Journal of Sound and Vibration (1999) **228**(1), 218–225
Article No. jsvi.1999.2380, available online at http://www.idealibrary.com on IDEAL®

DISTURBANCE PROPAGATION PATHS IN A CONSTRAINED AXIALLY MOVING STRING

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(*Received* 21 *December* 1998, *and in final form* 7 *June* 1999)

1. INTRODUCTION

The determination of disturbance propagation paths in distributed parameters systems is an important issue related to a wide range of vibration problems, such as transient, impact loads and passive and active control implementations. The purpose of the present study is to show the convenience of visualizing the disturbance propagation paths along continuous systems using a wave approach, based on a simulation of periodic waveguide dynamics [1]. This concept is illustrated by its applications to an axially moving string system coupled to a stationary guide and a point controller. Since the string is not dispersive, Laplace transform technique can be used to determine the exact response of this combined system subjected to transient loads. However, it is worth mentioning that this method is more general and its application is restricted to structures with regular geometry rather than one-dimensional systems. A recent study of the transient response of the translating string [2] used the (distributed) transfer function methods [3] to trace disturbance propagation paths along the string under general boundary conditions. Although the authors have carefully pointed out the advantages of their method with respect to Green's function modal expansion approach [4] and by some applications in control design [5], many of the peculiar properties of the wave propagation phenomenon are not well outlined. Furthermore, the disturbance propagation paths are especially difficult to trace when the string is subjected to multiple excitations and constraints.

Section 2 describes the wave transfer matrix method [1] and its interpretation for time domain applications. For brevity, details regarding the wave dynamics of the translating string are omitted, since these are described in reference [1]. In Section 3, an example of a string system coupled to point constraints, represented by a spring-mass-dashpot system and a point controller is studies. The response is given in the frequency domain and is then converted into time domain by use of Laplace transform package of *Mathematica.*

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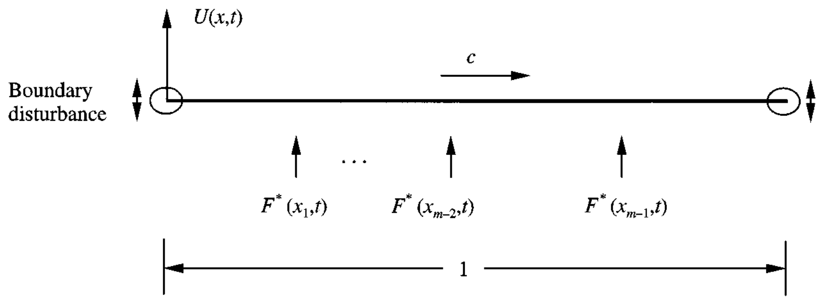

Figure 1. A schematic diagram of the axially moving string under arbitrarily spaced excitations.

2. DISTURBANCE PROPAGATION PATHS IN TRANSLATING STRING

Consider a translating string moving at constant (sub-critical) speed c between two boundaries separated by a distance $l = 1$, as shown in Figure 1. The string is subjected to point-wise excitations in terms of applied forces $F^*(x_i, i)$ and/or prescribed motions $U^*(x_i, t)$ along its length. The string's transverse deflection, denoted by $U^*(x, t)$ is viewed as comprising the sum of positive and negative displacement wave components $V^+(x, t)$ and $V^-(x, t)$. According to the wave transfer matrix approach [1], wave states of continuous systems under multiple excitations can be factorized as free and forced wave fields. The free wave field can be expressed with reference to the incoming wave at either end of the system, while the forced field resembles the infinite system behaviour in that it is independent of the boundary conditions. Thus, the string wave dynamics can be expressed through the right boundary incoming wave as

$$\begin{Bmatrix} v^+(x, s) \\ v^-(x, s) \end{Bmatrix} = \mathbf{T}(x, 1, s)\begin{Bmatrix} v^+(1, s) \\ r_1 v^+(1, s) \end{Bmatrix} + \sum_{x_k \geqslant x} \mathbf{T}(x, x_k, s)\begin{bmatrix} (1+c)/2 & -1/(2s) \\ (1-c)/2 & 1/(2s) \end{bmatrix} \times \begin{Bmatrix} u^*(x_k, s) \\ f^*(x_k, s) \end{Bmatrix}, \tag{1}$$

