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NON-LINEAR FORCED VIBRATIONS OF AN INHOMOGENEOUS LAYER

İ COŞKUN

Faculty of Civil Engineering, Yıldız Technical University, 80750 *Yıldız, Istanbul, Turkey*

AND

H. ENGIN AND M. ERTAÇ ERGÜVEN

Faculty of Civil Engineering, Istanbul Technical University, 80626 *Maslak, Istanbul, Turkey*

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The non-linear vibrations of an inhomogeneous soil layer which is subjected to a harmonic motion along its bottom are investigated in this study. The Ramberg–Osgood model is transformed to a suitable form to obtain an analytical solution and it is assumed that the shear modulus of the layer varies with depth. The governing equation is a non-linear partial differential equation. Because of weak non-linearity, the displacement and forcing frequency are expanded into perturbation series by using the Lindstedt–Poincaré technique, and it is assumed that the response has the same periodicity as the forcing. Then, the zeroeth and the first order linear equations of motion and boundary conditions are obtained. Different types of solutions are obtained for the zeroeth order equation depending on the inhomogeneity parameter α. The orthogonality condition of Millman–Keller [1] is used to extract secular terms which are important in the resonance region. Then, the variation of the amplitude at the top versus the forcing frequency Ω is investigated for some values of inhomogeneity and perturbation parameters.

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1. INTRODUCTION

In the recent years, engineers and geologists have become increasingly aware of the need for evaluating the effect of soil conditions on the ground surface accelerations and amplifications during earthquakes. Because the changes with soil conditions in the form of the response spectra can have significant effects on the lateral forces on structures, it would seem to be desirable that the nature of the soil conditions underlying a site should be taken into account, either qualitatively or quantitatively in evaluating the lateral forces for design purposes. Non-linear vibrations of continuous systems have largely been investigated in the last two decades. This type of problem is encountered in the vibrations of lumped-mass systems, plates, shells, beams, water and radio waves and motions of gases [2]. The source of non-linearity

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may be geometric, inertial, material or some combinations of these. The geometric non-linearity comes from the non-linear strain–displacement relation. Finite displacements, rotations and large curvatures are causes of this type of non-linearity. The inertial non-linearity occurs in systems with single or continuous mass. The materially non-linear effects (or physical non-linearity) come from the non-linear stress–strain relations. In this study, only the material non-linearity is considered. In many cases, the ground motions developed near the surface of a soil layer during an earthquake may be attributed primarily to the upward propagation of shear waves from an underlying rock formation. If the ground surface, the rock surface, or the boundaries between different soil layers are inclined, analyses of the response of the soil deposit can only be made by techniques such as the finite-element method or boundary-element method. If the ground surface, the rock surface, and the boundaries between soil layers are essentially horizontal, however, the lateral extent of the deposit has no influence on the response, and the deposit may be considered as a series of semi-infinite layers. In such cases the ground motions induced by a seismic excitation at the base are only the result of shear deformations in the soil, and the layer may be considered as a one-dimensional shear beam. Methods of analyzing the response of such layers are described in detail elsewhere [3, 4]. To determine the response of a soil deposit having irregularly varying soil properties, to use a lumped-mass type of analysis is convenient [4, 5]. Non-linearity of soil can be taken into account by using the force–displacement characteristics of springs between the masses.

Non-linearity may take place in the governing equations and stress boundary conditions. In the literature, a lot of non-linear governing equations have been proposed such as the Duffing, Van der Pol, Korteweg-de Vries, Sine-Gordon, the non-linear Schrödinger equation [6], and the idealization of the Ramberg–Osgood model [7, 8]. Because these equations are partial differential equations (lumped-mass system expected) the spectrum contains an infinite number of modes. A few modes depending on the degree of non-linearity may give sufficient knowledge about the response of systems. In some cases, the first mode proves to be enough. There are some interesting phenomenae in non-linear vibrations which cannot be seen in linear vibrations; such as shifts in frequency, dispersion due to the amplitude, generation of harmonics, removal of singularities in resonance, jump from one state to another, increase or decrease of amplitude. A lot of methods have been developed by several authors to solve non-linear vibration problems. Bojadziev and Lardner [9, 10] and Bojadziev [10] used the Krylov–Bogoliubov–Mitropolsky (KBM) technique for longitudinal non-linear vibrations of bars. Ablowitz *et al.* [7, 8] investigated the non-linear vibrations of a layer which is forced along its bottom by using the Lindstedt–Poincaré technique and the method of multiple scales. Engin *et al.* [12] investigated the same problem for a multilayered system by using the orthogonality of the zeroeth order solution to the right-hand side of the first order equation of motion. Sridhar *et al.* [13, 14] studied non-linear symmetric and asymmetric responses of circular plates by using the Galerkin procedure.

In this paper, the non-linear undamped vibrations of an inhomogeneous soil layer subjected to a harmonic motion along its bottom are investigated. The layer lies on a bed rock. It is assumed that the linear part of the shear modulus of the layer varies with depth. Transforming the Ramberg–Osgood model to a suitable form for analytical solution, a non-linear partial differential equation is obtained as the governing equation. The method of solution of this problem is based on the work of Engin *et al.* [12]. In this study, the solution is expanded to an inhomogeneous medium, which was given by Engin *et al.* [15] only for the inhomogeneity parameter α, $\alpha = 0{\cdot}5$. The influence of the characteristics of the base rock motion and soil conditions on ground response are shown by figures.

2. FUNDAMENTAL EQUATIONS

Figure 1 shows a soil layer of thickness d lying on a bed rock. The bed rock makes as harmonic horizontal motion $a\cos\Omega t$, so the bottom of the layer has to make the same motion because of the continuity of displacements at the interface. Since the thickness of the layer is constant, only shear strains occur in the system. As a result, shear vibrations of the layer are investigated. For this layer, the kinematic relation is

$$\gamma = \frac{\partial U}{\partial y}, \tag{1}$$

where γ is the shear strain, $U(y, t)$ is the lateral displacement of the layer which is assumed to be small, y is the co-ordinate measured from the top and t is the time.

A lot of constitutive equations have been proposed for soil-type medium. Among these, the Ramberg–Osgood model is widely used as it can be transformed to a suitable form for analytical solution [12]. Rearranging this transformed model and considering the relation between the stresses and strains, the non-linear stress–strain relation can be written as

$$\tau = G_{max}\gamma - G_1\gamma^3, \tag{2}$$

where τ is the shear stress, G_{max} and G_1 are, respectively, the linear and non-linear shear moduli.

