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A UNIVERSAL MATRIX PERTURBATION TECHNIQUE FOR STRUCTURAL DYNAMIC MODIFICATION USING SINGULAR VALUE DECOMPOSITION

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An investigation is made of an efficient matrix perturbation technique for structural dynamic modification in this paper. This paper is developed by performing a subspace condensation and an orthogonal decomposition procedure to obtain lower order perturbed eigensolutions. The matrix singular value decomposition approach is then employed to compute the higher order perturbations of eigensolutions. This method, with higher accuracy and simpler procedure, is universally applicable to all three cases of eigenvalues in the unperturbed system: distinct, repeated and closely spaced eigenvalues. Results of the application of the method to examples are shown and compared with numerical results to demonstrate the validity of the technique.

1. INTRODUCTION

The dynamic behavior of many physical systems is completely determined by obtaining the eigensolutions of an eigenvalue problem. Eigensolution calculation for any but the smallest systems is an expensive and time-consuming process, which means that the calculation is a major contributor to the computational expense of structural dynamic analysis. This is especially so when an iterative design or reanalysis process is performed until a satisfactory design is achieved. Therefore, a quick and efficient, even if approximate, evaluation of changes in eigensolutions as a result of variations in parameters of modified system is very valuable [1]. The matrix perturbation technique (MPT) is just one of the most powerful tools, through which the results of eigensolutions in the modified systems can be obtained directly from the results of the original/unperturbed systems without having to repeat an entire analysis if the design changes are small. Thus substantial cost and time savings can be realized.

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Since the earlier study by Rayleigh on structural parameter modification [2], many investigations have been sparked of the MPT. Caughey applied the MPT to the design of subsystems in large structures [3] and dynamic problems [4]. Romstad *et al.* investigated the general perturbation formulations employing a power-series approach [5]. Chen and Wada, and Chen and Garbe proposed analysis-test correlation criteria for structural dynamic systems [6] and analytical model improvement [7] using the MPT. Chen and Wada developed an MPT for structural dynamic analysis [8]. Stetson Meirovitch and Ryland put forward MPTs for holographic vibration analysis [9] and gyroscopic systems [10]. Jones studied the effect of small changes in mass and stiffness on the natural frequencies [11]. Rizai and Bernard analyzed the dynamic effects of redesign [12]. Baldwin and Hutton presented a review on perturbation analysis and studied the natural modes of modified structures [13]. Wang discussed the eigenvalue reanalysis of locally modified structures [14]. Hu developed a MPT for the analysis of systems with repeated eigenvalues [15]. For the case of closely spaced (nearly equal) eigenvalues, Hu first proposed an MPT starting from subspace eigensolutions [15], and more recently Chen *et al.* further discussed the same topic [16]. Chen, Liu and Zhao presented an improved MPT by implementing the concept of subspace [17]. Such an improved method possesses the ability to deal with the systems with distinct, repeated, and closely spaced eigenvalues, and so can be regarded as a universal method. In these studies, the modal expansion method is used to obtain the perturbations of eigensolutions.

In structural vibration analysis, the modal expansion method is extremely efficient [18]. However, for many systems, especially for large complicated structures, quite frequently it is difficult to obtain all the modes (eigenvectors), and so the truncated modal method should be employed [18, 19]. This, under some circumstances, inevitably leads to significant errors. To avoid this drawback, in the present paper, an efficient MPT is presented by using matrix singular value decomposition (SVD) [20]. Such a treatment improves the accuracy of calculation and efficiency of analysis without having to use the complete modal expansion. The proposed procedure is another universal MPT in analogy to that developed by Chen *et al.* [17], in which the complete expansion of eigenvectors is utilized and hence it will necessarily fail if only some of the eigenvectors are available. That is to say, the present method not only can give adequately accurate perturbed eigensolutions for all the three cases of eigenvalues, but also imposes no restrictions on the number of available eigenvectors. An example is presented to verify the validity of the proposed MPT. The results calculated by this method are extremely close to the numerical results obtained by the QR method [18].