where s is the Laplace variable and lower-case symbols (i.e. v^+, v^-, u^* and f^*) have been used to distinguish them from their time domain counterparts. The wave transfer matrix $\mathbf{T}(x_m, x_n, s) = \mathrm{diag}\,[\mathrm{e}^{\gamma_1(x_m - x_n)}\ \mathrm{e}^{\gamma_2(x_m - x_n)}]$, in which $\gamma_1 = -s/(1+c)$ and $\gamma_2 = s/(1-c)$ are the phase constants associated with the $v^+(x, s)$ and $v^-(x, s)$, respectively. The right boundary incoming wave $v^+(1, s)$ in equation (1) is given by

$$v^+(1, s) = [1 - r_0\,\mathrm{e}^{\gamma_1 - \gamma_2} r_1]^{-1} \sum_{x_k \leqslant 1} [\{\mathrm{e}^{\gamma_1(1 - x_k)} + r_0\,\mathrm{e}^{\gamma_1 - \gamma_2 x_k}\}\, f^*(x_k, s)/(2s) + \{-(1+c)\,\mathrm{e}^{\gamma_1(1 - x_k)} + (1-c)\, r_0\,\mathrm{e}^{\gamma_1 - \gamma_2 x_k}\}\, u^*(x_k, s)/2], \tag{2}$$

where $r_0(r_1)$ is the reflection coefficient of the left (right) boundary'

The denominator of $v^+(1, s)$ can be expanded as

$$[1 - r_0\,\mathrm{e}^{(\gamma_1 - \gamma_2)}\, r_1]^{-1} = 1 + r_0\, r_1\,\mathrm{e}^{(\gamma_1 - \gamma_2)} + (r_0\, r_1)^2\,\mathrm{e}^{2(\gamma_1 - \gamma_2)} + \cdots\,. \tag{3}$$

The above equations reveal the "global" behaviour of $v^+ (1, s)$ in the sense that it conveys all the information regarding the dynamics of the entire system during its continuous propagation upstream and downstream directions. Consequently, the response in equation (1) may be interpreted in physical terms as the sum of a reverberant field through $v^+ (1, s)$ plus the direct field. The is true for $v^- (x, s)$ since it is made up of the contribution of reflected $v^+ (1, s)$ and the direct field of excitations at $x_k \geqslant x$. However, this is not the case for $v^+ (x, s)$, since no positive wave emanating from x_k will directly contribute to the response at points positioned to the left of it ($x_k > x$). In other words, the response at $v^+ (x, s)$, for excitations departing from $x_k > x$, should entirely be contained in the first (reverberant) part of solution (1). To remove this apparent contradiction, let us consider the example of Figure 2. This figure shows the paths of the positive wave component generated from a disturbance located at x_1 and detected as a positive wave at a point x ($x < x_1$). It is clear that such excitation will first hit the right boundary (therefore, resulting in $v^+ (1, s)$) before reflecting at both boundaries and reappearing as $v^+ (x, s)$. If this disturbance propagates beyond the point x ($x < x_1$), then it circumnavigates the system returning to its sources: in other words, it reappears again as $v^+ (1, s)$. However, the reappearance of this wave component as $v^+ (1, s)$ includes, unavoidably, a further contribution of the same source in the positive direction. Therefore, if we want to describe $v^+ (x, s)$ in terms of $v^+ (1, s)$, we have to discard this extra source. Indeed, this is what equation (1) does. It actually makes a "short cut" in describing $v^+ (x, s)$ in terms of $v^+ (1, s)$ and compensates for the differences between they by an "artificial direct field", instead of following the entire paths of the disturbances along the system. This can be verified by extracting the contribution of the disturbance $f^* (x_1, s)$ from equation (1). Thus, it is found that

$$\begin{aligned} v^+ (x, s) &= e^{\gamma_1 (x-1)} v^+ (1, s) - e^{\gamma_1 (x - x_1)} f^* (x_1, s)/(2s) \\ &= e^{\gamma_1 (x-1)} [v^+ (1, s) - e^{\gamma_1 (x - x_1)} f^* (x_1, s)/(2s)], \quad x_1 > x. \end{aligned} \tag{4}$$