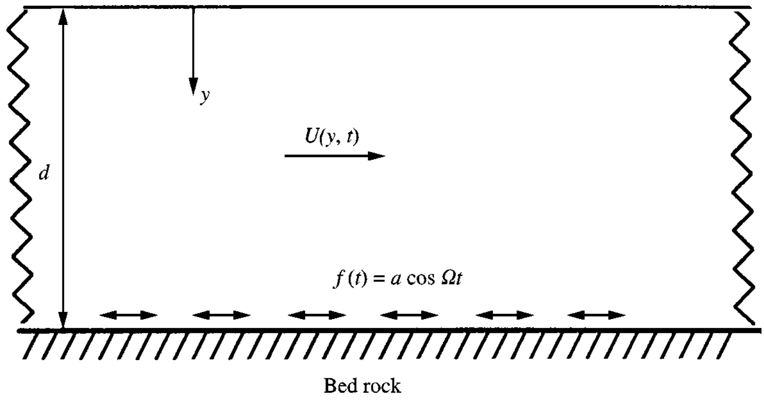

Figure 1. Soil layer on a bed rock.

Some factors in relation to the undrained deformation properties of clays show that it is appropriate to give some attention to their implications. Almost all clays show that the shape of the deformed surface is likely to be very sensitive to the stiffening of the soil with depth. Variation of some kind of rocks with depth and from place to place is examined in terms of the drained modulus of elasticity E. For a number of sites, depth profiles of moduli and other data are presented; these are related to lithology and weathering which are related in turn to the geology. The values of E can be obtained by laboratory tests, field tests and analysis of case histories. For instance, in triassic rocks and over-consolided clays, the modulus profile is inhomogeneous and the modulus E increases approximately linearly with depth. After 1970 the effect of inhomogeneity began to be considered in soil structure interaction analysis, both static and dynamic, soon after Gibson [16] introduced a special type of inhomogeneity with linearly varying stiffness with depth in an incompressible half-space with vanishing modulus at the surface. This so-called Gibson soil is adopted for modelling water-saturated normally consolidated clays. Experimental evidence shows that irrespective of the nature of sedimentary strata the depth variation of the shear modulus G follows a power law $G \propto z^n$ with $n \leqslant 1$, where z is the depth below soil surface. Based on the fact that clays are almost fully saturated the value $n = 1$ has been widely used for normally consolidated clays and for sand. For cohesionless soils, however, the exponent n is between 0·25 and 0·4 depending on the soil under consideration. In the early stages of analytical treatment of wave propagation problems in inhomogeneous soils exponents $n > 1$ have been used, mainly due to the resulting simplifications in the equations involved. In addition to the shear modulus, specific values for the Poission ratio were also often assumed solely for analytical convenience.

Considering these reasons, it is assumed that G_{max} varies with depth as

$$G_{max} = G_0\left(1 + k\frac{y}{d}\right)^{\alpha}, \tag{3}$$

where G_0 is the shear modulus at the top, k and α are the inhomogeneity parameters [16–18]. According to this, the non-linear stress-strain relation becomes

$$\tau = G_0\left(1 + k\frac{y}{d}\right)^{\alpha}\frac{\partial U}{\partial y} - G_1\left(\frac{\partial U}{\partial y}\right)^3. \tag{4}$$

Neglecting body forces, the equation of motion for the layer can be written as

$$\frac{\partial \tau}{\partial y} = \rho\frac{\partial^2 U}{\partial t^2}, \tag{5}$$

where ρ is the mass density. Substituting equation (4) into equation (5), the following differential equation is obtained:

$$G_0\left(1 + k\frac{y}{d}\right)^{\alpha}\frac{\partial^2 U}{\partial y^2} + G_0\frac{k}{d}\alpha\left(1 + k\frac{y}{d}\right)^{\alpha - 1}\frac{\partial U}{\partial y} - 3G_1\left(\frac{\partial U}{\partial y}\right)^2\frac{\partial^2 U}{\partial y^2} = \rho\frac{\partial^2 U}{\partial t^2}, \tag{6}$$

where the term $(\partial U/\partial y)^2\ (\partial^2 U/\partial y^2)$ indicates the non-linearity. The layer is subjected to a harmonic motion $a\cos\Omega t$ at the bottom $y = d$ and traction free at the surface $y = 0$. Thus, the boundary conditions can be written as

$$\tau|_{y=0} = 0, \qquad U(y,t)|_{y=d} = a\cos\Omega t. \tag{7}$$

In this study, weak non-linearity is considered. As a result, the response of the layer is assumed to be harmonic with the period $2\pi/\Omega$, as the forcing term. Thus, the periodicity condition is written as [12]

$$U(y,t) = U(y, t + 2\pi/\Omega). \tag{8}$$

It is convenient to rewrite the above equations in terms of dimensionless quantities, which are defined as follows:

$$u = U/a, \qquad s = \Omega t, \qquad \xi = y/d, \qquad \varepsilon = 3\frac{G_1}{G_0}\left(\frac{a}{d}\right)^2, \qquad q_0^2 = \frac{\rho d^2}{G_0}\Omega_0^2, \tag{9}$$

where u is the dimensionless displacement, s and ξ are the dimensionless time and co-ordinate respectively. The parameters Ω_0, $\varepsilon(\varepsilon \ll 1)$ and q_0 denote the frequency for the linear analysis of the layer, perturbation parameter and dimensionless wave number respectively. The non-linear governing equation (6), boundary conditions (7) and periodicity condition (8) can now be written in the non-dimensional form

$$(1+k\xi)^\alpha\frac{\partial^2 u}{\partial \xi^2} + k\alpha(1+k\xi)^{\alpha-1}\frac{\partial u}{\partial \xi} - \varepsilon\left(\frac{\partial u}{\partial \xi}\right)^2\frac{\partial^2 u}{\partial \xi^2} = q_0^2\left(\frac{\Omega}{\Omega_0}\right)^2\frac{\partial^2 u}{\partial s^2}, \tag{10}$$

$$(1+k\xi)^\alpha\frac{\partial u}{\partial \xi} - \frac{\varepsilon}{3}\left(\frac{\partial u}{\partial \xi}\right)^3\Bigg|_{\xi=0} = 0, \qquad u(\xi,s)|_{\xi=1} = \cos s, \tag{11}$$

