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VIBRATION AND BUCKLING OF DEEP BEAM-COLUMNS ON TWO-PARAMETER ELASTIC FOUNDATIONS

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Natural frequencies and buckling stresses of a deep beam-column on two-parameter elastic foundations are analyzed by taking into account the effect of shear deformation, depth change (the transverse displacement w can vary in the depth direction of beam-columns) and rotatory inertia. By using the method of power series expansion of displacement components, a set of fundamental dynamic equations of a one-dimensional higher order theory for thin rectangular beam-columns subjected to axial stress is derived through Hamilton's principle. Several sets of truncated approximate theories are applied to solve the eigenvalue problems of a simply supported deep elastic beam-column. In order to assure the accuracy of the present theory, convergence properties of the minimum natural frequency and the buckling stress are examined in detail. It is noted that the present approximate theories can predict the natural frequencies and buckling stress of deep beam-columns on elastic foundations accurately compared with the Timoshenko beam theory and the classical beam theory.

1. INTRODUCTION

Vibration and buckling problems of beams or beam-columns on elastic foundations occupy an important place in many fields of structural and foundation engineering. In many of the soil-structure interaction problems the elastic foundation has been usually modelled by a Winkler foundation for mathematical simplicity. However, it has been shown that the behavior of foundation materials in engineering practice cannot be represented by this foundation model which consists of independent linear elastic springs. In order to find a physically close and mathematically simple foundation model, Pasternak proposed a so-called two-parameter foundation model with shear interactions. The first foundation parameter is the same as the Winkler foundation model and the second one is the stiffness of the shearing layer in the Pasternak foundation model. While the interaction between springs has not been considered in the Winkler foundation model, the Pasternak foundation model has taken into account the interaction between springs for homogeneous elastic foundations. A more realistic and generalized representation of the elastic foundation can be accomplished by the

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way of a two-parameter foundation model. The physical meaning of the second generalized foundation parameter has different definitions depending on the foundation model [1].

The free flexural vibration of such structures has been extensively covered by many investigators. For the buckling problem of beam-columns on elastic foundations, only a few studies are available in the literature. Most of them have been done within the scope of the classical Bernoulli–Euler beam theory to investigate the vibration and buckling behavior of beams on elastic foundations. This theory leads to a significant overprediction of the natural frequencies and buckling stresses of deep beams due to the neglect of the effects of transverse shear deformation, depth change and rotatory inertia [2,3]. For deep beams with small length-to-depth ratio and/or beams in which higher modes may appear, the Timoshenko beam theory which takes into account the effects of shear deformation and rotatory inertia has been applied in the analysis. Wang and Stephens [4] studied the effects of Pasternak foundations on natural frequencies of finite Timoshenko beams. The effects of rotatory inertia, shear deformation and foundation constants on the natural frequencies of the beams with different end conditions were considered. Yokoyama [5] developed a finite element procedure for analyzing the flexural vibrations of a Timoshenko beam-column on two-parameter elastic foundations. The frequency parameter of a hinged–hinged Timoshenko beam-column without axial force has been compared with that of an elementary Euler–Bernoulli beam-column. The free vibration frequencies of Timoshenko beams on two-parameter elastic foundation were examined by Rosa [6] for two different foundation models of the second foundation parameter. In the first model, the second foundation parameter is assumed to be a function of the flexural rotation, whereas in the second model, it is assumed to be a function of the global cross-section rotation. The inherent deficiency of the Timoshenko beam theory is the presence of a correction factor, the so-called Timoshenko shear correction coefficient κ^2, which is introduced to correct the contradictory shear stress distribution over the cross-section of the beam and cannot be found from within the assumptions of the theory itself. The shear correction coefficient should be adjusted for studying the higher mode vibrational behavior of beams because the dynamic shear strain distribution may differ significantly from the parabolic form of the static shear strain distribution.