The upper parts of the above equation is taken straight from equation (1), while the lower part of it shows exactly what we were trying to explain in Figure 2, substituting equation (2) into equation (4), gives

$$\begin{aligned} v^+ (x, s) &= \left[\frac{e^{\gamma_1 (x - x_1)} + r_0 e^{(\gamma_1 x - \gamma_2 x_1)}}{1 - r_0 r_1 e^{\gamma_1 - \gamma_2}} - e^{\gamma_1 (x - x_1)} \right] f^* (x_1, s)/(2s) \\ &= \frac{r_0 r_1 e^{\gamma_1 (1 + x - x_1) - \gamma_2} + r_0 e^{(\gamma_1 x - \gamma_2 x_1)}}{1 - r_0 r_1 e^{(\gamma_1 - \gamma_2)}} f^* (x_1, s)/(2s), \quad x_1 > x, \end{aligned} \tag{5}$$

where the "artificial direct field" is cancelled, giving rise to the true paths sketched in Figure 2. The second term on the numerater of equation (5) is the contribution of the negative wave of $f^*(x_1, s)$ after being reflected at the left boundary and appearing as a positive wave at the detection point. This way of describing the disturbance propagation paths can be better appreciated if one remembers that the first term on the upper part of equation (5) holds for the case $x_1 < x$.

As the reader may realize, the intention in the above discussion is primarily to describe how equation (1) factorizes the disturbance propagation paths. The importance of this, when each term in it exactly reflects the physics of the system, is the "local modelling ability" inherent in the solution. In other words, in some circumstances the information contained in equation (1) may be sufficient and the details regarding the reverberant response contained in equation (2) may be unnecessary. This point is further clarified in the next section

3. DISTURBANCE PROPAGATION PATHS IN A TRANSLATING STRING WITH A SPRING-MASS-DAMPER AND POINT CONTROLLER CONSTRAINTS

Consider the system shown in Figure 3, where the string is constrained by a stationary spring-mass-damper and a point controller positioned at x_1 and x_2 respectively. The quantities k, m and d are the constraint stiffness, mass and damping constants respectively. Disturbances are assumed to be applied at regions upstream of the guide position. In order to determine the modified propagation paths of this system, all constraints will be expressed in terms of the boundary incoming water $v^+(1, s)$.

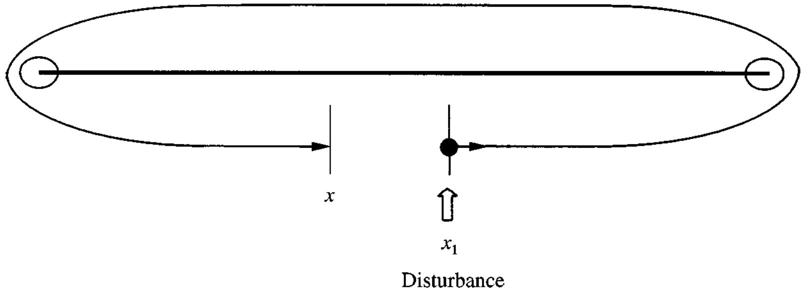

Figure 2. Propagation paths of the positive wave generated by a disturbance at x_1 detected at $x < x_1$.

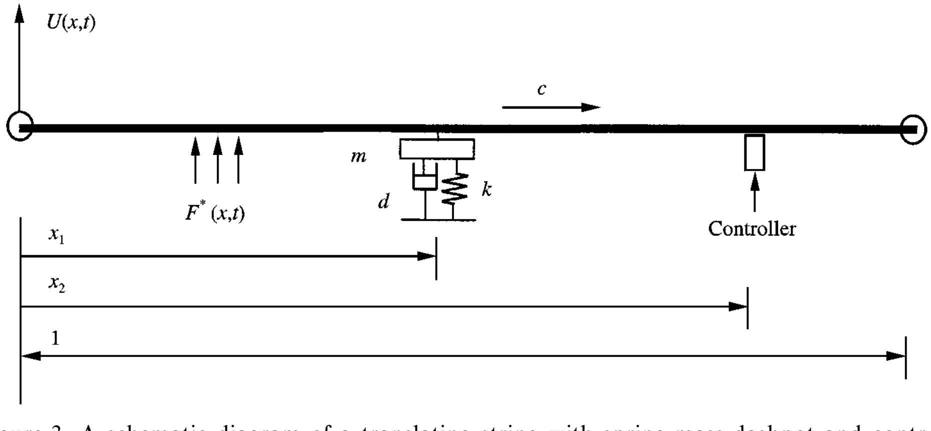

Figure 3. A schematic diagram of a translating string with spring-mass-dashpot and controller constraints at points x_1 and x_2.