$$u(\xi,s) = u(\xi, s+2\pi). \tag{12}$$

3. SOLUTION

Due to the non-linear term $(\partial u/\partial \xi)^2\,(\partial^2 u/\partial \xi^2)$, to find an exact solution of the governing equation (10) is impossible. For this reason, considering weak non-linearity, an approximate solution is sought by using the perturbation method. Among a few variants of the perturbation method, the Lindsted–Poincaré technique seems to be a suitable one [2]. For the use of this technique, the displacement $u(\xi,s;\varepsilon)$ and forcing frequency $\Omega(\varepsilon)$ are expanded into the perturbation series as follows:

$$u(\xi,s;\varepsilon) = u_0(\xi,s) + \varepsilon u_1(\xi,s) + \cdots, \tag{13}$$

$$\Omega(\varepsilon) = \Omega_0(1 + \varepsilon\Omega_1 + \cdots), \tag{14}$$

where Ω_1 is the dimensionless frequency shift, u_0 and u_1 are the linear and the first order parts of the displacement respectively. With the substitution of these series into the governing equation (10) and boundary conditions (11), and equating the coefficients of the same powers of ε to zero, a system of linear partial differential equations is obtained replacing non-linear governing equation (10). Thus, the zeroeth order equation of motion and the boundary conditions become

$$(1 + k\xi)^{\alpha}\frac{\partial^2 u_0}{\partial \xi^2} + k\alpha(1 + k\xi)^{\alpha-1}\frac{\partial u_0}{\partial \xi} - q_0^2\frac{\partial^2 u_0}{\partial s^2} = 0, \tag{15}$$

$$\left.\frac{\partial u_0}{\partial \xi}\right|_{\xi=0} = 0, \qquad u_0(\xi, s)|_{\xi=1} = \cos s. \tag{16}$$

The first order equation of motion and the boundary conditions become

$$(1 + k\xi)^{\alpha}\frac{\partial^2 u_1}{\partial \xi^2} + k\alpha(1 + k\xi)^{\alpha-1}\frac{\partial u_1}{\partial \xi} - q_0^2\frac{\partial^2 u_1}{\partial s^2} = 2q_0^2\Omega_1\frac{\partial^2 u_0}{\partial s^2} + \frac{\partial^2 u_0}{\partial \xi^2}\left(\frac{\partial u_0}{\partial \xi}\right)^2, \tag{17}$$

$$\left.\frac{\partial u_1}{\partial \xi}\right|_{\xi=0} = 0, \qquad u_1(\xi, s)|_{\xi=1} = 0, \tag{18}$$

where u_0 and u_1 have the period 2π. The periodicity conditions for the zeroeth and first order terms can be written by using equation (12) as

$$u_0(\xi, s) = u_0(\xi, s + 2\pi), \qquad u_1(\xi, s) = u_1(\xi, s + 2\pi). \tag{19}$$

3.1. ZEROETH ORDER SOLUTION

Equation (15) is homogeneous and linear so the method of separation of variables can be used in the form

$$u_0(\xi, s) = \bar{u}_0(\xi)\cos s. \tag{20}$$

By substituting equation (20) into equations (15) and (16), the zeroeth order equation of motion and the boundary conditions are obtained as

$$(1 + k\xi)^{\alpha}\frac{\mathrm{d}^2\bar{u}_0}{\mathrm{d}\xi^2} + k\alpha(1 + k\xi)^{\alpha-1}\frac{\mathrm{d}\bar{u}_0}{\mathrm{d}\xi} + q_0^2\bar{u}_0 = 0, \tag{21}$$

$$\left.\frac{\mathrm{d}\bar{u}_0}{\mathrm{d}\xi}\right|_{\xi=0} = 0, \qquad \bar{u}_0(\xi, s)|_{\xi=1} = 1. \tag{22}$$

Equation (21) has different solutions depending on different values of the inhomogeneity parameter α [19]. The general solution function of equation (21) can be written as

$$\bar{u}_0 = (1 + k\xi)^m Z_p[(q_0/rk)(1 + k\xi)^r], \tag{23}$$

where p, m and r are real constants and given by

$$p = \frac{1-\alpha}{2-\alpha}, \qquad m = \frac{1-\alpha}{2}, \qquad r = \frac{2-\alpha}{2}. \tag{24}$$

In equation (23), Z_p represents the Bessel functions of order p. Solution (23) has several forms with respect to the values of α. Some of them are studied below.

3.1.1. *The case for $\alpha < 2$ and p a rational number*

In this case, the zeroeth order solution $\bar{u}_0$ can be written from equation (23) in the form

$$\bar{u}_0(\xi) = (1 + k\xi)^m \left[A_1 \mathrm{J}_p\left(\frac{q_0}{rk}(1 + k\xi)^r\right) + A_2 \mathrm{J}_{-p}\left(\frac{q_0}{rk}(1 + k\xi)^r\right)\right], \qquad (25)$$

where J_p and J_{-p} are the first kind Bessel functions of order p, A_1 and A_2 are the constants of integration which are to be determined from the boundary conditions. Using boundary conditions (22), the integration constants can be obtained as

$$A_1 = \frac{\mathrm{Z}_2(1 + k)^{-m}}{\mathrm{Z}_1\mathrm{Z}_4 + \mathrm{Z}_2\mathrm{Z}_3}, \qquad A_2 = \frac{\mathrm{Z}_1(1 + k)^{-m}}{\mathrm{Z}_1\mathrm{Z}_4 + \mathrm{Z}_2\mathrm{Z}_3}, \qquad (26)$$

where

$$\mathrm{Z}_1 = \mathrm{J}_{p-1}\left(\frac{q_0}{rk}\right), \qquad \mathrm{Z}_2 = \mathrm{J}_{-p+1}\left(\frac{q_0}{rk}\right), \qquad \mathrm{Z}_3 = \mathrm{J}_p\left(\frac{q_0}{rk}(1 + k)^r\right),$$

$$\mathrm{Z}_4 = \mathrm{J}_{-p}\left(\frac{q_0}{rk}(1 + k)^r\right). \qquad (27)$$

Now, the zeroeth order solution has been obtained in terms of the unknown parameter q_0. The amplitude of the vibration at the surface can be obtained from equation (25) as

$$A = \bar{u}_0(\xi)|_{\xi=0} = A_1 \mathrm{J}_p\left(\frac{q_0}{rk}\right) + A_2 \mathrm{J}_{-p}\left(\frac{q_0}{rk}\right). \qquad (28)$$

where A can be considered as the amplitude magnification factor. As it is known, there are some special cases in which the values of q_{0n} make A very large. These important cases are called as resonance. The values of q_{0n}, $n = 1, 2, \ldots$, are the roots of the equation

$$\mathrm{J}_{p-1}\left(\frac{q_0}{rk}\right)\mathrm{J}_{-p}\left(\frac{q_0}{rk}(1 + k)^r\right) + \mathrm{J}_{-p+1}\left(\frac{q_0}{rk}\right)\mathrm{J}_p\left(\frac{q_0}{rk}(1 + k)^r\right) = 0. \qquad (29)$$

Using the roots of equation (29), one can calculate the natural frequencies from equation (9) as

$$\Omega_{0n} = q_{0n}\left(\frac{G_0}{\rho d^2}\right)^{1/2}, \qquad n = 1, 2, \ldots . \qquad (30)$$

It is understood that the number of natural frequencies are infinite.