2. LOWER ORDER PERTURBATIONS OF EIGENSOLUTIONS

The original/unperturbed eigenvalue problem and corresponding normalization condition are expressed by

$$K_0 x_{i0} = \lambda_{i0} M_0 x_{i0}, \quad x_{i0}^{\mathrm{T}} M_0 x_{i0} = \delta_{ij}, \quad i, j = 1, 2, \ldots, n, \tag{1a, b}$$

where K_0 and M_0 are $n \times n$ real-symmetric stiffness and mass matrices respectively; λ_{i0}, x_{i0} are the ith eigenvalue and the associated eigenvector, and

$$\delta_{ij} = 1 \quad \text{if } i = j, \quad \delta_{ij} = 0 \quad \text{if } i \neq j \tag{1c}$$

is the Kronecker delta.

Without loss of generality, assume that the n eigenvalues satisfy

$$\lambda_{10} < \lambda_{20} < \cdots < \lambda_{j0} \cong \lambda_{k0} < \cdots < \lambda_{n0}, \tag{1d}$$

which means the unperturbed system possesses $k - j + 1$ repeated or closely spaced eigenvalues.

In structural dynamic modification, the design changes in a structural system may be reflected by the variations in K_0 and M_0. Regardless of the reasons, the net effect is that the matrices K_0 and M_0 are different from the original ones. Since these changes are usually small compared to the entire system, the two updated matrices relative to that in equation (1a) can be expressed as

$$K = K_0 + K_1, \quad M = M_0 + M_1, \tag{2a, b}$$

in which K_1 and M_1 are the $n \times n$ real-symmetric matrices representing the corresponding changes of K_0 and M_0 respectively, and both are small relative to K_0 and M_0. Therefore, K_1 and M_1 are first order perturbed matrices. As the design changes approach zero, $K_1 \to 0$, $M_1 \to 0$ and $K \to K_0$, $M \to M_0$.

By analogy with equations (1a) and (1b), considering equations (2a) and (2b), the perturbed eigenvalue problem and the associated normalization condition can be written as

$$(K_0 + K_1)x_i = \lambda_i(M_0 + M_1)x_i, \quad i, j = 1, 2, \ldots, n, \tag{3a}$$

$$x_i^{\mathrm{T}}(M_0 + M_1)x_i = \delta_{ij}, \quad i, j = 1, 2, \ldots, n, \tag{3b}$$

where λ_i and x_i are the ith eigenvalue and eigenvector of the perturbed system respectively.

First, we choose several eigenvectors to span an eigensubspace Φ. For practical use, the eigenvectors corresponding to those repeated or closely spaced eigenvalues are usually chosen for the analysis [15, 17], i.e.,

$$\Phi = [x_{j0}, \ldots, x_{k0}]. \tag{4}$$

Generally speaking, the variation of an eigenvector may be significant, but the angle between the unperturbed and perturbed eigensubspaces can be regarded as small. This is discussed in Hu's book [15]. Therefore, an orthogonal decomposition of the perturbed eigenvector, say, x_i, with respect to the unperturbed eigensubspace

can be made as

$$x_i = \Phi q_i + \delta x_i, \quad \Phi^{\mathrm{T}} M_0 \delta x_i = \Phi^{\mathrm{T}} K_0 \delta x_i = 0, \quad i = j \sim k, \tag{5a, b}$$

where q_i is a $k - j + 1$-dimensional column vector and δx_i is a first order $k - j + 1$ column vector. Both are to be determined.

The variational principle corresponding to the perturbed system (3a) is

$$\lambda_i = \underset{q_j,\, \delta x_j}{st} \frac{(\Phi q_i + \delta x_i)^{\mathrm{T}}(K_0 + K_1)(\Phi q_i + \delta x_i)}{(\Phi q_i + \delta x_i)^{\mathrm{T}}(M_0 + M_1)(\Phi q_i + \delta x_i)}, \quad i = j \sim k, \tag{6}$$

where the symbol '*st*" means "stationary value".

