse support stiffnesses (k_c, k_r, k_s, k_n), and rotational support stiffnesses (k_{cu}, k_{ru}, k_{su}) (Figure 1). The natural frequency sensitivity to a certain stiffness is found to be uniquely associated with the modal strain energy occurring in that spring. To demonstrate the procedure, let the sun–planet mesh stiffness k_{sn} be the varied parameter.

4.1. TUNED SYSTEM

Consider the case where all sun–planet mesh stiffnesses $k_{sn} = k_{sp}$ are altered equally so the perturbed system remains tuned. The derivatives of the mass and stiffness matrices with respect to k_{sp} are

$$\mathbf{M}' = \mathbf{0}, \quad \mathbf{K}' = \frac{1}{k_{sp}} \sum_{n=1}^{N} \begin{bmatrix} \ddots & & & & \\ & \mathbf{K}_{s1}^n & \cdots & \mathbf{K}_{s2}^n & \\ & \vdots & \ddots & \vdots & \\ & (\mathbf{K}_{s2}^n)^{\mathrm{T}} & \cdots & \mathbf{K}_{s3}^n & \\ & & & & \ddots \end{bmatrix} = \sum_{n=1}^{N} \frac{\partial \mathbf{K}}{\partial k_{sn}}, \tag{21}$$

where all submatrices of $\partial \mathbf{K}/\partial k_{sn}$ are zero except the four that involve k_{sn}. $\mathbf{K}_{s1}^{n}, \mathbf{K}_{s2}^{n}, \mathbf{K}_{s3}^{n}$ are coupling terms between the sun and planet n and depend linearly on k_{sp}. Their details are given in reference [6].

For a rotational mode, the eigensensitivities are obtained by substitution of equation (21) into equations (4)–(6):

$$\frac{\partial \lambda_i}{\partial k_{sp}} = \sum_{n=1}^{N} (\delta_{sn}^{i})^2, \tag{22}$$

$$\frac{\partial \phi_i}{\partial k_{sp}} = \sum_{k=1,k\neq i}^{L} \sum_{n=1}^{N} \frac{\delta_{sn}^{k}\delta_{sn}^{i}}{\lambda_i - \lambda_k}\, \phi_k, \tag{23}$$

$$\frac{\partial^2 \lambda_i}{\lambda k_{sp}^2} = \sum_{k=1,k\neq i}^{L} \frac{2}{\lambda_i - \lambda_k}\left(\sum_{n=1}^{N} \delta_{sn}^{k}\delta_{sn}^{i}\right)^2, \tag{24}$$

where δ_{sn}^{i} is the spring deformation of the sun–planet n mesh in mode ϕ_i given by $\delta_{sn}^{i} = y_s \cos(\psi_n - \alpha_s) - x_s \sin(\psi_n - \alpha_s) - \eta_n \cos \alpha_s - \zeta_n \sin \alpha_s + u_s + u_n$. The algebraic relation

$$\phi_i^{\mathrm{T}} \frac{\partial \mathbf{K}}{\partial k_{sp}} \phi_k = \delta_{sn}^{k}\delta_{sn}^{i}, \quad n = 1, \ldots, N \tag{25}$$

is used in obtaining equations (22)–(24). The rotational mode property (18) dictates that all sun–planet mesh deformations are equal, i.e., $\delta_{sn}^{j} = \delta_{s1}^{i}$, so equation (22) becomes

$$\frac{\partial \lambda_i}{\partial k_{sp}} = N(\delta_{s1}^{i})^2. \tag{26}$$

In equation (23), ϕ_i' is expressed as a modal expansion of the eigenvectors, and the contribution of each eigenvector is readily obtained from its coefficient. When two eigenvalues λ_i and λ_k are nearly equal, the influence of ϕ_k on ϕ_i' is dominant because the denominator in its coefficient is small. In such cases, the second derivative λ_i'' is also large, and the natural frequency changes rapidly with k_{sp}. For close eigenvalues with strong coupling, frequency loci veering and mode localization may occur [13, 14].

The translational mode eigenvalues $\lambda_{1,2}$ do not separate because the perturbed system remains tuned. Thus, the matrix $\mathbf{D}$ in equation (8) has degenerate eigenvalues $\lambda_{1,2}'$. Accordingly, the unperturbed eigenvectors $\phi_{1,2}$ cannot be uniquely determined from the procedure associated with equation (8), and $\Phi = [\phi_1, \phi_2]$ are an arbitrary orthogonal ($\Phi^{\mathrm{T}}\mathbf{M}\Phi = \mathbf{I}_{2\times 2}$) pair of translational modes of the unperturbed system. From equation (8) and (25), $\lambda_{1,2}'$ are the eigenvalues of

$$\mathbf{D} = \Phi^{\mathrm{T}}\mathbf{K}'\Phi = \sum_{n=1}^{N} \begin{bmatrix} (\delta_{sn}^{1})^2 & \delta_{sn}^{1}\delta_{sn}^{2} \\ \delta_{sn}^{1}\delta_{sn}^{2} & (\delta_{sn}^{2})^2 \end{bmatrix}. \tag{27}$$

Use of the translational mode property (19) and algebraic operations yield $\sum_{n=1}^{N}(\delta_{sn}^{1})^2 = \sum_{n=1}^{N}(\delta_{sn}^{2})^2$ and $\sum_{n=1}^{N}\delta_{sn}^{1}\delta_{sn}^{2} = 0$ in $\mathbf{D}$. Thus, the eigenvalues of $\mathbf{D}$ (i.e., $\lambda_{1,2}'$) are degenerate and have the form (22) for $i = 1, 2$. From equation (16), $\lambda_{1,2}''$ are the eigenvalues of matrix $\mathbf{E}$ with elements

$$E_{ij} = 2\phi_i^{\mathrm{T}}\mathbf{K}'\mathbf{v}_j = \sum_{k=3}^{L}\sum_{n=1}^{N} \frac{2(\delta_{sn}^k)^2}{\lambda_j - \lambda_k}\,\delta_{sn}^i\delta_{sn}^j, \quad i,j = 1,2. \tag{28}$$

The translational mode properties result in $E_{11} = E_{22}$ and $E_{12} = E_{21} = 0$. Thus,

$$\frac{\partial^2\lambda_i}{\partial k_{sp}^2} = \sum_{k=3}^{L}\sum_{n=1}^{N} \frac{2(\delta_{sn}^k\delta_{sn}^i)^2}{\lambda_i - \lambda_k}, \quad i = 1,2. \tag{29}$$

Planet modes are also degenerate and the procedure is similar to that for translational modes. For planet modes $\Phi = [\phi_1, \ldots, \phi_m]$, the elements of the matrix $\mathbf{D}$ are $D_{ij} = \sum_{n=1}^{N}\delta_{sn}^i\delta_{sn}^j$ for i, $j = 1, \ldots, m$. Applying property (20) to calculate δ_{sn}^i results in $D_{11} = \cdots = D_{mm}$ and $D_{ij} = 0$, $i \neq j$. It follows that all λ_i' of a group of planet modes are equal and can also be expressed as equation (22) for $i = 1, \ldots, m$. In the same way, all λ_i'', $i = 1, \ldots, m$ are equal and of the form (29).