3.1.2. *The case for $\alpha < 2$ and p an integer number*

In this case, the zeroeth order solution $\bar{u}_0$ can be written from equation (23) in the form

$$\bar{u}_0(\xi) = (1 + k\xi)^m \left[B_1 \mathrm{J}_p\left(\frac{q_0}{rk}(1 + k\xi)^r\right) + B_2 \mathrm{Y}_p\left(\frac{q_0}{rk}(1 + k\xi)^r\right)\right], \qquad (31)$$

where Y_p is the second kind Bessel function of order p, B_1 and B_2 are the constants of integration. From boundary conditions (22), B_1 and B_2 can be obtained as

$$B_1 = -\frac{\mathrm{Z}_2(1+k)^{-m}}{\mathrm{Z}_1\mathrm{Z}_4 - \mathrm{Z}_2\mathrm{Z}_3}, \qquad B_2 = \frac{\mathrm{Z}_1(1+k)^{-m}}{\mathrm{Z}_1\mathrm{Z}_4 - \mathrm{Z}_2\mathrm{Z}_3},$$

where

$$\mathrm{Z}_1 = \mathrm{J}_{p-1}\left(\frac{q_0}{rk}\right), \qquad \mathrm{Z}_2 = \mathrm{Y}_{p-1}\left(\frac{q_0}{rk}\right), \qquad \mathrm{Z}_3 = \mathrm{J}_p\left(\frac{q_0}{rk}(1+k)^r\right),$$

$$\mathrm{Z}_4 = \mathrm{Y}_p\left(\frac{q_0}{rk}(1+k)^r\right). \tag{32}$$

The amplitude of the vibration at the surface can be obtained from equation (31) as

$$A = \bar{u}_0(\xi)|_{\xi=0} = B_1\mathrm{J}_p\left(\frac{q_0}{rk}\right) + B_2\mathrm{Y}_p\left(\frac{q_0}{rk}\right). \tag{33}$$

The values of q_{0n}, $n = 1, 2, \ldots$, are the roots of the equation $\mathrm{Z}_1\mathrm{Z}_4 - \mathrm{Z}_2\mathrm{Z}_3 = 0$, and the natural frequencies of the layer can be obtained by using equation (30). It is noted that when $\alpha = 1$, that is $p = 0$, $G_{\max}$ varies linearly with depth. In this case, the solution of equation (21) becomes

$$\bar{u}_0(\xi) = C_1\mathrm{J}_0\left(\frac{2q_0}{k}\sqrt{1+k\xi}\right) + C_2\mathrm{Y}_0\left(\frac{2q_0}{k}\sqrt{1+k\xi}\right), \tag{34}$$

where J_0 and Y_0 are the zeroeth order first and second kind Bessel functions respectively; C_1 and C_2 are the constants of integration. Using boundary conditions (22), these constants can be obtained as

$$C_1 = -\mathrm{Y}_1\left(\frac{2q_0}{k}\right)\Big/\Delta_1, \qquad C_2 = \mathrm{J}_1\left(\frac{2q_0}{k}\right)\Big/\Delta_1, \tag{35}$$

$$\Delta_1 = \mathrm{J}_1\left(\frac{2q_0}{k}\right)\mathrm{Y}_0\left(\frac{2q_0}{k}\sqrt{1+k}\right) - \mathrm{Y}_1\left(\frac{2q_0}{k}\right)\mathrm{J}_0\left(\frac{2q_0}{k}\sqrt{1+k}\right), \tag{36}$$

where J_1 and Y_1 are the first order Bessel functions. The amplitude of the vibration at the surface can be obtained from equation (34) as

$$A = \bar{u}_0|_{\xi=0} = C_1\mathrm{J}_0\left(\frac{2q_0}{k}\right) + C_2\mathrm{Y}_0\left(\frac{2q_0}{k}\right). \tag{37}$$

The values of q_{0n}, $n = 1, 2, \ldots$ are the roots of the equation

$$\mathrm{J}_1\left(\frac{2q_0}{k}\right)\mathrm{Y}_0\left(\frac{2q_0}{k}\sqrt{1+k}\right) - \mathrm{Y}_1\left(\frac{2q_0}{k}\right)\mathrm{J}_0\left(\frac{2q_0}{k}\sqrt{1+k}\right) = 0. \tag{38}$$

The natural frequencies of the layer can be obtained by using equation (30).

3.1.3. *The case for* $\alpha = 2$

In this case, equation (21) becomes as follows:

$$(1 + k\xi)^2 \frac{d^2 \bar{u}_0}{d\xi^2} + 2k(1 + k\xi) \frac{d\bar{u}_0}{d\xi} + q_0^2 \bar{u}_0 = 0. \tag{39}$$

This is an Euler equation of the second order [18]. After some manipulations, solution of this equation can be obtained as

$$\bar{u}_0(\xi) = D_1(1 + k\xi)^{p_1 - \frac{1}{2}} + D_2(1 + k\xi)^{-p_1 - \frac{1}{2}}, \tag{40}$$

where p_1 may be an imaginary number and is given by

$$p_1^2 = \frac{1}{4} - \frac{q_0^2}{k^2}. \tag{41}$$

Here, D_1 and D_2 are the constants of integration and they can be obtained by using boundary conditions (22) as follows:

$$D_1 = (p_1 + 1/2)/\Delta_2, \qquad D_2 = (p_1 - 1/2)/\Delta_2,$$

$$\Delta_2 = [(p_1 + 1/2)(1 + k)^{p_1} + (p_1 - 1/2)(1 + k)^{-p_1}]/\sqrt{1 + k}. \tag{42}$$