Suppose the objective of the controller constraint is the minimization of vibration in regions upstream of the actuator position. Thus, in principle, is fulfilled by cancelling the negative wave component at the actuator location. Thus, with reference to equation (1), this wave component is given by

$$v^- (x_2, s) = r_1 \, e^{\gamma_2 (x_2 - 1)} \, v^+ (1, s) + f^* (x_2, s)/(2s), \tag{6}$$

which consists of the contribution from the reverberant field through $v^+ (1, s)$ plus the control force, $f^* (x_2, s)$, direct field. Note that no information regarding the dynamics of regions upstream of the controller position is explicitly involved in equation (6). Thus, the cancellation of this wave component required that

$$f^* (x_2, s)/(2s) = -r_1 \, e^{\gamma_2 (x_2 - 1)} \, v^+ (1, s), \quad x_2 \leqslant 1. \tag{7}$$

Although the above control law appears very simple, the detection of (reflected) $v^+ (1, s)$ is difficult. This is due to the fact that it cannot be sensed directly but must be inferred from physical measurements. This in turn complicates the control procedure [6, 7]. This is because if the measurement is made in terms, say, of a displacement sensor, this will lead to the detection of both negative and positive wave components at the sensor position. On the other hand, the controller will suppress then negative wave and generate a wave component, with the same amplitude, in the positive direction. The consequence of this is the creation of additional "closed-loop paths" between the actuator position and the system end (the right end in the present case).

Since the right boundary incoming wave cannot be sensed directly, it is desirable to express the controller force in equation (7) in terms of physical measurements. Suppose that $u(x_3, s)$ is the displacement at the sensor position, then for $x_3 \geqslant x_2$,

$$u(x_3, s) = v^+ (x_3, s) + v^- (x_3, s) = (e^{\gamma_1 (x_3 - 1)} + r_1 \, e^{\gamma_2 (x_3 - 1)}) \, v^+ (1, s). \tag{8}$$

By expressing $v^+ (1, s)$ in terms of $u(x_3, s)$ and substituting the result in equation (7), the control force can be given in terms of the sensor displacement as

$$f^*(x_2, s) = -\frac{2sr_1 \, e^{\gamma_1 (1 - x_3) - \gamma_2 (1 - x_2)}}{1 - (-1) \, r_1 e^{(\gamma_1 - \gamma_2)(1 - x_1)}} \, u(x_3, s). \tag{9}$$

From equation (1), the displacement at the spring-mass-damper position is given by

$$u(x_1, s) = (e^{\gamma_1 (x_1 - 1)} + r_1 \, e^{\gamma_2 (x_1 - 1)}) \, v^+ (1, s) + (e^{\gamma_2 (x_1 - x_2)} - e^{\gamma_1 (x_1 - x_2)}) f^* (x_2, s)/(2s), \tag{10}$$

or in terms of $v^+ (1, s)$ alone, with the use of equation (7), as

$$u(x_1, s) = e^{\gamma_1 (x_1 - 1)} \, (1 + r_1 \, e^{(\gamma_1 - \gamma_2)(1 - x_2)}) \, v^+ (1, s). \tag{11}$$

The guide reaction force can be given as

$$f^* (x_1, s) = -(k + sd + s^2 m) \, u(x_1, s). \tag{12}$$

If the system is subjected to left end prescribed motion $u^*(0, s)$, then by substituting equations (7), (11) and (12) to equation (2), the modified paths of the right boundary incoming wave becomes

$$v^+(1, s) = \frac{a_2\ \mathrm{e}^{\gamma_1}}{[\{1 - a_1\ r_0\,\mathrm{e}^{(\gamma_1 - \gamma_2)x_1}\}\{1 - (-1)\ r_1\,\mathrm{e}^{(\gamma_1 - \gamma_2)(1 - x_2)}\}]}\ u^*(0, s), \qquad (13)$$

where

$$a_1 = -(k + sd + s^2 m)/(k + 2s + sd + s^2 m) \quad \text{and} \quad a_2 = 2s/(k + 2s + sd + s^2 m) \qquad (14)$$

are the reflection and transmission coefficients at the spring-mass-damper position [8].