Equation (22), which is valid for all three types of vibration modes, can be related to the modal strain energy in ϕ_i. The total modal strain energy U is the summation of the individual support and mesh strain energies:

$$U = \tfrac{1}{2}\,\phi_i^{\mathrm{T}}\mathbf{K}\phi_i = U_c + U_{cu} + U_r + U_{ru} + U_s + U_{su} + \sum_{n=1}^{N}(U_n + U_{rn} + U_{sn}), \tag{30}$$

$$U_h = \tfrac{1}{2}\,k_h(x_h^2 + y_h^2), \quad U_{hu} = \tfrac{1}{2}\,k_{hu}u_h^2, \quad h = c, r, s,$$

$$U_n = \tfrac{1}{2}\,k_n[(\delta_{nr}^i)^2 + (\delta_{nt}^i)^2], \quad \delta_{nr}^i = y_c\sin\psi_n + x_c\cos\psi_n - \zeta_n,$$

$$\delta_{nt}^i = y_c\cos\psi_n - x_c\sin\psi_n - \eta_n + u_c,$$

$$U_{sn} = \tfrac{1}{2}\,k_{sn}(\delta_{sn}^i)^2, \quad \delta_{sn}^i = y_s\cos(\psi_n - \alpha_s) - x_s\sin(\psi_n - \alpha_s) - \eta_n\cos\alpha_s - \zeta_n\sin\alpha_s + u_s + u_n,$$

$$U_{rn} = \tfrac{1}{2}\,k_{rn}(\delta_{rn}^i)^2, \quad \delta_{rn}^i = y_r\cos(\psi_n + \alpha_r) - x_r\sin(\psi_n + \alpha_r) - \eta_n\cos\alpha_r + \zeta_n\sin\alpha_r + u_r - u_n,$$

where (1) U_h, U_{hu}, $h = c, r, s$ are the strain energies in the translational and rotational support springs, respectively, of the carrier, ring and sun; and (2) U_n, U_{rn}, U_{sn}, $n = 1, \ldots, N$ are the strain energies in the nth planet bearing, ring–planet mesh and sun–planet mesh. With the definition of strain energy U_{sn} in mode ϕ_i equation (22) becomes

$$\frac{\partial\lambda_i}{\partial k_{sp}} = \frac{2}{k_{sp}}\sum_{n=1}^{N} U_{sn}, \quad \frac{\partial\omega_i}{\partial k_{sp}} = \frac{1}{\omega_i k_{sp}}\sum_{n=1}^{N} U_{sn}. \tag{31}$$

Equation (31) allows one to obtain the natural frequency sensitivity to sun–planet mesh stiffness by inspection of the modal strain energy distribution.

As an example, consider a planetary gear used in the transmission of a U.S. Army OH-58 helicopter. The nominal model parameters are listed in Table 1. The natural frequencies from equation (3) are shown in Figure 2(a) for a range of k_{sp}. The strain energies of each spring are calculated according to equation (30) and their distribution in mode 16 (a translational mode) is shown in Figures 2(b,c) for two

TABLE 1

Parameters of an example system with four planets

	Sun	Ring	Carrier	Planet
Mass (kg)	0·4	2·35	5·43	0·66
I/r^2 (kg)	0·39	3·00	6·29	0·61
Base diameter (mm)	77·4	275·0	176·8	100·3
Mesh stiffness (N/m)		$k_{sp} = k_{rp} = 5 \times 10^8$		
Bearing stiffness (N/m)		$k_p = k_s = k_r = 10^8$		
Torsional stiffness (N/m)		$k_{ru} = 10^9 \quad k_{su} = k_{cu} = 0$		
Pressure angle		$\alpha_s = \alpha_r = 24{\cdot}6^\circ$		

cases: $k_{sp} = 70$ N/μm and $k_{sp} = 500$N/μm. The associated vibration mode ϕ_{16} is also shown for these two cases. Little strain energy is stored in the sun–planet meshes U_{sn} for case I, while substantial strain energy results in case II. Consequently, ω_{16} is more sensitive to k_{sp} in case II than in case I. This conclusion is consistent with the larger slope of the ω_{16} locus for case II in Figure 2(a). In fact, natural frequency sensitivities to *all* stiffnesses can be obtained quantitatively directly from the strain energy distribution using equation (31) and analogous relations (32) and (33) below.

When other system stiffnesses are changed such that the system remains tuned, analogous calculations lead to compact relations similar to equations (22) and (31):

$$\frac{\partial \lambda_i}{\partial k_h} = x_h^2 + y_h^2 = \frac{2}{k_h} U_h, \quad \frac{\partial \lambda_i}{\partial k_{hu}} = u_h^2 = \frac{2}{k_{hu}} U_{hu}, \quad h = c, r, s, \tag{32}$$

$$\frac{\partial \lambda_i}{\partial k_p} = \sum_{n=1}^{N} (\delta_{nr}^i)^2 + (\delta_{nt}^i)^2 = \frac{2}{k_p} \sum_{n=1}^{N} U_n, \quad \frac{\partial \lambda_i}{\partial k_{rp}} = \sum_{n=1}^{N} (\delta_{rn}^i)^2 = \frac{2}{k_{rp}} \sum_{n=1}^{N} U_{rn}. \tag{33}$$

These relations apply for all three types of vibration modes. Expressions for ϕ_i' and λ_i'' for all of the stiffness parameters are collected in Appendix B. Recalling the special properties of the vibration modes, equations (31)–(33) imply the following.

(1) Rotational modes are independent of the transverse support stiffnesses of the carrier, ring, and sun because these components have no deformation of, and hence no modal strain energy in, their transverse support springs. Thus, use of a "floating" sun, ring, or carrier (i.e., low stiffness support) has no impact on rotational modes.
(2) Translational modes are similarly independent of the rotational support stiffnesses of the carrier, ring, and sun,
(3) Planet modes are insensitive to all carrier, ring, and sun support stiffnesses.

4.2. MISTUNED SYSTEM

In practical planetary gears, mistuning may be caused by differing mesh stiffnesses between planets due to differing numbers of teeth in contact, manufacturing