Equating Δ_2 to zero, one obtains the non-dimensional natural frequency of the layer as the roots of the following equation:

$$\Delta_2 = 0 \rightarrow \frac{\sqrt{(1/4) - (q_0^2/k^2)} - \frac{1}{2}}{\sqrt{(1/4) - (q_0^2/k^2)} + \frac{1}{2}} + (1 + k)^{\sqrt{1 - 4q_0^2/k^2}} = 0. \tag{43}$$

In this case, the amplitude of the vibration at the surface can be obtained from equation (40)as

$$A = \bar{u}_0(\xi)|_{\xi = 0} = D_1 + D_2. \tag{44}$$

3.1.4. *The case for* $\alpha = 3$

In this case, equation (21) becomes as follows:

$$(1 + k\xi)^3 \frac{d^2 \bar{u}_0}{d\xi^2} + 3k(1 + k\xi)^2 \frac{d\bar{u}_0}{d\xi} + q_0^2 \bar{u}_0 = 0. \tag{45}$$

Because of the variable coefficients, this equation is solved by using the power series. $\xi = -1/k$ is the singular point of the equation. If k is chosen as $k > -1$, equation (45) does not have any singularity inside the layer. To keep away from the singular point, power series solution is carried out at the center of the layer, $\xi = 1/2$. The series solution of equation (45) can be taken as

$$\bar{u}_0(\xi) = \sum_{n=0}^{\infty} A_n(\xi - 1/2)^n. \tag{46}$$

The coefficients A_n can be written in terms of the independent unknowns A_0 and A_1 as follows:

$$A_n = A_0 f_n + A_1 g_n, \quad n \geqslant 2. \tag{47}$$

The recurrence formulas for f_n and g_n are given by

$$\begin{aligned} f_n = &- \{f_{n-1} a_2 (n-1)^2 + f_{n-2}[a_3(n-2)(n-1) + q_0^2] \\ &+ f_{n-3}[n^3 - 9n^2 + 29n - 33]\}/[n(n-1)a_1], \quad n \geqslant 3, \\ g_n = &- \{g_{n-1} a_2 (n-1)^2 + g_{n-2}[a_3(n-2)(n-1) + q_0^2] \\ &+ g_{n-3}[n^3 - 9n^2 + 29n - 33]\}/[n(n-1)a_1], \quad n \geqslant 3, \end{aligned}$$

$$f_0 = 1, \quad g_0 = 0, \quad f_1 = 0, \quad g_1 = 1, \quad f_2 = -q_0^2/2a_1, \quad g_2 = -a_2/2a_1, \tag{48}$$

where

$$a_1 = (1 + k/2)^3, \quad a_2 = 3k(1 + k/2)^2, \quad a_3 = 3k^2(1 + k/2). \tag{49}$$

Now, the solution of equation (45) can be written in the form

$$\bar{u}_0(\xi) = A_0 \sum_{n=0}^{\infty} f_n (\xi - 1/2)^n + A_1 \sum_{n=1}^{\infty} g_n (\xi - 1/2)^n, \tag{50}$$

where A_0 and A_1 are the constants yet to be determined. Using boundary conditions (22), the constants A_0 and A_1 can be obtained as

$$A_0 = -\frac{1}{\Delta_3} \sum_{n=1}^{\infty} n g_n (-1/2)^{n-1}, \qquad A_1 = \frac{1}{\Delta_3} \sum_{n=2}^{\infty} n f_n (-1/2)^{n-1},$$

$$\Delta_3 = \sum_{n=2}^{\infty} n f_n (-1/2)^{n-1} \sum_{n=1}^{\infty} g_n (1/2)^n - \sum_{n=0}^{\infty} f_n (1/2)^n \sum_{n=1}^{\infty} n g_n (-1/2)^{n-1}. \tag{51}$$

The dimensionless natural frequencies are the roots of $\Delta_3 = 0$, and the amplitude of the vibration at the surface is

$$A = \bar{u}_0|_{\xi=0} = A_0 \sum_{n=0}^{\infty} f_n (-1/2)^n + A_1 \sum_{n=1}^{\infty} g_n (-1/2)^n. \tag{52}$$

3.2. FIRST ORDER SOLUTION

Now, a solution is going to be searched for the first order equation (17). As it is seen, it is a non-homogeneous partial differential equation with variable coefficients and has the term $(\partial^2 u_0/\partial \xi^2)(\partial u_0/\partial \xi)^2$ at the right-hand side. This term includes the products of various types of functions which are not suitable for analytic solution in most of the cases. For this reason, only the numerical solution of this equation is possible. The frequency shift Ω_1, which is not known yet, appears at the right-hand side of equation (17). Boundary conditions (18) and periodicity conditions (19) are not sufficient to obtain this parameter. Instead of the solution of equation (17), the frequency shift is obtained by using the orthogonality condition introduced in references [1, 12]. For this purpose, both sides of equation (17) is multiplied by $u_0(\xi, s)$ and integrated over the time and co-ordinate. Using boundary conditions (18) and periodicity conditions (19), one obtains the following equation:

$$\int_0^{2\pi}\int_0^1 \left\{(1+k\xi)^\alpha \frac{\partial^2 u_1}{\partial \xi^2} + k\alpha(1+k\xi)^{\alpha-1}\frac{\partial u_1}{\partial \xi} - q_0^2 \frac{\partial^2 u_1}{\partial s^2}\right\} u_0 \, \mathrm{d}\xi \, \mathrm{d}s \equiv 0. \tag{53}$$

As a result of this, the orthogonality condition is obtained as

$$\int_0^{2\pi}\int_0^1 \left[2q_0^2 \Omega_1 \frac{\partial^2 u_0}{\partial s^2} + \frac{\partial^2 u_0}{\partial \xi^2}\left(\frac{\partial u_0}{\partial \xi}\right)^2\right] u_0 \, \mathrm{d}\xi \, \mathrm{d}s = 0. \tag{54}$$

By the substitution of equation (20) into this equation, frequency shift is obtained as

$$\Omega_1 = \frac{3}{8q_0^2}\frac{\int_0^1 \bar{u}_0 (\mathrm{d}^2\bar{u}_0/\mathrm{d}\xi^2)(\mathrm{d}\bar{u}_0/\mathrm{d}\xi)^2 \, \mathrm{d}\xi}{\int_0^1 \bar{u}_0^2 \, \mathrm{d}\xi}. \tag{55}$$