As clearly shown in the denominator of equation (13), there are two sources of possible resonance in $v^+(1, s)$. The first is due to the round trip path that occurs between the string's left boundary and the spring-mass-damper position. The second source is introduced by the action of the controller itself. The controller, by suppressing the string's right boundary reflected wave, effectively acts as a perfectly reflecting device (with reflection coefficient equal to -1). The latter source can be avoided by placing the controller at the right boundary ($x_2 = 1$), since it effectively generated only one wave component that destructively interacts with the reflected $v^+(1, s)$.

As the disturbance propagation paths are described in terms of $v^+(1, s)$, then the second step is to determine the physical response along the entire system.

With equations (7), (11) and (12), the response at $x \in [x_2, 1]$ is

$$u(x, s) = (\mathrm{e}^{\gamma_1(x-1)} + r_1\,\mathrm{e}^{\gamma_2(x-1)})\ v^+(1, s), \qquad (15)$$

where $v^+(1, s)$ is given in equation (13).

Similarly, the response at $x \in [x_1, x_2]$ is

$$u(x, s) = \mathrm{e}^{\gamma_1(x-1)}(1 + r_1\ \mathrm{e}^{(\gamma_1 - \gamma_2)(1 - x_2)})\ v^+(1, s),$$

$$= \frac{a_2\ \mathrm{e}^{\gamma_1 x}}{1 - a_1 r_0\ \mathrm{e}^{(\gamma_1 - \gamma_2)x_1}}\ u^*(0, s). \qquad (16)$$

where it is clear that the negative-going wave is cancelled by the controller.

The response at $x \in [0, x_1]$ is

$$u(x, s) = (1/a_2)\ \mathrm{e}^{\gamma_1(x-1)}(1 - a_1 r_0\,\mathrm{e}^{(\gamma_1 - \gamma_2)(x_1 - x)})\ (1 + r_1\ \mathrm{e}^{(\gamma_1 - \gamma_2)(1 - x_2)})\ v^+(1, s)$$

$$= \frac{\mathrm{e}^{\gamma_1 x}\ (1 - a_1 r_0\,\mathrm{e}^{(\gamma_1 - \gamma_2)(x_1 - x)})}{1 - a_1 r_0\,\mathrm{e}^{(\gamma_1 - \gamma_2)x_1}}\ u^*(0, s). \qquad (17)$$

Notice that, as long as $x_2 \in (x_1, 1)$, solutions (16) and (17) are independent of the controller location. For $x_3 = x_2 = 1$ (collocated boundary control), equation (15) is meaningful only for $x = 1$.

Before performing the inverse Laplace transform of the solution each term in the denominators is expanded in a manner similar to equation (3). The time domain response is then obtained numerically using the Laplace transform package of