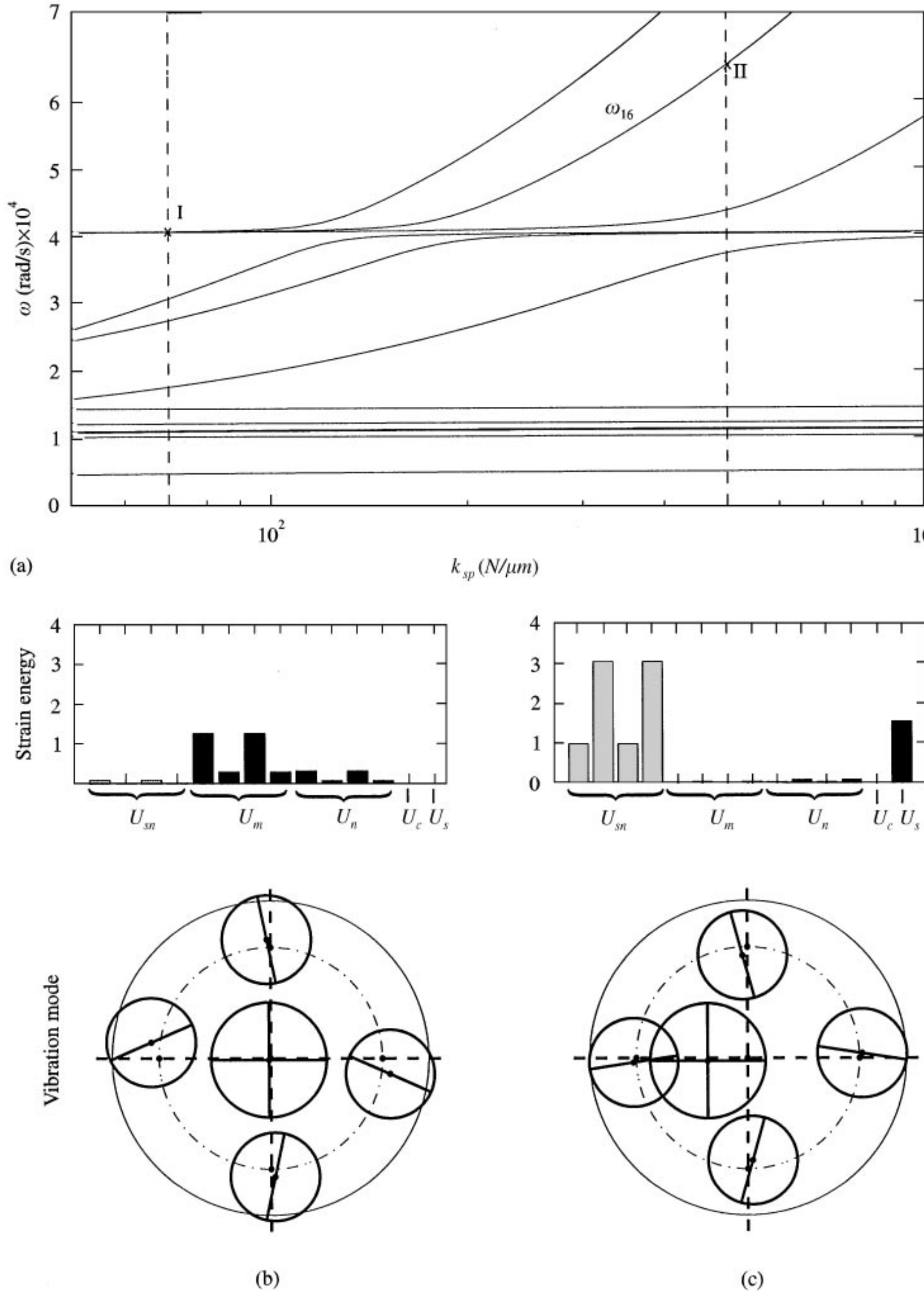

Figure 2. (a) Natural frequency versus the sun–planet stiffness k_{sp}. The strain energy distributions and vibration modes of ϕ_{16} are shown in (b) for case I and (c) for case II. All U are defined in equation (30). The natural frequency sensitivity is associated with U_{sn} by relation (31). The carrier movement is not shown in the vibration modes. Dashed lines are the equilibrium positions and solid lines are the deflected positions. Dots represents the component centers. Parameters are taken from Table 1.

variations, and assembly errors. To study the effects of mistuning on the free vibration, we now examine the eigensensitivity to parameter variations that differ between the planets. Consider an example with only the first sun–planet mesh stiffness k_{s1} varying from the nominal (unperturbed) value k_{sp}. The derivatives of the mass and stiffness matrices with respect to k_{s1} are $\mathbf{M}' = \mathbf{M}'' = \mathbf{K}'' = \mathbf{0}$, $\mathbf{K}' = \partial\mathbf{K}/\partial k_{s1}$.

The eigensensitivities of the rotational modes are obtained from equations (4)–(6) and (25):

$$\frac{\partial\lambda_i}{\partial k_{s1}} = (\delta_{s1}^i)^2 = \frac{2}{k_{s1}} U_{s1}, \quad \frac{\partial\phi_i}{\partial k_{s1}} = \sum_{k=1,k\neq i}^{L} \frac{\delta_{s1}^k\delta_{s1}^i}{\lambda_i-\lambda_k}\phi_k, \quad \frac{\partial^2\lambda_i}{\partial k_{s1}} = \sum_{k=1,k\neq i}^{L} \frac{2(\delta_{s1}^k\delta_{s1}^i)^2}{\lambda_i-\lambda_k}. \tag{34–36}$$

Equation (34) relates λ_i' to the modal strain energy in the first sun–planet mesh. Equations (34)–(36) are similar to (22)–(24) without the summation over n because the varying parameter is located only at the first sun–planet mesh.

For translational modes, we begin with an arbitrary pair of orthonormal, unperturbed vibration modes $\Gamma = [\gamma_1, \gamma_2]$ and use equation (8) to calculate eigenvalue derivatives. The notation Δ_{s1}^i is introduced to represent the sun–planet 1 mesh spring deformation in mode γ_i; this definition is analogous to δ_{s1}^i for ϕ_i. The matrix $\mathbf{D}$ and its eigenvalues (i.e., $\lambda_{i,2}'$) are

$$\mathbf{D} = \Gamma^{\mathrm{T}}\mathbf{K}'\Gamma = \begin{bmatrix} (\Delta_{s1}^1)^2 & \Delta_{s1}^1\Delta_{s1}^2 \\ \Delta_{s1}^1\Delta_{s1}^2 & (\Delta_{s1}^2)^2 \end{bmatrix}, \tag{37}$$

$$\frac{\partial\lambda_1}{\partial k_{s1}} = (\Delta_{s1}^1)^2 + (\Delta_{s1}^2)^2, \quad \frac{\partial\lambda_2}{\partial k_{s1}} = 0. \tag{38}$$

λ_1' can be further simplified if the arbitrary modes $\Gamma = \gamma_1, \gamma_2$ in equation (37) are replaced by the specific modes $\Phi = [\phi_1, \phi_2] = \Gamma\mathbf{A}$, where $\mathbf{A}$ consists of the eigenvectors of $\mathbf{D}$ and $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}_{2\times 2}$. Thus, equations (37) and (38) are replaced by

$$\mathbf{D}_{new} = \mathbf{A}^{\mathrm{T}}\mathbf{D}\mathbf{A} = \begin{bmatrix} \lambda_1' & 0 \\ 0 & \lambda_2' \end{bmatrix} = \Phi^{\mathrm{T}}\mathbf{K}'\Phi = \begin{bmatrix} (\delta_{s1}^1)^2 & \delta_{s1}^1\delta_{s1}^2 \\ \delta_{s1}^1\delta_{s1}^2 & (\delta_{s1}^2)^2 \end{bmatrix}, \tag{39}$$

$$\frac{\partial\lambda_1}{\partial k_{s1}} = (\delta_{s1}^1)^2, \quad \frac{\partial\lambda_2}{\partial k_{s1}} = (\delta_{s1}^2)^2 = 0. \tag{40}$$

From equations (9)–(15), the eigenvector and second eigenvalue derivatives are

$$\frac{\partial\phi_1}{\partial k_{s1}} = \sum_{k=3}^{L} \frac{\delta_{s1}^k\delta_{s1}^1}{\lambda_1-\lambda_k}\phi_k, \quad \frac{\partial\phi_2}{\partial k_{s1}} = \mathbf{0}, \tag{41}$$

$$\frac{\partial^2\lambda_1}{\partial k_{s1}^2} = \sum_{k=3}^{L} \frac{2(\delta_{s1}^k\delta_{s1}^1)^2}{\lambda_1-\lambda_k}, \quad \frac{\partial^2\lambda_2}{\partial k_{s1}^2} = 0. \tag{42}$$