Because of the difficulties in obtaining these integrals analytically, a numerical method of solution, for example Simpson's rule, has been used. After integration, the frequency shift Ω_1 is obtained dependent on Ω_0. Then, using the first two terms in series (14), one obtains the following relation:

$$\Omega = \Omega_0[1 + \varepsilon\Omega_1(\Omega_0)]. \tag{56}$$

Since the physical and geometrical properties of the layer and the amplitude of the forcing are known, using equation (9) and substituting the solutions $\bar{u}_0$ and equation (55) into equation (56), a relation is obtained between Ω and Ω_0 as follows:

$$\Omega = \Omega(\Omega_0). \tag{57}$$

If Ω_0 is chosen, one may get into the resonance region, when $\Omega_0 \cong \Omega_{0n}$. As a result of this, $\bar{u}_0(\xi)$ increases indefinitely and perturbation series (13) and (14) diverge. Instead of this, if the forcing frequency Ω is chosen, the parameter Ω_0 can be calculated as the first root of non-linear algebraic equation (57). Using this root in one of amplitude equations (28), (33), (37), (44) or (52), one can get a relation between the amplitude magnification factor A and the forcing frequency Ω as follows:

$$A = A(\Omega). \tag{58}$$

In this way, a finite amplitude is obtained even in the resonance region.

4. NUMERICAL RESULTS AND DISCUSSION

The numerical results obtained with the above formulation are presented in this section for the inhomogeneity parameters α and k, and the non-linearity parameter ε. The variation of the ratio of shear modulus in linear case to the shear modulus at the surface with respect to depth for parameters k and α are given in Figures 2(a) and 2(b) respectively. From the figures, it is observed that the parameter α is more

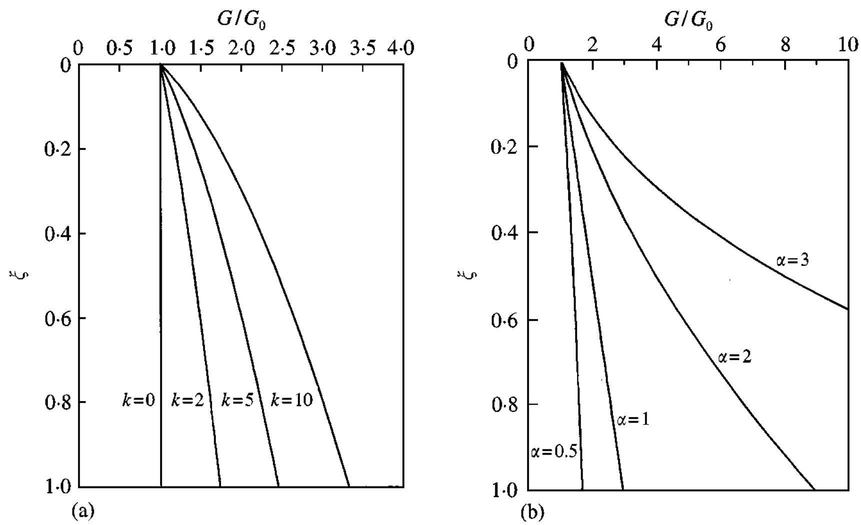

Figure 2. (a) Shear modulus ratio G/G_0 versus depth of the layer ξ for $\alpha = 0{\cdot}5$ at various values of inhomogeneity parameter k. (b) Shear modulus ratio G/G_0 versus depth of the layer ξ for $k = 2$ at various values of inhomogeneity parameter α.

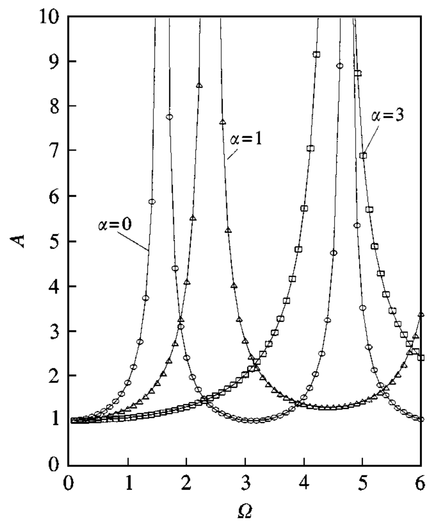

Figure 3. Amplitude magnification factor A versus forcing frequency Ω for linear case ($\varepsilon = 0$) at various values of inhomogeneity parameter α for $k = 2$.

efficient than the parameter k, and especially in case of $\alpha = 3$, the bottom of the layer behaves almost as a rigid medium. This case has a great effect on the amplitude magnification factor A which can be seen from the latter results and is an expected attitude. The variation of the amplitude magnification A with respect to the forcing frequency Ω in linear case for $\varepsilon = 0$, $k = 2$ and different values of α is given in Figure 3. In the homogeneous medium, resonance cases occur when dimensionless frequency is equal to odd multiples of $\pi/2$. But in the inhomogeneous medium, for positive coefficients, resonance occurs in some frequencies which are larger than the ones for homogeneous medium. In this case, the response curves displace to the right, and the effect of the inhomogeneity away from the resonance region becomes small. The amplitude magnification factor–forcing frequency relations for a fixed non-linearity parameter and various values of α and k are given in Figures 4(a)–4(d). The influence of the parameter k upon the amplitude magnification factor A and resonance regions for different values of α can be seen clearly in these figures. In all cases, whatever α is, with the increase of k, resonance regions displace to the right and A increases. In obtaining these results, the solution