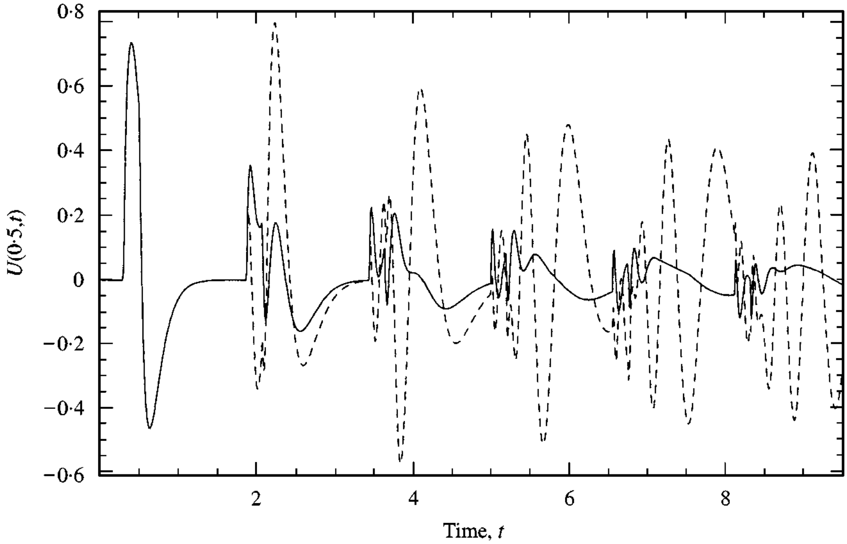

Figure 4. Response measured at the central constraint of the axially moving string subjected to a rectangular displacement pulse applied at $x = 0$; with the controller constraint (solid lines); without control (dashed lines). $k = 10$, $m = 0{\cdot}1$, $d = 0$ and $c = 0{\cdot}6$.

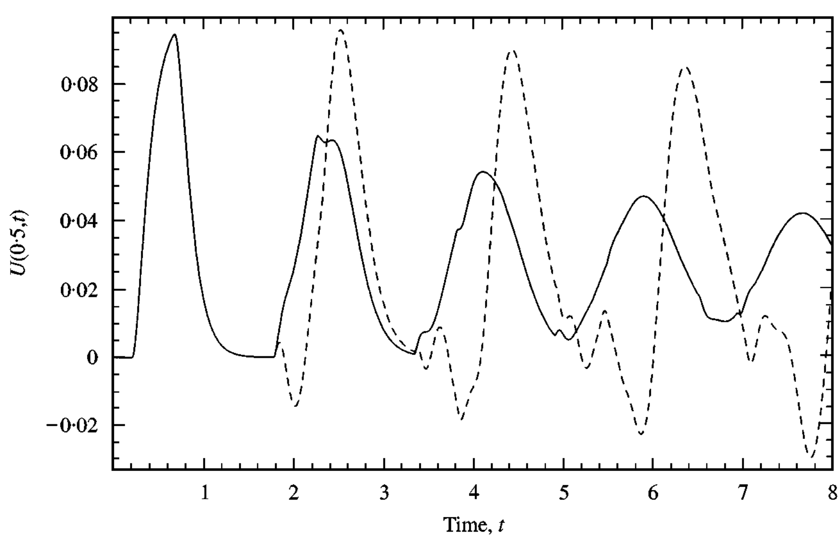

Figure 5. Response measured at the central constraint of the axially moving string subjected to a unit step load applied at $x = 0{\cdot}15$; with the controller constraint (solid lines); without control (dashed lines). $k = 10$, $m = 0{\cdot}1$, $d = 0$ and $c = 0{\cdot}6$.

Mathematica. The following simulation results illustrate the response of the string with fixed boundaries ($r_0 = r_1 = -1$) and a central dynamic constraint ($x_1 = 0{\cdot}5$) with $k = 10$. $m = 0{\cdot}1$ and $d = 0$. Figure 4 gives the response of the system to a rectangular displacement pulse of unit magnitude and duration $\tau = 0{\cdot}2$. In Figure 5, the system is subjected to a unit step load applied at $x = 0{\cdot}15$. In both figures the response is measured at the spring-mass system position and the string transport speed $c = 0{\cdot}6$. In these figures, the dashed lines (solid lines) are the system's response without (with) the controller constraint. In both cases the response with controller constraint is reduced.

4. CONCLUSION

This work shows that disturbance propagation paths in continuous systems can be visualized, in a simple and compact way, by simulating periodic waveguide dynamics. This is illustrated through a simple example consisting of an axially moving string constrained by a spring-mass-damper stationary system and by a point controller. The advantage of factorizing the response into free and forced fields is shown and its interpretation for time domain analysis is given. By using Laplace transform technique, plots characterizing the exact response of the string system subjected to transient loads are also provided.

ACKNOWLEDGMENTS

This work was carried out while the author was a Post-doctoral Research Fellow at DISG under the guidance of Prof. Vestroni. The author acknolwedges the referee for his useful comments.

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