The behavior of ϕ_1 and ϕ_2 is shown in Figure 3 for the example of Table 1. A pair of translational mode natural frequencies separate as a disorder $\varepsilon = k_{s1}/k_{sp} - 1$ is introduced. The modal strain energy distributions in the four sun–planet

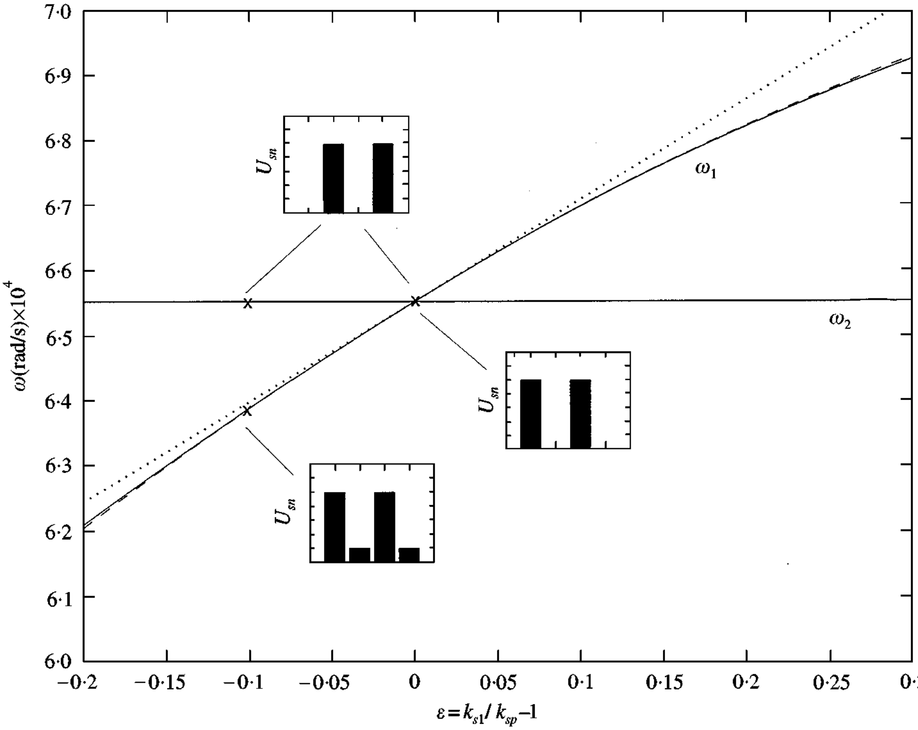

Figure 3. Influence of the disorder $\varepsilon = k_{s1}/k_{sp} - 1$ in the first sun–planet mesh stiffness on the natural frequencies. The strain energy distributions in the four sun–planet meshes are shown in bar plots for $\varepsilon = 0, \ -0{\cdot}1$. Linear ($\cdots$) and quadratic (– –) approximations agree well with the exact loci (——). Parameters are taken from Table 1.

meshes are shown for $\varepsilon = 0, \ -0{\cdot}1$. ϕ_1 is sensitive to k_{s1} because of the high strain energy in the first sun–planet mesh. ϕ_2 has no strain energy in the first sun–planet mesh and is independent of k_{s1}. The linear $(\omega_2 + \varepsilon k_{sp}\omega_2')$ and quadratic $(\omega_2 + \varepsilon k_{sp}\omega_2' + \frac{1}{2}\varepsilon^2 k_{sp}^3 \omega_2'')$ approximations of the loci are shown in Figure 3 and agree well with the exact loci. These two loci intersect at $\varepsilon = 0$ when there is only one disorder in the perturbed system. If one more disorder $\varepsilon_2 = k_{s2}/k_{sp} - 1 = 0{\cdot}1$ is added at the second mesh (Figure 4), the two loci suddenly change direction and veer away. For an initially tuned (cyclically symmetric) system, two independent varying parameters (e.g., k_{s1} and k_{s2}) are necessary to break the symmetry of both ϕ_1 and ϕ_2 and cause frequency loci veering [15].

If the number of planets $N = 4$ or 5, the planet modes have multiplicity $m = 1, 2$ and their eigensensitivities can be obtained from equations (34)–(36) or (40)–(42). When $N > 5$, eigensolutions of matrix $\mathbf{D}$ in equation (8) are difficult to achieve in closed form, but can be obtained numerically.

5. EIGENSENSITIVITY TO GEAR MASS AND INERTIA

The parameters of interest consist of masses (m_c, m_r, m_s, m_p) and moments of inertia (I_c, I_r, I_s, I_p) for the carrier, ring, sun, and planets. The eigensensitivity to

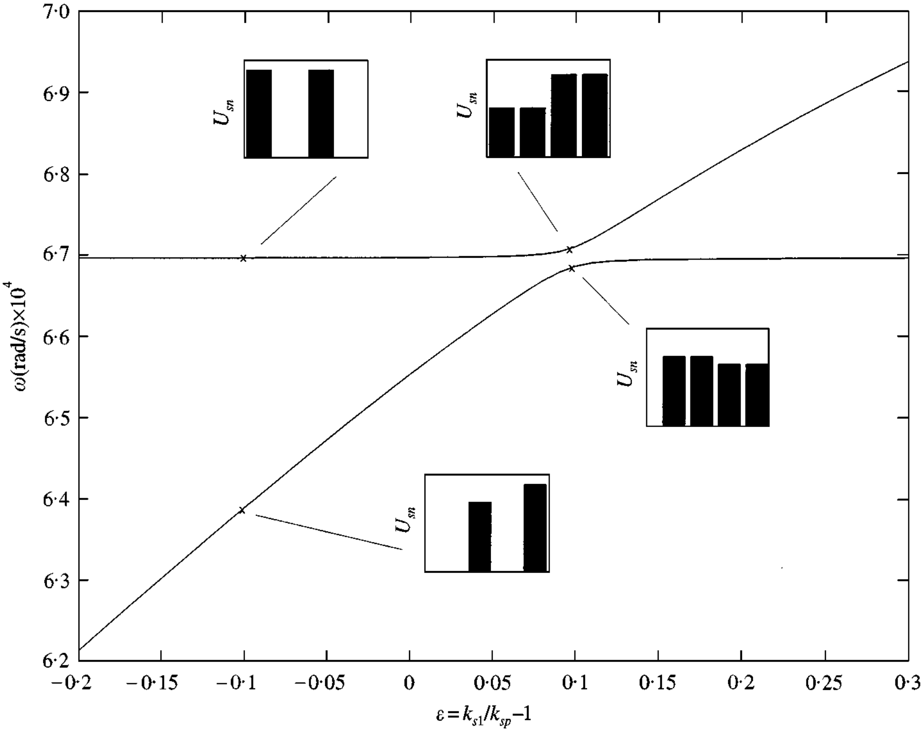

Figure 4. Influence of two disorders on natural frequencies. Another disorder $\varepsilon_2 = k_{s2}k_{sp} - 1 = 0{\cdot}1$ at the second sun–planet mesh is added to the system shown in Figure 3. The frequency loci do not intersect but veer away. Parameters are taken from Table 1.