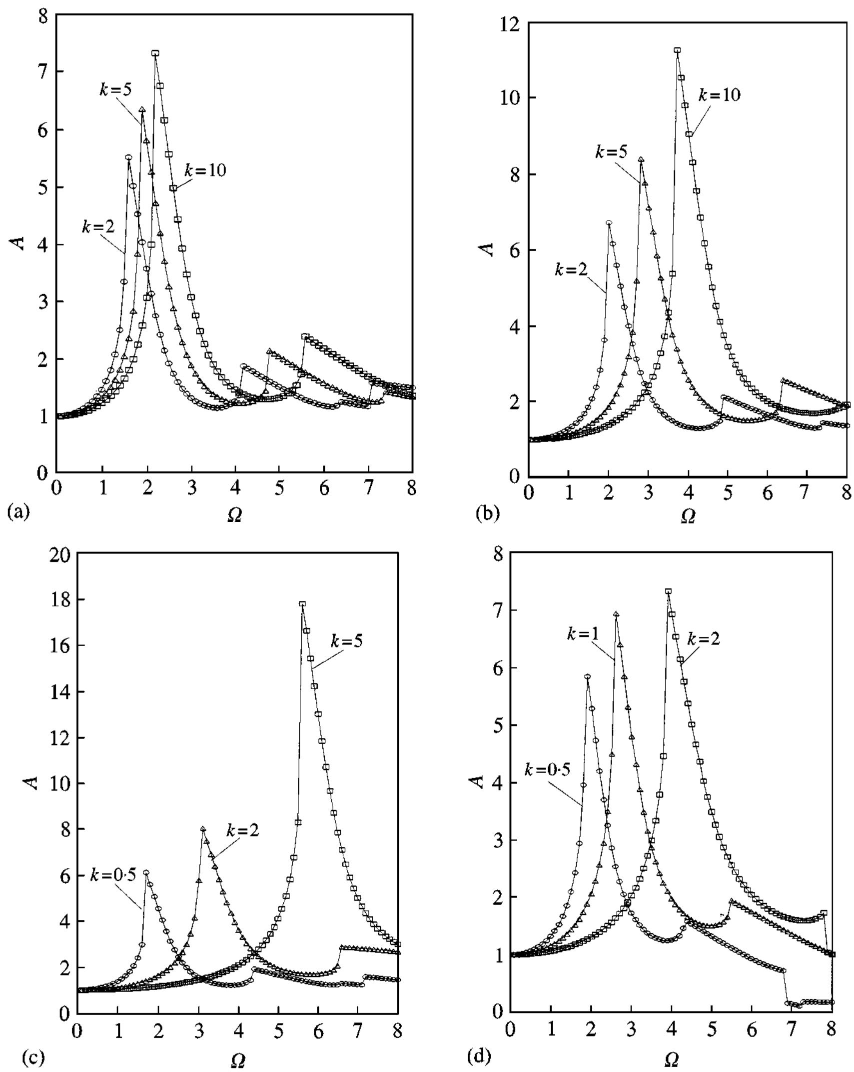

Figure 4. Amplitude magnification factor A versus forcing frequency Ω for non-linear case ($\varepsilon = 0{\cdot}05$) at various values of inhomogeneity parameter k. (a) $\alpha = 0{\cdot}5$; (b) $\alpha = 1$; (c) $\alpha = 2$; (d) $\alpha = 3$.

for the linear case has been used. In these solution functions, there exists dimensionless wave number q_0, which is not known yet. In the non-linear case, orthogonality condition (54) is used instead of the solution of first order equation (17). In equation (55), which has been obtained by using the orthogonality condition, q_0 is still an unknown parameter. The curves of $\Omega = \Omega(\Omega_0)$ for the homogeneous case are given in Figure 5. In this figure, it is observed that $|\Omega| \to \infty$ for some special values of q_0 such as $\pi/2$, $3\pi/2$. But the parameter q_0 is calculated as the root of the non-linear equation (56) by using the Newton–Raphson technique after choosing Ω. The o–a, b–c and d–e curve segments in this figure correspond to

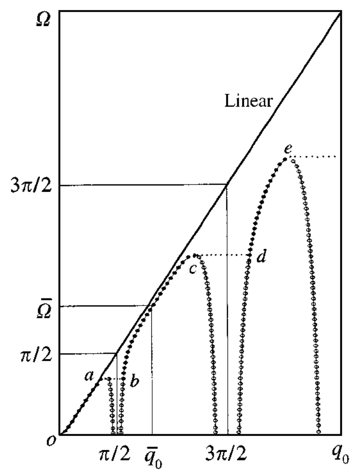

Figure 5. Forcing frequency Ω versus dimensionless wave number q_0 for homogeneous case.

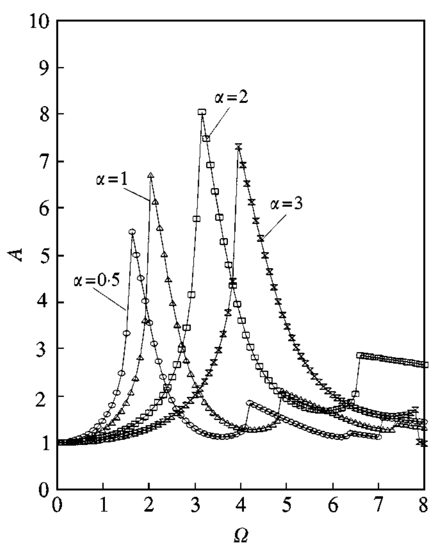

Figure 6. Amplitude magnification factor A versus forcing frequency Ω for non-linear case ($\varepsilon = 0{\cdot}05$) at various values of inhomogeneity parameter α for $k = 2$.

different choices of Ω, therefore, the resonances values q_{0n} jump. To avoid such an inconvenience the resonances are removed with the elimination of these special values. With the choice of Ω and calculation of q_{0n} the relation $A = A(\Omega)$ is reached at the end. Using the obtained relation the A values are obtained from the equations (52), (44), (37), (33) and (28). It is noteworthy that the A values may be infinite in linear case whereas they remain finite even in the resonance region for non-linear case. Thus, it can be concluded that the non-linearity and damping effects are similar.

The A–Ω response curves for $k = 2$, $\varepsilon = 0{\cdot}05$ and different values of α are given in Figure 6. From the figure, it can be observed that the effect of α on A and resonance