Higher order shear-deformable theories have been developed for beams with rectangular cross-sections that account for the strain distribution through the depth to satisfy the stress-free boundary conditions on the upper and lower surfaces without the need for a shear correction coefficient. In retaining the parabolic distribution of the tranverse shear strain, a shear deformation theory for rectangular beams that accounts for the shear free boundary conditions on the lateral surfaces of the beam was proposed by Levinson [7, 8]. Without requiring the specification of a shear correction coefficient in the Timoshenko beam theory, the theory allows the cross-sections both to rotate relative to the neutral axis and to warp into a non-planar surface. Although the shear strain vanishes on the upper and lower surfaces of a beam, the boundary conditions for shear and normal stresses may not be satisfied in the theory. By expanding the axial displacement component as a cubic function of the beam depth co-ordinate, Heyliger and Reddy [9] have also developed a higher order beam finite element by using a variationally consistent higher order beam theory. The theory accounts for the shear strain vanished on the upper and lower surfaces of the beam. Since the normal displacement is assumed to be constant through the depth of a beam, the depth change of the beam is not allowed and the stress boundary conditions for normal stress on the upper and lower surfaces of the beam are not satisfied. These theories further assume that the in-plane cross-sectional stresses are negligible and that the cross-section does not deform in its own plane.

As an extension of the classical beam theory, a one-dimensional higher order theory has been developed for a deep beam and has been applied to the vibration and stability problems of a very deep beam by Matsunaga [2, 3]. Natural frequencies and buckling stresses of deep beams subjected to axial stresses have been analyzed by using the one-dimensional higher order theory. Remarkable effects of transverse shear deformations and depth changes have been noticed in the results. However, higher order theories of beams which take into account the complete effects of shear deformations, depth changes and rotatory inertia have not been investigated in the problem of beam-columns on elastic foundations.

This paper presents a one-dimensional higher order theory of deep elastic beam-columns resting on elastic foundations which can take into account the complete effects of both shear deformations and depth changes. Several sets of the governing equations of truncated approximate theories are applied to the analysis of vibration and buckling problems of a simply supported deep elastic beam-column on two-parameter elastic foundations subjected to axial stresses. Based on the power series expansions of displacement components, a fundamental set of equations of a one-dimensional higher order beam-column theory is derived through Hamilton's principle. Natural frequencies and buckling stresses of a beam-column on two-parameter elastic foundation subjected to axial stresses are obtained by solving the eigenvalue problem numerically. Convergence properties of the present numerical solutions are shown to be accurate for the natural frequencies and buckling stresses with respect to the order of approximate theories. A comparison of the present results is made with previously published results. The present results obtained by various sets of approximate theories are considered to be accurate enough for deep beam-columns with small length-to-depth ratio. It is noticed that the one-dimensional higher order theory, in the present paper, can predict the natural frequencies and buckling stresses of simply supported deep beam-columns on two-parameter elastic foundations accurately when compared with the Timoshenko beam theory and the classical beam theory.

2. GOVERNING EQUATIONS OF BEAM-COLUMNS ON ELASTIC FOUNDATIONS

Consider a straight uniform beam-column of length L resting on a two-parameter elastic foundation (Figure 1), having a rectangular cross-section of

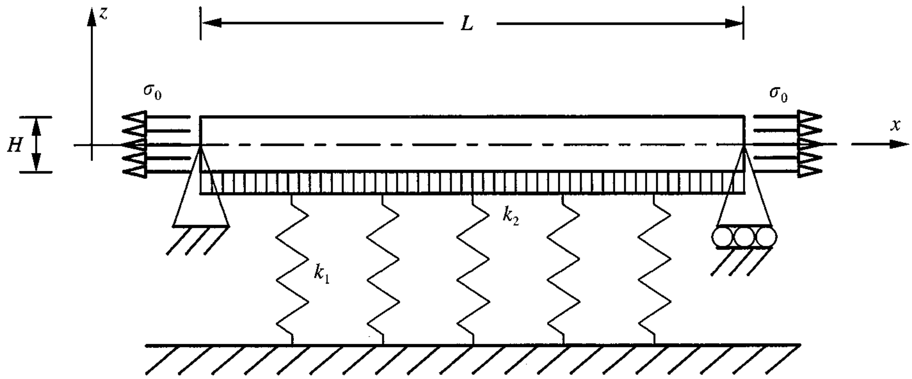

Figure 1. Co-ordinate and geometry of beam-column on two-parameter elastic foundation.