these parameters follows the same procedure as described previously. When the perturbed system remains tuned, the eigenvalue derivatives for the three types of modes are

$$\frac{\partial \lambda_i}{\partial m_h} = -\lambda_i(x_h^2 + y_h^2) = -\frac{2}{m_h}T_h, \quad \frac{\partial \lambda_i}{\partial I_h} = -\frac{\lambda_i}{r_h^2}u_h^2 = -\frac{2}{I_h}T_{hu}, \quad h = c, r, s, \tag{43}$$

$$\frac{\partial \lambda_i}{\partial m_p} = -\lambda_i \sum_{n=1}^{N} (\zeta_n^2 + \eta_n^2) = -\frac{2}{m_p}\sum_{n=1}^{N} T_n, \quad \frac{\partial \lambda_i}{\partial I_p} = -\frac{\lambda_i}{r_p^2}\sum_{n=1}^{N} u_n^2 = -\frac{2}{I_p}\sum_{n=1}^{N} T_{nu}, \tag{44}$$

where T_h, T_{hu}, $h = c, r, s$ and T_n, T_{nu} are the modal translational and rotational kinetic energies, respectively, of the carrier, ring, sun, and planets. The total modal kinetic energy T is

$$T = \tfrac{1}{2}\lambda_i \phi_i^{\mathrm{T}} \mathbf{M} \phi_i = T_c + T_{cu} + T_r + T_{ru} + T_s + T_{su} + \sum_{n=1}^{N} (T_n + T_{nu}), \tag{45}$$

where each term of equation (45) is defined implicitly in equations (43) and (44). Expressions for ϕ_i' and λ_i'' for all of the mass and inertia parameters are collected in Appendix B. Figure 5(a) shows an example plot of the natural frequencies versus the

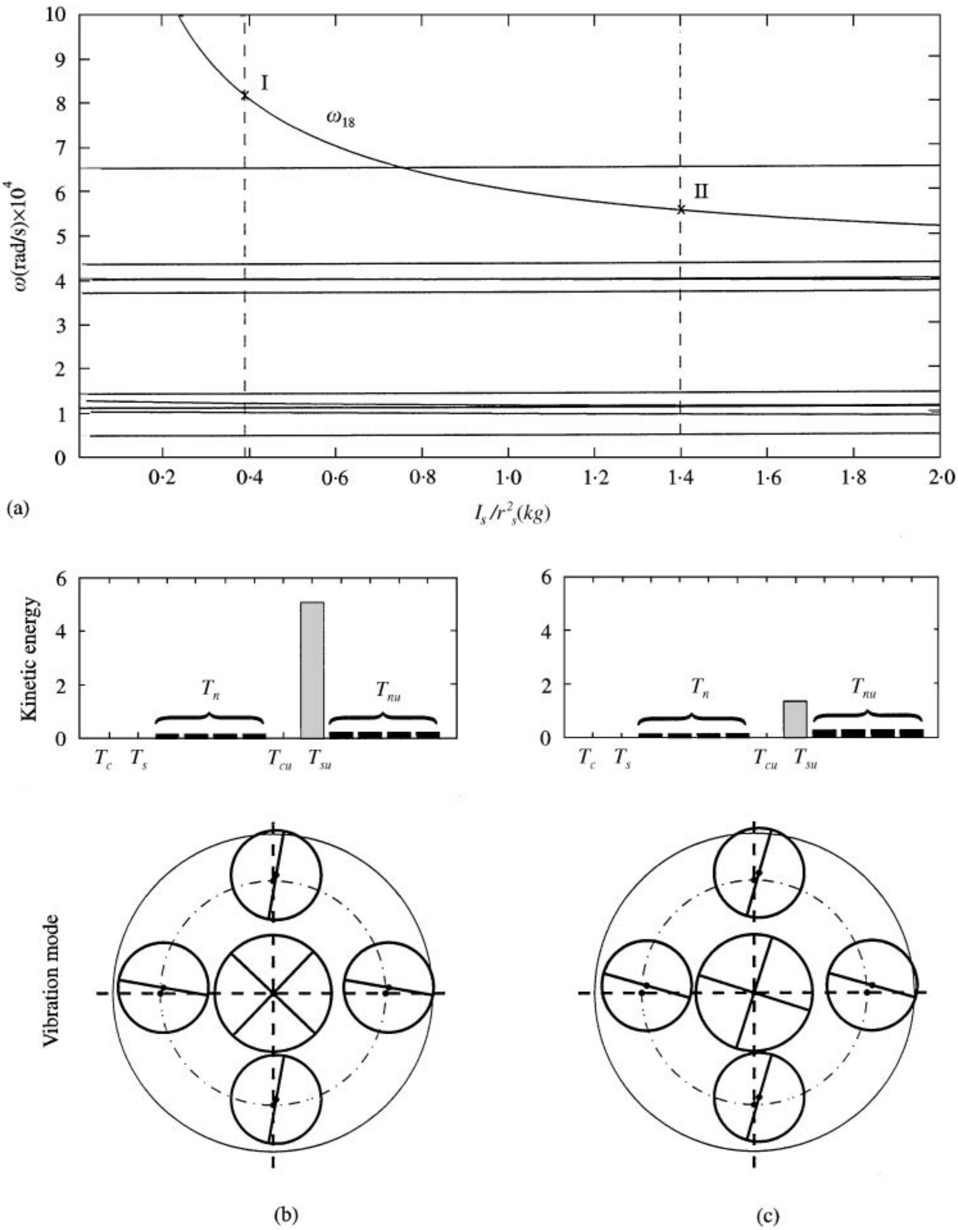

Figure 5. (a) Natural frequency versus the sun moment of inertia I_s. The kinetic energy distributions and vibration modes of ϕ_{16} are shown in (b) for case I and (c) for case II. The T are defined in equation (45). The natural frequency sensitivity to I_s is associated with the sun rotational kinetic energy T_{su}. The carrier movement is not shown in the vibration modes. Dashed lines are the equilibrium positions and solid lines are the deflected positions. Dots represents the component centers. Parameters are taken from Table 1.

sun moment of inertia I_s. Most natural frequencies are insensitive to changes in I_s. The kinetic energy distribution and vibration modes of mode 18 (rotational mode) are shown in Figures 5(b, c) for cases I and II. The sun has more rotational kinetic energy T_{su} in case I than in case II, so ω_{18} locus has larger slope in case I.

Equations (43) and (44) allow quantitative calculation of natural frequency sensitivity to all masses and moments of inertia directly from the modal kinetic energy distributions. Considering the properties of each class of vibration modes, some conclusions are immediate from equations (43) and (44):

(1) Rotational modes are independent of the masses of the carrier, ring, and sun because such modes have no translations of these components.
(2) Translational modes are similarly independent of the inertias of the carrier, ring, and sun.
(3) Planet modes are independent of both masses and inertias of the carrier, ring, and sun.