Neglecting small quantities of second order and using equation (5b), equation (6) reduces to

$$\mu_i = \underset{q_j}{st} \frac{q_i^{\mathrm{T}} \Phi^{\mathrm{T}}(K_0 + K_1)\Phi q_i}{q_i^{\mathrm{T}} \Phi^{\mathrm{T}}(M_0 + M_1)\Phi q_i}, \quad i = j \sim k, \tag{7}$$

where μ_i is an approximation to λ_i, precise to the first order.

Transforming equation (7) into an algebraic equation results in

$$\bar{K} q_i = \mu_i \bar{M} q_i, \quad i = j \sim k, \tag{8}$$

where $\bar{K} = \Phi^{\mathrm{T}}(K_0 + K_1)\Phi$, $\bar{M} = \Phi^{\mathrm{T}}(M_0 + M_1)\Phi$.

Inserting equation (5a) into equation (3b), using equation (5b), and neglecting second order quantities, we obtain

$$q_i^{\mathrm{T}} \bar{M} q_i = 1, \quad i = j \sim k \tag{9}$$

Equations (8) and (9) comprise an eigenvalue problem of the reduced system, from which the $k - j + 1$ eigensolutions μ_i and q_i, and hence the lower order perturbations of the eigensolutions λ_i and x_i, can be quickly determined because of $k - j + 1 \ll n$ in general.

The obtained mathematical procedure may be summed up in the following statement, i.e., to find the approximate solution for the perturbed system by the Ritz method in the subspace spanned by the eigenvectors corresponding to the repeated or closely spaced eigenvalues of the unperturbed system. Since there is only a small angle between the unperturbed and the corresponding perturbed subspaces [15], Φq_i has an error of the first order and hence μ_i has only an error of the second order according to the Rayleigh's quotient theorem.

3. HIGHER ORDER PERTURBATIONS OF EIGENSOLUTIONS

In some cases, the accuracy of the lower order perturbations obtained in section 2 may be acceptable. However, quite often such accuracy may be unacceptable, and

thus higher order perturbations are required. To this end, the perturbed eigensolutions are expressed as

$$\lambda_i = \mu_i + \lambda_{i2} + \lambda_{i3} + \cdots, \quad x_i = \Phi q_i + x_{i1} + x_{i2} + x_{i3} + \cdots, \quad i = j \sim k, \tag{10a, b}$$

where $\lambda_{i2}, \lambda_{i3}$ are the second and third order perturbations of λ_i respectively; x_{i1}, x_{i2} and x_{i3} are the first, second and third order perturbations of x_i respectively. Comparing equation (10b) with equation (5a) yields

$$\delta x_i = x_{i1} + x_{i2} + x_{i3} + \cdots, \quad i = j \sim k. \tag{11}$$

Substituting equations (10a) and (5a) into equation (3a), we have

$$(K_0 + K_1)(\Phi q_i + \delta x_i) = (\mu_i + \lambda_{i2} + \lambda_{i3} + \cdots)(M_0 + M_1)(\Phi q_i + \delta x_i), \quad i = j \sim k. \tag{12}$$

Premultiplying equation (12) by $(\Phi q_i)^{\mathrm{T}}$, collecting terms of the same order, and using equations (5b), (8) and (11), we have

$$O(\varepsilon^2): \quad \lambda_{i2} q_i^{\mathrm{T}} \Phi^{\mathrm{T}} M_0 \Phi q_i = q_i^{\mathrm{T}} \Phi^{\mathrm{T}} (K_1 - \mu_i M_1) x_{i1}, \quad i = j \sim k, \tag{13a}$$

$$O(\varepsilon^3): \quad \lambda_{i3} q_i^{\mathrm{T}} \Phi^{\mathrm{T}} M_0 \Phi q_i = q_i^{\mathrm{T}} \Phi^{\mathrm{T}} (K_1 - \mu_i M_1) x_{i2} - \lambda_{i2} q_i^{\mathrm{T}} \Phi^{\mathrm{T}} M_1 \Phi q_i, \quad i = j \sim k. \tag{13a}$$

By inserting equation (1b) into equations (13a) and (13b), we obtain the second and third perturbations of eigenvalues

$$\lambda_{i2} = \frac{q_i^{\mathrm{T}} \Phi^{\mathrm{T}} (K_1 - \mu_i M_1)}{q_i^{\mathrm{T}} q_i} x_{i1}, \quad i = j \sim k, \tag{14a}$$

$$\lambda_{i3} = \frac{q_i^{\mathrm{T}} \Phi^{\mathrm{T}} [(K_1 - \mu_i M_1) x_{i2} - \lambda_{i2} M_1 \Phi q_i]}{q_i^{\mathrm{T}} q_i}, \quad i = j \sim k, \tag{14b}$$

respectively. However, they have not been completely determined because x_{i1}, x_{i2} remain unknown.