depth H and width B which is assumed to be sufficiently small relative to the depth. A Cartesian co-ordinate system (x, y, z) is defined on the central axis of the beam-column, where the x-axis is taken along this axis, with the y-axis in the width direction and the z-axis in the depth direction. Assuming that the deformations of the beam taken place in the x–z plane, the displacement components in a beam-column can be expressed as

$$u \equiv u(x, z; t), \quad v \equiv v(x, z; t) = 0, \quad w \equiv w(x, z; t), \tag{1}$$

where t denotes time. The displacement components may be expanded into power series of the depth co-ordinate z as follows:

$$u = \sum_{n=0}^{\infty} \overset{(n)}{u} z^n, \quad w = \sum_{n=0}^{\infty} \overset{(n)}{w} z^n, \tag{2}$$

where $n = 0, 1, 2, \ldots, \infty$.

Based on this expression of the displacement components, a set of the linear governing equations of a one-dimensional higher order theory of beam-column can be summarized in the following.

2.1. STRAIN–DISPLACEMENT RELATIONS

Strain components may also be expanded as follows:

$$\varepsilon_{xx} = \sum_{n=0}^{\infty} \varepsilon_{xx}^{(n)} z^n, \quad \gamma_{xz} = \gamma_{zx} = \sum_{n=0}^{\infty} \gamma_{xz}^{(n)} z^n, \quad \varepsilon_{zz} = \sum_{n=0}^{\infty} \varepsilon_{zz}^{(n)} z^n \tag{3}$$

and strain–displacement relations can be written as [10]

$$\varepsilon_{xx}^{(n)} = \overset{(n)}{u}_{,x}, \quad \gamma_{xz}^{(n)} = \gamma_{zx}^{(n)} = \frac{1}{2}\left\{(n+1)\overset{(n+1)}{u} + \overset{(n)}{w}_{,x}\right\}, \quad \varepsilon_{zz}^{(n)} = (n+1)\overset{(n+1)}{w}, \tag{4}$$

where a comma denotes partial differentiation with respect to the co-ordinate subscripts that follow.

2.2. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Under the assumption of plane strain or plane stress in the width direction, by introducing stress components $\sigma_{xx}, \tau_{xz} = \tau_{zx}$ and σ_{zz}, Hamilton's principle is applied to derive the equations of motion and natural boundary conditions of a beam-column on two-parameter elastic foundations. In order to treat vibration and stability problems of a beam subjected to uniformly distributed axial stress $\sigma_0 = \sigma_0(z)$, additional work due to this stress which is assumed to remain unchanged during vibrating and/or buckling is taken into consideration. It is also assumed that stresses are free on the upper surface of the beam-column, the lower surface of which is supported on an elastic foundation.

The principle for the present problems may be expressed as follows:

$$\int_{t_1}^{t_2}\left[\int_V \{\sigma_{xx}\delta\varepsilon_{xx} + 2\tau_{xz}\delta\gamma_{xz} + \sigma_{zz}\delta\varepsilon_{zz} + \sigma_0(u_{,x}\delta u_{,x} + w_{,x}\delta w_{,x}) - \rho(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w})\}\,\mathrm{d}V\right.$$
$$\left. + \int_S \{k_1 w\delta w|_{z=-H/2} + k_2 w_{,x}\delta w_{,x}|_{z=-H/2}\}\,\mathrm{d}S\right]\mathrm{d}t = 0, \tag{5}$$

where the over dot indicates partial differentiation with respect to time and ρ denotes the mass density; $\mathrm{d}V$, the volume element; $\mathrm{d}S$, the element of area of the external surface; k_1, the first foundation parameter which is referred to as the Winkler foundation stiffness; k_2, the second generalized foundation parameter.