6. EIGENSENSITIVITY TO OPERATING SPEED

In high-speed applications (e.g., aircraft engines), gyroscopic effects may significantly alter the system stability and dynamic behavior. Eigenvalue derivatives evaluated for $\Omega_c = 0$ are calculated to assess the influence of operating speed on the natural frequency spectrum. The eigenvalue problem (2) is the standard form for a gyroscopic system. For practical operating speeds (i.e., subcritical) the eigenvalues remain purely imaginary. Suppose a zero speed natural frequency ω_i has multiplicity m and the arbitrarily chosen independent eigenvectors are $\Gamma = [\gamma_1, \ldots, \gamma_m]$ with normalization $\Gamma^{\mathrm{T}}\mathbf{M}\Gamma = \mathbf{I}_{m\times m}$. While eigenvectors for $\Omega_c \neq 0$ are complex, the γ_i are real. Analogous to the derivation of equation (8), differentiation of equation (2) with respect to Ω_c and evaluation at $\Omega_c = 0$ yield $(\mathbf{K} - \omega_i^2\mathbf{M})\phi_i' = (2\omega_i\omega_i'\mathbf{M} - j\omega_i\mathbf{G})\Gamma\mathbf{a}_i = \mathbf{f}$, where $\phi_i = \Gamma\mathbf{a}_i$. Applying solvability and normalization conditions results in an $m \times m$ Hermitian eigenvalue problem

$$\mathbf{D}\mathbf{a}_i = \omega_i'\mathbf{a}_i, \quad \mathbf{D} = \mathrm{j}\Gamma^{\mathrm{T}}\mathbf{G}\Gamma/2. \tag{46}$$

The natural frequency sensitivities ω_i' are obtained from the eigenvalues of equation (46) for the three classes of vibration modes. Rotational mode natural frequencies are distinct and equation (46) becomes a scalar equation. Hence $\omega_i' = 0$ because $\gamma_i^{\mathrm{T}}\mathbf{G}\gamma_i = 0$ for real γ_i and skew-symmetric $\mathbf{G}$. For translational modes γ_1 and γ_2, $\mathbf{D}$ and its eigenvalues are

$$\mathbf{D} = \frac{\mathrm{j}}{2}\begin{bmatrix} 0 & \gamma_1^{\mathrm{T}}\mathbf{G}\gamma_2 \\ -\gamma_1^{\mathrm{T}}\mathbf{G}\gamma_2 & 0 \end{bmatrix}, \quad \omega_{1,2}' = \pm\,\gamma_1^{\mathrm{T}}\mathbf{G}\gamma_2/2. \tag{47}$$

For a group of planet modes $\gamma_1, \ldots, \gamma_m$, property (20) guarantees $D_{ij} = \gamma_i^{\mathrm{T}}\mathbf{G}\gamma_j = 0$ for $i \neq j$. $D_{ii} = 0, i = 1, \ldots, m$ because of the skew-symmetry of $\mathbf{G}$. Thus, $\mathbf{D} = \mathbf{0}$ and all planet mode natural frequency sensitivities vanish, i.e., $\omega_i' = 0$.

Equation (47) can be used to approximate the frequency loci $\tilde{\omega}_i = \Omega_c\omega_i'$. The result $\omega_i' = 0$ for rotational and planet modes at $\Omega_c = 0$ indicates the natural frequencies of these modes are scarcely affected by operating speed. Figure 6 shows the first 10 frequency loci versus Ω_c for the gear system in Table 1. The rotational mode (ω_4, ω_8) and planet mode (ω_5) loci are nearly constant. Translational mode frequencies ($\omega_{2,3}, \omega_{6,7}, \omega_{9,10}$) split as Ω_c is increased from zero. In this example, ω_2 and ω_3 at $\Omega_c = 600$ rad/s deviate about 10% from the zero speed value. Typical

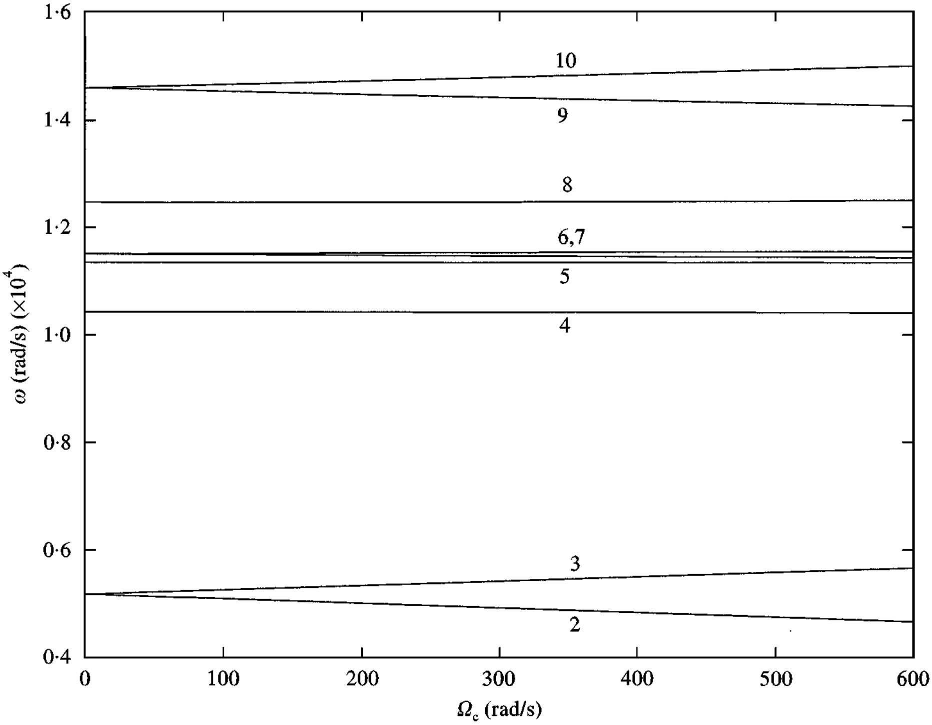

Figure 6. Influence of the carrier rotation speed Ω_c on natural frequencies. $\omega_1 = 0$ is a rigid-body mode. Parameters are taken from Table 1.

helicopter carrier speeds are less than 100 rad/s. For applications with high speed (e.g., turbofan and turbprop engine systems), heavy component masses, and compliant stiffnesses, the gyroscopic effects can be more significant. If a natural frequency locus has large slope and decreases to zero in the range of operating speed, the stability and system behavior are dramatically impacted.

7. SUMMARY AND CONCLUSIONS

This work systematically investigates the natural frequency and vibration mode sensitivities to key planetary gear design parameters including all support and mesh stiffnesses, component masses, moments of inertia, and operating speed. The main results are:

(1) The well-defined vibration mode properties lead to simple, exact formulae relating eigenvalue derivatives to modal strain and kinetic energies. These formulae allow one to obtain the natural frequency sensitivity directly from modal energy distributions and effectively adjust the system frequencies. Both tuned (cyclically symmetric) and mistuned parameter variations are considered. The eigensensitivities help identify the parameters critical to planetary gear vibration and assess design choices such as the use of “floating” components on low stiffness supports.

(2) Rotational modes are independent of the transverse support stiffnesses and masses of the carrier, ring, and sun. Translational modes are independent of the rotational support stiffnesses and moments of inertia of the carrier, ring, and sun. Planet modes are remarkably insensitive to all support stiffnesses, mass, and moments of inertia of the carrier, ring, and sun.
(3) The impact of gyroscopic effects are estimated by the natural frequency sensitivity to the operating speed. Rotational and planet mode natural frequencies are insensitive to the operating speed. The degenerate zero speed translational mode natural frequencies split into distinct ones as the operating speed increases. Translational mode natural frequency sensitivity to operating speed increases with component inertia and decreases with system stiffness.

ACKNOWLEDGMENT

The authors would like to thank Timothy L. Krantz of the Army Research Lab at NASA Glenn Research Center for his support and advice on the project. This material is based upon work supported by the NASA Glenn Research Center under grant NAG3-1979 and the U.S. Army Research Office under grant DAAG55-98-1-0398.