Substituting equation (11) into equation (12), and ignoring fourth order terms, we obtain

$$O(\varepsilon): \quad (K_0 - \mu_i M_0) x_{i1} = -[(K_0 + K_1) - \mu_i (M_0 + M_1)] \Phi q_i, \quad i = j \sim k, \tag{15a}$$

$$O(\varepsilon^2): \quad (K_0 - \mu_i M_0) x_{i2} = -(K_1 - \mu_i M_1) x_{i1} + \lambda_{i2} M_0 \Phi q_i, \quad i = j \sim k, \tag{15b}$$

$$O(\varepsilon^3): \quad (K_0 - \mu_i M_0)x_{i3} = -(K_1 - \mu_i M_1)x_{i2} + \lambda_{i2}(M_0 x_{i1} + M_1 \Phi q_i) + \lambda_{i3} M_0 \Phi q_i, \quad i = j \sim k. \tag{15c}$$

Introducing equation (10b) into equation (3b) yields

$$(\Phi q_i + x_{i1} + x_{i2} + x_{i3})^{\mathrm{T}}(M_0 + M_1)(\Phi q_i + x_{i1} + x_{i2} + x_{i3}) = 1, \quad i = j \sim k. \tag{16}$$

Expanding equation (16) and using equation (9), we obtain

$$O(\varepsilon): \quad (\Phi q_i)^{\mathrm{T}} M_0 x_{i1} = 0, \quad i = j \sim k, \tag{17a}$$

$$O(\varepsilon^2): \quad (\Phi q_i)^{\mathrm{T}} M_0 x_{i2} = -(\Phi q_i)^{\mathrm{T}} M_1 x_{i1} - \tfrac{1}{2} x_{i1}^{\mathrm{T}} M_0 x_{i1}, \quad i = j \sim k, \tag{17b}$$

$$O(\varepsilon^3): \quad (\Phi q_i)^{\mathrm{T}} M_0 x_{i3} = -(\Phi q_i)^{\mathrm{T}} M_1 x_{i2} - \tfrac{1}{2} x_{i1}^{\mathrm{T}} M_1 x_{i1} - x_{i1}^{\mathrm{T}} M_0 x_{i2}, \quad i = j \sim k, \tag{17c}$$

Up till now, we have obtained all the necessary fundamental equations governing the determination of higher order perturbations, i.e., λ_{i2}, λ_{i3}, x_{i1}, x_{i2}, x_{i3}.

As pointed out previously, μ_i is an approximation of λ_i, and so $(K_0 - \mu_i M_0)$ is singular or is nearly singular; x_{i1}, x_{i2}, x_{i3} in equations (15a), (15b) and (15c), and then λ_{i2}, λ_{i3} in equations (14a) and (14b) cannot be determined by the classical solution procedure of linear algebraic equations. On the other hand, the perturbations of eigensolutions have to satisfy equations (17a), (17b) and (17c). Here again, x_{i1}, x_{i2}, x_{i3} and then λ_{i2}, λ_{i3} cannot be determined by the classical solution procedure because the number of equations (15a) and (17a), (15b) and (17b), or (15b) and (17b) is greater than that of the unknowns. To overcome the difficulty, now the matrix SVD technique is used to solve the equations.