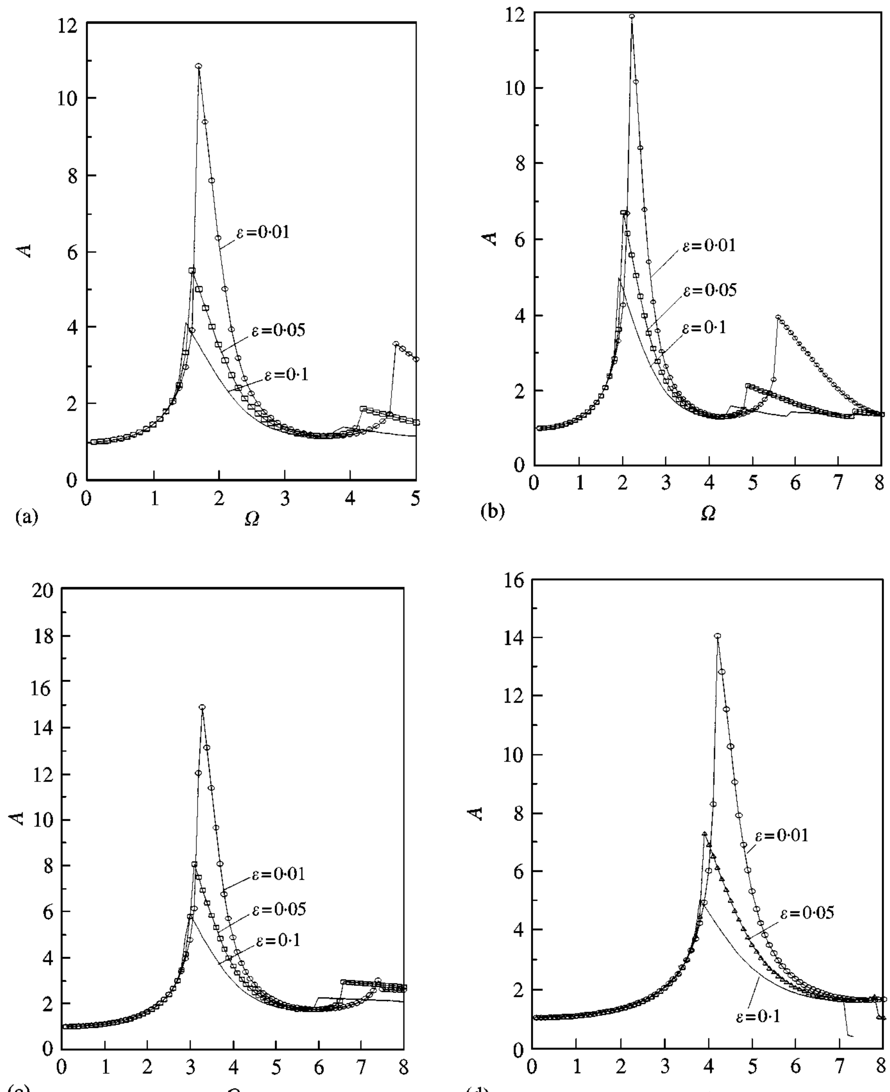

Figure 7. Amplitude magnification factor A versus forcing frequency Ω for non-linear case at various values of non-linearity parameter ε for $k = 2$. (a) $\alpha = 0{\cdot}5$; (b) $\alpha = 1$; (c) $\alpha = 2$; (d) $\alpha = 3$.

regions are similar to the previous case. The effect of the non-linearity on the relation between A and Ω for various values of α and k are given in Figures 7(a)–7(d). From the figures, it is obviously seen that, especially in the resonance region, the A values decrease with the increase of the non-linearity parameter ε. In this case, one can conclude that the non-linearity is effective in the resonance region. The effect of non-linearity on A for homogeneous and inhomogeneous media are given in Figures 8(a) and 8(b) respectively. From the figures, it can be observed that the non-linearity is very effective in the resonance regions for both media. The variation of dimensionless wave number q_0 with respect to the parameter k for $\alpha = 0{\cdot}5; 1; 2; 3$ is given in Figure 9. The variation is slow for $\alpha = 0{\cdot}5$

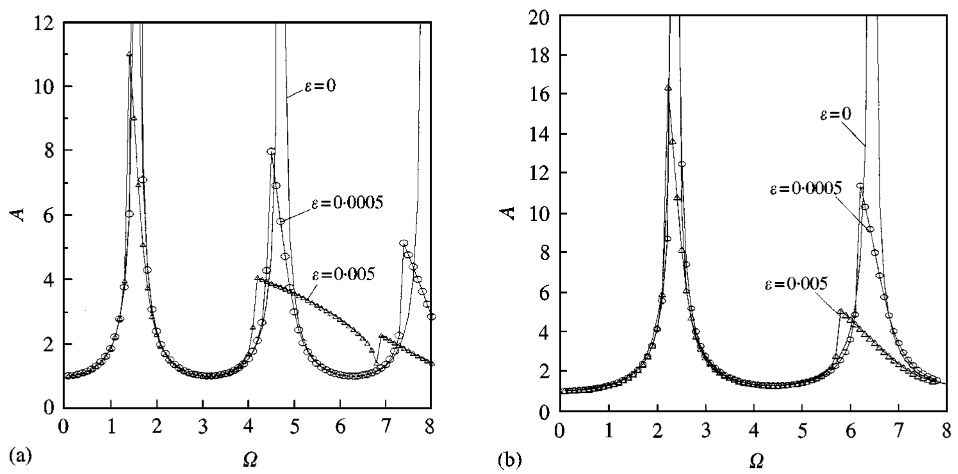

Figure 8. Amplitude magnification factor A versus forcing frequency Ω. (a) For homogeneous medium ($k = 0$), (b) for inhomogeneous medium $k = 2$ and $\alpha = 1$.

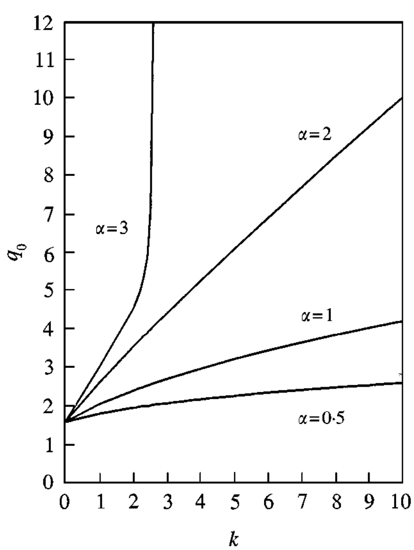

Figure 9. Dimensionless wave number q_0 versus inhomogeneity parameter k at various values of inhomogeneity parameter α.

and $\alpha = 1$ whereas it is fast for $\alpha = 2$. For $\alpha = 3$, the effect of k decreases when it is greater than 2.

5. CONCLUSIONS

The forced vibrations of an inhomogeneous layer have been obtained by using a perturbation method. The effects of the inhomogeneity parameters α and k and the non-linearity parameter ε on the behaviour of the layer have been investigated. The Bessel functions have been used in the solution for the different values of inhomogeneity parameter α. The resonance regions and the amplitudes at the surface, in the linear and non-linear cases, vary depending on the inhomogeneity parameters. The non-linearity parameter, which is dependent on the physical and geometrical properties of the layer and the amplitude of the forcing, has also a great influence on the response of the layer. In the linear case ($\varepsilon = 0$), resonance occurs for some special forcing frequencies, but in the non-linear case, resonances are removed with the elimination of these special frequencies. In this case, the amplitudes at the surface decrease and it is concluded that the non-linearity causes damping.

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