REFERENCES

1. M. Botman 1976 *ASME Journal of Engineering for Industry* 811–815. Epicyclic gear vibations.
2. F. Cunliffe, J. D. Smith and D. B. Welbourn 1974 *ASME Journal of Engineering for Industry* 578–584. Dynamic tooth loads in epicyclic gears.
3. A. Kahraman 1994 *Journal of Sound and Vibration* **173**, 125–130. Natural modes of planetary gear trains.
4. A. Saada and P. Velex, 1995 *ASME Journal of Mechanical Design* **117**, 241–247. An extended model for the analysis of the dynamic behavior of planetary trains.
5. J. L. Frater, R. August and F. B. Oswald 1983 *NASA Technical Memorandum* 83428. Vibration in planetary gear systems with unequal planet stiffnesses.
6. J. Lin and R. G. Parker 1999 *ASME Journal of Vibration and Acoustics* **121**, 316–321. Analytical characterization of the unique properties of planetary gear free vibration.
7. R. L. Fox and M. P. Kapoor 1968. *AIAA Journal* **6**, 2426–2429. Rates of change of eigenvalues and eigenvetors.
8. R. B. Nelson 1976 *AIAA Journal* **14**, 1201–1205. Simplified calculation of eigenvector derivatives.
9. W. C. Mills-Curran 1988 *AIAA Journal* **26**, 867–871. Calculation of eigenvector derivatives for structures with repeated eigenvalues.
10. M. L. Bernard and A. J. Bronowicki 1994 *AIAA Journal* **32**, 1500–1506. Modal expansion method for eigensensitivity with repeated roots.
11. M. I. Friswell 1996 *Journal of Vibration and Acoustics* **118**, 390–397. The derivatives of repeated eigenvalues and their associated eigenvectors.
12. R. Courant and D. Hilbert 1953 *Methods of Mathematical Physics*. New York: Interscience Publishers.
13. N. C. Perkins and C. D. Mote Jr 1986 *Journal of Sound and Vibration* **106**, 451–463 Comments on curve werring in eigenvalue problems.

14. C. Pierre 1988. *Journal of Sound and Vibration* **126**, 485–502. Mode localization and eigenvalue loci veering phenomena in disordered structures.
15. G. S. Happawana, O. D. I. Nwokah, A. K. Bajaj, M. Azene 1998. *Journal of Sound and Vibration* **211**, 761–789. Free and forced response of mistuned linear cyclic systems: a singular perturbation approach.

APPENDIX A: NOMENCLATURE

I_h	moments of inertia, $h = c, r, s, p$
k_h	transverse support stiffness, $h = c, r, s, p$
k_{hu}	torsional support stiffness, $h = c, r, s, p$
k_{sn}, k_{rn}	nth sun–planet and ring–planet mesh stiffness
k_{sp}	nominal sun–planet mesh stiffness in tuned system
L	total number of degrees of freedom $= 3(N + 3)$
m	multiplicity of eigensolution
m_h	component masses, $h = c, r, s, p$
N	number of planets
r_c	radius of the circle passing through planet centers
r_h	gear base radii, $h = r, s, p$
u_h	component rotations, $h = c, r, s, p$
x_h, y_h	component translations, $h = c, r, s$
α_s, α_r	sun–planet and ring–planet mesh pressure angles
δ	deformation of an elastic element
δ^i_{sn}	deformation of the nth sun–planet mesh in mode ϕ_i
λ_i	ith eigenvalue $\lambda_i = \omega_i^2$
ω_i	ith natural frequency
Ω_c	carrier operating speed
ϕ_i	ith vibration mode
ψ_n	position angle of the nth planet ($\psi_1 = 0$)
ζ_n, η_n	radial and tangential motion of the nth planet
$(\ )'$, $(\ )''$	first and second derivatives

APPENDIX B

The superscripts k and i in x, y, ζ, η, u and δ indicate they are from the vibration modes $\phi_{k,i}$.

Tuned system: ϕ_i' and λ_i'' for a rotational mode ϕ_i are

$$\frac{\partial \phi_i}{\partial k_{rp}} = \sum_{k=1,k\neq i}^{L} \sum_{n=1}^{N} \frac{\delta^k_{rn}\delta^i_{rn}}{\lambda_i - \lambda_k}\phi_k, \quad \frac{\partial^2 \lambda_i}{\partial k_{rp}^2} = \sum_{k=1,k\neq i}^{L} \frac{2}{\lambda_i - \lambda_k}\left(\sum_{n=1}^{N} \delta^k_{rn}\delta^i_{rn}\right)^2,$$

$$\frac{\partial \phi_i}{\partial k_p} = \sum_{k=1,k\neq i}^{L} \sum_{n=1}^{N} \frac{\delta^k_{nr}\delta^i_{nr} + \delta^k_{nt}\delta^i_{nr}}{\lambda_i - \lambda_k}\phi_k,$$

$$\frac{\partial^2 \lambda_i}{\partial k_p^2} = \sum_{k=1,k\neq i}^{L} \frac{2}{\lambda_i - \lambda_k}\left(\sum_{n=1}^{N} \delta^k_{nr}\delta^i_{nr} + \delta^k_{nt}\delta^i_{nt}\right)^2,$$

$$\frac{\partial\phi_i}{\partial k_h} = \sum_{k=1,k\neq i}^{L} \frac{x_h^k x_h^i + y_h^k y_h^i}{\lambda_i - \lambda_k}\phi_k, \quad \frac{\partial^2\lambda_i}{\partial k_h^2} = \sum_{k=1,k\neq i}^{L} \frac{2(x_h^k x_h^i + y_h^k y_h^i)^2}{\lambda_i - \lambda_k}, \quad h = c, r, s,$$

$$\frac{\partial\phi_i}{\partial k_{hu}} = \sum_{k=1,k\neq i}^{L} \frac{u_h^k u_h^i}{\lambda_i - \lambda_k}\phi_k, \quad \frac{\partial^2\lambda_i}{\partial k_{hu}^2} = \sum_{k=1,k\neq i}^{L} \frac{2(u_h^k u_h^i)^2}{\lambda_i - \lambda_k}, \quad h = c, r, s,$$

$$\frac{\partial\phi_i}{\partial m_p} = \sum_{k=1,k\neq i}^{L}\sum_{n=1}^{N} \frac{-\lambda_i(\zeta_n^k\zeta_n^i + \eta_n^k\eta_n^i)}{\lambda_i - \lambda_k}\phi_k - \frac{\phi_i}{2}\left[\sum_{n=1}^{N}(\zeta_n^i)^2 + (\eta_n^i)^2\right],$$

$$\frac{\partial^2\lambda_i}{\partial m_p^2} = \sum_{k=1,k\neq i}^{L} \frac{2\lambda_i^2}{\lambda_i - \lambda_k}\left(\sum_{n=1}^{N}\zeta_n^k\zeta_n^i + \eta_n^k\eta_n^i\right)^2 + 2\lambda_i\left[\sum_{n=1}^{N}(\zeta_n^i)^2 + (\eta_n^i)^2\right]^2,$$

$$\frac{\partial\phi_i}{\partial I_p} = \sum_{k=1,k\neq i}^{L}\sum_{n=1}^{N} \frac{-\lambda_i u_n^k u_n^i}{r_p^2(\lambda_i - \lambda_k)}\phi_k - \frac{\phi_i}{2r_p^2}\sum_{n=1}^{N}(u_n^i)^2,$$

$$\frac{\partial^2\lambda_i}{\partial I_p^2} = \sum_{k=1,k\neq i}^{L} \frac{2\lambda_i^2}{r_p^4(\lambda_i - \lambda_k)}\left(\sum_{n=1}^{N}u_n^k u_n^i\right)^2 + \frac{2\lambda_i}{r_p^4}\left[\sum_{n=1}^{N}(u_n^i)^2\right]^2,$$