The axial stress σ_0 is assumed to be expanded a follows:

$$\sigma_0 = \sum_{\ell=0}^{\infty} \overset{(\ell)}{\sigma}_0 z^{\ell}, \tag{6}$$

where $\ell = 0, 1, 2, \dots, \infty$.

By performing the integration over the area of cross-section of the beam-column and the variation as indicated in equation (5), the equations of motion are obtained as follows:

$$\delta\overset{(n)}{u}:\ \overset{(n)}{N}_{,x} - n\overset{(n-1)}{Q} + \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\overset{(\ell)}{\sigma}_0\overset{(m)}{u}_{,xx} f(n+m+\ell+1) = \rho\sum_{m=0}^{\infty}\overset{(m)}{\ddot{u}} f(n+m+1),$$

$$\delta\overset{(n)}{w}:\ \overset{(n)}{Q}_{,x} - n\overset{(n-1)}{T} - \sum_{m=0}^{\infty}[k_1\overset{(m)}{w} - k_2\overset{(m)}{w}_{,xx}](-H/2)^{n+m}$$
$$+ \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\overset{(\ell)}{\sigma}_0\overset{(m)}{w}_{,xx} f(n+m+\ell+1) = \rho\sum_{m=0}^{\infty}\overset{(m)}{\ddot{w}} f(n+m+1), \tag{7}$$

where $n, m = 0, 1, 2, \dots, \infty$.

The stress resultants are defined as follows:

$$\overset{(n)}{N} = \int_{-H/2}^{+H/2}\sigma_{xx}z^n\,\mathrm{d}z, \quad \overset{(n)}{Q} = \int_{-H/2}^{+H/2}\tau_{xz}z^n\,\mathrm{d}z, \quad \overset{(n)}{T} = \int_{-H/2}^{+H/2}\sigma_{zz}z^n\,\mathrm{d}z. \tag{8}$$

The following function is defined as

$$f(k) \equiv \int_{-H/2}^{+H/2} z^{k-1}\,\mathrm{d}z = \frac{1}{k}\left(\frac{H}{2}\right)^k [1-(-1)^k] = \begin{cases} 0 & (k\text{: even}) \\ \dfrac{2}{k}\left(\dfrac{H}{2}\right)^k & (k\text{: odd}), \end{cases} \tag{9}$$

where k is an integer.

For the equations of boundary conditions at the ends on the central axis, the following quantities:

$$\begin{aligned} &\overset{(n)}{u} \text{ or } \overset{(n)}{N} + \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \overset{(\ell)}{\sigma}_0 \overset{(m)}{u}_{,x} f(n+m+\ell+1), \\ &\overset{(n)}{w} \text{ or } \overset{(n)}{Q} + \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \overset{(\ell)}{\sigma}_0 \overset{(m)}{w}_{,x} f(n+m+\ell+1), \end{aligned} \tag{10}$$

are to be prescribed.

2.3. CONSTITUTIVE RELATIONS

For elastic and homogeneous isotropic materials, the two-dimensional constitutive relations can be written as

$$\sigma_{xx} = 2\mu\varepsilon_{xx} + \bar{\lambda}(\varepsilon_{zz} + \varepsilon_{xx}), \quad \tau_{xz} = 2\mu\gamma_{xz}, \quad \sigma_{zz} = 2\mu\varepsilon_{zz} + \bar{\lambda}(\varepsilon_{xx} + \varepsilon_{zz}). \tag{11}$$