Combining equations (15a) and (17a) yields

$$A x_{i1} = b, \quad i = j \sim k, \tag{18}$$

where

$$A = \begin{bmatrix} K_0 - \mu_i M_0 \\ (\Phi q_i)^{\mathrm{T}} M_0 \end{bmatrix}, \quad b = \begin{bmatrix} -[(K_0 + K_1) - \mu_i (M_0 + M_1)]\Phi q_i \\ 0 \end{bmatrix}, \quad i = j \sim k, \tag{19a, b}$$

Simply speaking, for a given $m \times n$ real matrix A, there exist orthogonal matrices U and V, such that

$$A = U W V^{\mathrm{T}}, \tag{20a}$$

and

$$W = \text{diag.}(\omega_1, \omega_2, \ldots, \omega_n), \quad \omega_1 \geqslant \omega_2 \geqslant \cdots \geqslant \omega_n \geqslant 0, \tag{20b, c}$$

are the non-negative square roots of n eigenvalues of symmetric matrix $A^{\mathrm{T}}A$. This is the so-called SVD of a real matrix.

Assume that $A = UWV^{\mathrm{T}}$ is available, the least-squares solutions of the set of linear algebraic equations, i.e., equation (18), are

$$x_{i1} - A^{+}b, \quad i = j \sim k, \tag{21}$$

where superscript " + " denotes the generalized inverse, and

$$A^{+} = VW^{+}U^{\mathrm{T}}, \quad W^{+} = \text{diag.}[\omega_1^{+}, \omega_2^{+}, \ldots, \omega_n^{+}], \tag{22a, b}$$

in which $\omega_i^{+} = 0$ if $\omega_i = 0$, $\omega_i^{+} = 1/\omega_i$ if $\omega_i \neq 0$.

Therefore, we have obtained x_{i1} and hence λ_{i2} be recalling equation (14a).

Similarly, by combining equations (15b) and (17b), we have

$$Ax_{i2} = c, \quad i = j \sim k, \tag{23a}$$

$$c = \begin{bmatrix} -(K_1 - \mu_i M_1)x_{i1} + \lambda_{i2} M_0 \Phi q_i \\ -(\Phi q_i)^{\mathrm{T}} M_1 x_{i1} - \frac{1}{2} x_{i1}^{\mathrm{T}} M_0 x_{i1} \end{bmatrix}, \quad i = j \sim k. \tag{23b}$$

Using the same pattern, we have

$$x_{i2} = A^{+}c, \quad i = j \sim k. \tag{24}$$

As for x_{i3}, we have

$$x_{i3} = A^{+}d, \quad i = j \sim k, \tag{25}$$

in which

$$d = \begin{bmatrix} -(K_1 - \mu_i M_1)x_{i2} + \lambda_{i2}(M_0 x_{i1} + M_1 \Phi q_i) + \lambda_{i3} M_0 \Phi q_i \\ -(\Phi q_i)^{\mathrm{T}} M_1 x_{i1} - \frac{1}{2} x_{i1}^{\mathrm{T}} M_1 x_{i1} - x_{i1}^{\mathrm{T}} M_0 x_{i2} \end{bmatrix}, \quad i = j \sim k. \tag{26}$$

The perturbations of eigensolutions of the fourth order or higher, albeit of little practical value, can also be found by the same procedure, provided that fourth order terms in the associated expansions are retained.

For the case of distinct eigenvalues, i.e., when $i \neq j \sim k$, the perturbations of eigensolutions can be found by the classical MPT. In fact, however, the proposed method is completely applicable to such case. At this moment, we only need to choose $\Phi = x_{i0} (l \neq j \sim k)$.

From the above derivations and formulas, it can be seen that: firstly, for the higher order perturbations of eigenvectors, the complete modal expansion has not

been use. This treatment is especially useful and effective when only some (usually the lower order) of the unperturbed eigenvectors have been obtained. Secondly, the SVD is expensive in general. However, the matrix A to be decomposed in equation (18) is the same in the first, second, and third order perturbation formulas of eigenvectors; see equations (21), (24), and (25). That is to say, the SVD of the matrix is required only one time for all order perturbations. Thirdly, since $k - j + 1$ is generally much less than n (the number of degree of freedom of a system), the proposed technique is less expensive. Finally, the present method is applicable to all the three cases: distinct, repeated, and closely spaced eigenvalues. These, if not more, are the main reasons why this paper has been presented.