$$\frac{\partial\phi_i}{\partial m_h} = \sum_{k=1,k\neq i}^{L} \frac{-\lambda_i(x_h^k x_h^i + y_h^k y_h^i)}{\lambda_i - \lambda_k}\phi_k - \frac{\phi_i}{2}[(x_h^i)^2 + (y_h^i)^2],$$

$$\frac{\partial^2\lambda_i}{\partial m_h^2} = \sum_{k=1,k\neq i}^{L} \frac{2\lambda_i^2(x_h^k x_h^i + y_h^k y_h^i)^2}{\lambda_i - \lambda_k} + 2\lambda_i[(x_h^i)^2 + (y_h^i)^2]^2, \quad h = c, r, s,$$

$$\frac{\partial\phi_i}{\partial I_h} = \sum_{k=1,k\neq i}^{L} \frac{-\lambda_i u_h^k u_h^i}{r_h^2(\lambda_i - \lambda_k)}\phi_k - \frac{\phi_i}{2r_h^2}(u_h^i)^2,$$

$$\frac{\partial^2\lambda_i}{\partial I_h^2} = \sum_{k=1,k\neq i}^{L} \frac{2(\lambda_i u_h^k u_h^i)^2}{r_h^4(\lambda_i - \lambda_k)} + \frac{2\lambda_i}{r_h^4}(u_h^i)^4, \quad h = c, r, s.$$

For translational and planet modes, ϕ_i' cannot be uniquely determined; λ_i'' can be numerically obtained from equation (16).

Mistuned system: Only one parameter is altered at the first planet. The case of varying k_{r1} is the same as equations (34)–(42), except subscript $s1$ is replaced by $r1$. For other planet parameters, the rotational mode derivatives are

$$\frac{\partial\lambda_i}{\partial k_1} = (\delta_{1r}^i)^2 + (\delta_{1t}^i), \quad \frac{\partial\phi_i}{\partial k_1} = \sum_{k=1,k\neq i}^{L} \frac{\delta_{1r}^k\delta_{1r}^i + \delta_{1t}^k\delta_{1t}^i}{\lambda_i - \lambda_k}\phi_k,$$

$$\frac{\partial^2\lambda_i}{\partial k_1^2} = \sum_{k=1,k\neq i}^{L} \frac{2(\delta_{1r}^k\delta_{1r}^i + \delta_{1t}^k\delta_{1t}^i)^2}{\lambda_i - \lambda_k},$$

$$\frac{\partial\lambda_i}{\partial m_1} = -\lambda_i[(\zeta_1^i)^2 + (\eta_1^i)^2],$$

$$\frac{\partial\phi_i}{\partial m_1} = \sum_{k=1,k\neq i}^{L} \frac{-\lambda_i(\zeta_1^k\zeta_1^i + \eta_1^k\eta_1^i)}{\lambda_i - \lambda_k}\phi_k - \frac{\phi_i}{2}[(\zeta_1^i)^2 + (\eta_1^i)]^2,$$

$$\frac{\partial^2\lambda_i}{\partial m_1^2} = \sum_{k=1,k\neq i}^{L} \frac{2\lambda_i^2(\zeta_1^k\zeta_1^i + \eta_1^k\eta_1^i)^2}{\lambda_i - \lambda_k} + 2\lambda_i[(\zeta_1^i)^2 + (\eta_1^i)^2]^2,$$

$$\frac{\partial\lambda_i}{\partial I_1} = -\lambda_i \frac{(u_1^i)^2}{r_p^2}, \quad \frac{\partial\phi_i}{\partial I_1} = \sum_{k=1,k\neq i}^{L} \frac{-\lambda_i u_1^k u_1^i}{r_p^2(\lambda_i - \lambda_k)}\phi_k - \frac{\phi_i}{2r_p^2}(u_1^i)^2,$$

$$\frac{\partial^2\lambda_i}{\partial I_1^2} = \sum_{k=1,k\neq i}^{L} \frac{2(\lambda_i u_1^k u_1^i)^2}{r_p^4(\lambda_i - \lambda_k)} + 2\lambda_i \frac{(u_1^i)^4}{r_p^4}.$$

For translational modes $\phi_{1,2}$, which are obtained from the procedure associated with equation (8), the eigensensitivities are

$$\frac{\partial\lambda_i}{\partial k_1} = (\delta_{1r}^i)^2 + (\delta_{1t}^i)^2, \quad \frac{\partial^2\lambda_i}{\partial k_1^2} = \sum_{k=3}^{L} \frac{2(\delta_{1r}^k\delta_{1r}^i + \delta_{1t}^k\delta_{1t}^i)^2}{\lambda_i - \lambda_k}, \quad i = 1, 2,$$

$$\frac{\partial\phi_i}{\partial k_1} = \sum_{k=3}^{L} \frac{\delta_{1r}^k\delta_{1r}^i + \delta_{1t}^k\delta_{1t}^i}{\lambda_i - \lambda_k}\left(\phi_k + \frac{\delta_{1r}^j\delta_{1r}^j + \delta_{1t}^k\delta_{1t}^j}{\lambda_i' - \lambda_j'}\phi_j\right), \quad i, j = 1, 2, \quad i \neq j,$$

$$\frac{\partial\lambda_i}{\partial m_1} = -\lambda_i[(\zeta_n^i)^2 + (\eta_n^i)^2],$$

$$\frac{\partial^2\lambda_i}{\partial m_1^2} = \sum_{k=3}^{L} \frac{2\lambda_i^2}{\lambda_i - \lambda_k}(\zeta_1^k\zeta_1^i + \eta_1^k\eta_1^i)^2 + 2\lambda_i[(\zeta_1^i)^2 + (\eta_1^i)^2]^2, \quad i = 1, 2.$$

$$\frac{\partial\phi_i}{\partial m_1} = \sum_{k=3}^{L} \frac{\zeta_1^k\zeta_1^i + \eta_1^k\eta_1^i}{\lambda_i - \lambda_k}\left[\frac{\lambda_i^2(\zeta_1^k\zeta_1^j + \eta_1^k\eta_1^j)}{\lambda_i' - \lambda_j'}\phi_j - \lambda_i\phi_k\right] - \frac{\phi_i}{2}[(\zeta_1^i)^2 + (\eta_1^i)^2],$$

$$i, j = 1, 2, \quad i \neq j,$$

$$\frac{\partial\lambda_1}{\partial I_1} = -\frac{\lambda_1}{r_p^2}(u_1^1)^2, \quad \frac{\partial^2\lambda_1}{\partial I_1^2} = \sum_{k=3}^{L} \frac{2(\lambda_1 u_1^k u_1^1)^2}{r_p^4(\lambda_1 - \lambda_k)} + \frac{2\lambda_1(u_1^1)^4}{r_p^4}, \quad \frac{\partial\lambda_2}{\partial I_1} = \frac{\partial^2\lambda_2}{\partial I_1^2} = 0,$$

$$\frac{\partial\phi_1}{\partial I_1} = \sum_{k=3}^{L} \frac{-\lambda_1 u_1^k u_1^1}{r_p^2(\lambda_1 - \lambda_k)}\phi_k - \frac{(u_1^1)^2}{2r_p^2}\phi_1, \quad \frac{\partial\phi_2}{\partial I_1} = \mathbf{0}.$$