According to the assumptions of plane strain or stress states in the width direction of the beam-column, the coefficient $\bar{\lambda}$ is defined by

$$\bar{\lambda} = \begin{cases} \lambda & (\text{plane strain}), \\ \dfrac{2\mu\lambda}{2\mu+\lambda} & (\text{plane stress}), \end{cases} \tag{12}$$

where Lamé's constants λ and μ are defined by using Young's modulus E and the Poisson ratio v as follows:

$$\lambda = \frac{Ev}{(1+v)(1-2v)}, \quad \mu = \frac{E}{2(1+v)}. \tag{13}$$

2.4. STRESS RESULTANTS IN TERMS OF THE EXPANDED DISPLACEMENT COMPONENTS

Stress resultants can be expressed in terms of the expanded displacement components as follows:

$$\begin{aligned} \overset{(n)}{N} &= \sum_{m=0}^{\infty} [(\bar{\lambda}+2\mu)\overset{(m)}{u}_{,x} + \bar{\lambda}(m+1)\overset{(m+1)}{w}] f(n+m+1), \\ \overset{(n)}{Q} &= \sum_{m=0}^{\infty} \mu[(m+1)\overset{(m+1)}{u} + \overset{(m)}{w}_{,x}] f(n+m+1), \\ \overset{(n)}{T} &= \sum_{m=0}^{\infty} [(\bar{\lambda}+2\mu)(m+1)\overset{(m+1)}{w} + \bar{\lambda}\,\overset{(m)}{u}_{,x}] f(n+m+1). \end{aligned} \tag{14}$$

2.5. EQUATIONS OF MOTION IN TERMS OF THE EXPANDED DISPLACEMENT COMPONENTS

The equations of motion can be expressed in terms of the expanded displacement components by using equations (14) as

$$\delta \overset{(n)}{u}: \sum_{m=0}^{\infty} \Bigg[\Bigg\{ \Bigg[(\bar{\lambda} + 2\mu) \overset{(m)}{u}_{,x} + \bar{\lambda}(m+1) \overset{(m+1)}{w} \Bigg]_{,x} - \rho \overset{(m)}{\ddot{u}} \Bigg\} f(n+m+1)$$

$$- n\mu \Bigg[(m+1) \overset{(m+1)}{u} + \overset{(m)}{w}_{,x} \Bigg] f(n+m) + \sum_{\ell=0}^{\infty} \overset{(\ell)}{\sigma}_0 \overset{(m)}{u}_{,xx} f(n+m+\ell+1) \Bigg] = 0,$$

$$\delta \overset{(n)}{w}: \sum_{m=0}^{\infty} \Bigg[\Bigg\{ \mu \Bigg[(m+1) \overset{(m+1)}{u} + \overset{(m)}{w}_{,x} \Bigg]_{,x} - \rho \overset{(m)}{\ddot{w}} \Bigg\} f(n+m+1) - n \Bigg[(\bar{\lambda} + 2\mu)(m+1) \overset{(m+1)}{w}$$

$$+ \bar{\lambda} \overset{(m)}{u}_{,x} \Bigg] f(n+m) - [k_1 \overset{(m)}{w} - k_2 \overset{(m)}{w}_{,xx}](-H/2)^{n+m}$$

$$+ \sum_{\ell=0}^{\infty} \overset{(\ell)}{\sigma}_0 \overset{(m)}{w}_{,xx} f(n+m+\ell+1) \Bigg] = 0. \tag{15}$$

Since the lower plane of a deep beam-column is supported by an elastic foundation and the upper plane is stress free, the equations of motion cannot be separated into the axial and flexural problems.

2.6 MTH ORDER APPROXIMATE THEORY

Since the fundamental equations mentioned above are complicated, Mth ($M \geqslant 1$) order approximate theory may be considered for the present analysis. A similar set of the following combination of the selected terms of displacement components is suggested from the form of shear strain components in the second equation of equation (4) as follows:

$$u = \sum_{m=0}^{2M-1} \overset{(m)}{u} z^m, \quad w = \sum_{m=0}^{2M-2} \overset{(m)}{w} z^m, \tag{16}$$

where $m = 0, 1, 2,3, \dots, M$. The total number of the unknown displacement components is $(4M - 1)$.