4. AN EXAMPLE

Consider a six-degree-of-freedom system

$$K_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1{\cdot}05 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2{\cdot}06 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 0{\cdot}05 & 0 & 0 & 0 & 0 \\ 0{\cdot}05 & 0 & 0{\cdot}05 & 0 & 0 & 0 \\ 0 & 0{\cdot}05 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0{\cdot}05 & 0 \\ 0 & 0 & 0 & 0{\cdot}05 & 0 & 0{\cdot}05 \\ 0 & 0 & 0 & 0 & 0{\cdot}05 & 0 \end{bmatrix}$$

$$M_0 = I, \quad M_1 = 0,$$

in which I denotes the identity matrix.

The six groups of eigensolutions of the unperturbed system are listed in Table 1. It can be seen that there are three cases of eigenvalues, i.e., distinct, repeated and closely spaced eigenvalues.

By using the present method, the eigensolutions of the perturbed system can be easily obtained, which are listed in Table 2, and the exact numerical solutions computed by QR method are also listed in Table 2 for comparison.

TABLE 1

Six groups of eigensolutions of the unperturbed system

λ_{i0} $(i = 1\text{–}6)$	x_{i0}^{T} $(i = 1\text{–}6)$
1	(1, 0, 0, 0, 0, 0)
1·05	(0, 1, 0, 0, 0, 0)
2	(0, 0, 1, 0, 0, 0)
2	(0, 0, 0, 1, 0, 0)
2·06	(0, 0, 0, 0, 1, 0)
5	(0, 0, 0, 0, 0, 1)

TABLE 2
Six groups of eigensolutions of the perturbed system

	λ_{i0} $(i = 1–6)$	x_{i0}^{T} $(i = 1–6)$
QR method	0·968418	(0·84518, − 0·53385, 0·02588, 0, 0, 0)
	1·078951	(0·53447, 0·84394, − 0·04581, 0, 0, 0)
	1·971489	(0, 0, 0, 0·86867, − 0·049533, 0.00818)
	2·002631	(0·00262, 0·05255, 0·99861, 0, 0, 0)
	2·087661	(0, 0, 0, 0·49540, 0·86854, − 0·01491)
	5·000850	(0, 0, 0, 0·00028, 0·01700, 0·99986)
Present method (first order)	0·969098	(0·85065, − 0·52573, 0·02550, 0, 0, 0)
	1·080902	(0·52573, 0·85065, − 0·04628, 0, 0, 0)
	1·971691	(0, 0, 0, 0·87020, − 0·49270, 0·00813)
	2	(0, 0·5263, 1, 0, 0, 0)
	2·088310	(0, 0, 0, 0·49270, 0·87020, − 0·01494)
	5	(0, 0, 0, 0, 0·01701, 1)
Present method (second order)	0·968428	(0·84263, − 0·53807, 0·0255, 0, 0, 0)
	1·078933	(0·53951, 0·84086, − 0·04628, 0, 0, 0)
	1·971490	(0, 0, 0, 0·86805, − 0·49642, 0·00813)
	2·002631	(0·00263, 0·05263, 0.99861, 0, 0, 0)
	2·087659	(0, 0, 0, 0·49656, 0·86788, − 0·01494)
	5·000850	(0, 0, 0, 0·00028, 0·01701, 0·99986)

5. CONCLUDING REMARKS

From the above results as shown in Tables 2, it can be observed that, in all the three cases of distinct, repeated and closely spaced eigenvalues, the first order perturbed eigensolutions obtained by the proposed MPT have sufficient precision compare with the numerical solutions calculated by QR method. Furthermore, the second order approximations are nearly equal to the QR solutions, and thus the validity and universality of the method are completely verified.

Besides universality, validity and higher accuracy, the other attribute of the method is its simplicity in the derivation of the working procedure. These advantages mainly lie in two operations: taking the projection of a vector onto a subspace and performing the SVD, both being the most elegant and stable algorithms in numerical algebra. On the other hand, the method is especially effective if only some of the unperturbed eigensolutions are known or only several eigensolutions are of concern, because the complete modal expansion has not been used in the higher order perturbations.

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