In the above cases of $M = 1$, an assumption that the normal strain ε_{zz} is zero is inherently imposed. Another set of the governing equations of the lowest order approximate theory ($M = 1$) is derived using an assumption that the normal stress σ_{zz} is zero. For flexural problems, this theory corresponds to the Timoshenko beam theory with the shear correction coefficient $\kappa^2 = 1$. Under this assumption, if the shear strain γ_{xz} vanishes through the depth of a beam-column, the lowest order approximate theory reduces to the classical beam theory.

3. NAVIER SOLUTION FOR SIMPLY SUPPORTED BEAM-COLUMN

In order to show the applicability and reliability of the present one-dimensional higher order theories for the analysis of vibration and buckling problems of a deep elastic beam, a simply supported beam-column subjected to axial stress on a two-parameter elastic foundation is analyzed. In the following analysis, the axial stress is assumed to distribute uniformly in the depth direction. Only the first term of the expanded axial stress (6) is considered, i.e. $\sigma_0 = \sigma_0^{(0)}$.

Boundary conditions (10) can be expressed on the x-constant edges,

$$\overset{(n)}{u}_{,x} = 0, \quad \overset{(n)}{w} = 0. \tag{17}$$

Since a beam-column is in a state of uniform stresses, the axial stress is considered to be constant during vibrating and/or buckling. Following the Navier solution procedure, displacement components that satisfy the equations of boundary conditions (17) may be expressed as

$$\overset{(n)}{u} = \sum_{r=1}^{\infty} \overset{(n)}{u}_r \cos\frac{r\pi x}{L} \cdot e^{\mathrm{i}\omega t}, \quad \overset{(n)}{w} = \sum_{r=1}^{\infty} \overset{(n)}{w}_r \sin\frac{r\pi x}{L} \cdot e^{\mathrm{i}\omega t}, \tag{18}$$

where the displacement mode number $r = 1, 2, 3, \ldots, \infty$, ω denotes the circular frequency and i, the imaginary unit.

The equations of motion are rewritten in terms of the generalized displacement components $\overset{(n)}{u}_r$ and $\overset{(n)}{w}_r$.

The dimensionless axial or buckling stress Λ is defined as follows:

$$\Lambda = \frac{A\sigma_0}{P_c}, \tag{19}$$

where P_c is the minimum buckling load for the bending problem of a beam from the classical beam theory expressed by

$$P_c = \frac{\pi^2 EI}{L^2}, \quad I = \frac{BH^3}{12}. \tag{20}$$

The dimensionless frequency Ω is defined as follows:

$$\Omega = \omega L^2 \sqrt{\frac{\rho A}{EI}}, \quad A = BH. \tag{21}$$

The dimensionless elastic constants of foundations may also be defined as follows:

$$K_1 = \frac{k_1 L^4}{EI}, \quad K_2 = \frac{k_2 L^2}{\pi^2 EI}. \tag{22}$$

4. EIGENVALUE PROBLEM FOR VIBRATION AND BUCKLING PROBLEMS

The equations of motion (15) can be rewritten by collecting the coefficients for the generalized displacements of any fixed value r. The generalized displacement vector $\{\mathbf{U}\}$ is expressed as

$$\{\mathbf{U}\}^{\mathrm{T}} = \left\{ \overset{(0)}{u_r}, \ldots, \overset{(2M-1)}{u_r}; \overset{(0)}{w_r}, \ldots, \overset{(2M-2)}{w_r} \right\}. \tag{23}$$

The dynamic equation can be expressed as the following eigenvalue problem:

$$([\mathbf{K}] - \Omega^2[\mathbf{M}])\{\mathbf{U}\} = \{\mathbf{0}\}, \tag{24}$$

where matrix $[\mathbf{K}]$ denotes the stiffness matrix which may contain the terms of the axial stress, and matrix $[\mathbf{M}]$ denotes the mass matrix.

For buckling problems, the natural frequency vanishes and the stability equation can be expressed as the following eigenvalue problem:

$$([\mathbf{K}] + \Lambda[\mathbf{S}])\{\mathbf{U}\} = \{\mathbf{0}\}, \tag{25}$$

where matrix $[\mathbf{K}]$ denotes the stiffness matrix and matrix $[\mathbf{S}]$, the geometric-stiffness matrix due to the axial stress.

The power method [11] is used to obtain the numerical solution of the eigenvalue problems. The number of eigenvalues is the same as that of the components of the generalized displacement vector for each displacement mode number of r. Although all the eigenvalues and eigenvectors can be computed by this method, the dominant eigenvalue which corresponds to the minimum frequency is of great concern.

When the lowest frequency vanishes, the axial stress reduces to the critical buckling stress of the beam-column. The same buckling stresses are obtained by solving the stability equation (25) and also the dynamic equation (24) by increasing the axial compressive stresses.

5. NUMERICAL EXAMPLES AND RESULTS

5.1 NUMERICAL EXAMPLES

A simply supported deep beam-column on a two-parameter elastic foundation is analyzed. The effects of higher order deformations such as shear deformation, depth change and rotatory inertia on natural frequencies and buckling stresses of a deep beam-column subjected to axial stresses are studied through the numerical examples. The Poisson ratio is fixed at $\nu = 0{\cdot}3$. All the numerical results are obtained for the case of plane stress in the width direction and are shown in the dimensionless quantities.

5.2. CONVERGENCE OF THE LOWEST NATURAL FREQUENCY AND BUCKLING STRESS AND COMPARISON WITH THOSE OF EXISTING THEORIES

In order to verify the accuracy of the present solutions, the convergences of the lowest natural frequencies without axial stresses and the critical buckling stresses of the present approximate theories for the first displacement mode $r = 1$ are shown in Tables 1 and 2 respectively. A direct comparison of the present solutions with those

TABLE 1

Convergence of natural frequencies and comparison with previously published results

L/H	K_1	K_2	CBT	TBT	$M = 1^{\dagger}$	$M = 2$	$M = 3$	$M = 4$
2	0	0	9·8696	7·4127	7·6269	7·4701	7·4664	←
	10		10·3638	8·0106	8·2033	8·0133	8·0102	←
	10^2		14·0502	12·1084	12·2016	11·2841	11·2820	←
	10^3		33·1272	29·0828	29·3176	17·5505	17·5208	←
	10^4		100·4859	34·6364	36·6283	18·9936	18·9458	←
	10^5		316·3817	34·8004	36·8546	19·1246	19·0754	←
	0	1	13·9577	12·0106	12·1055	11·2156	11·2136	←
	10		14·3115	12·3836	12·4719	11·4743	11·4721	←
	10^2		17·1703	15·3153	15·3609	13·2711	13·2672	←
	10^3		34·5661	29·9225	30·2643	17·6944	17·6630	←
	10^4		100·9694	34·6383	36·6310	18·9950	18·9472	←
	10^5		316·5356	34·8004	36·8546	19·1246	19·0755	←
5	0	0	9·8696	9·2740	9·3430	9·2905	9·2903	←
	10		10·3638	9·7848	9·8497	9·7914	9·7912	←
	10^2		14·0502	13·5408	13.5851	13·4727	13·4726	←
	10^3		33·1272	32·5378	32·5469	31·6171	31·6171	←
	10^4		100·4859	98·5400	98·6078	53·8885	53·8874	←
	10^5		316·3817	177·2381	192·2551	54·0261	54·0250	←
	0	1	13·9577	13·4473	13·4920	13·3812	←	←
	10		14·3115	13·8045	13·8478	13·7307	←	←
	10^2		17·1703	16·6781	16·7119	16·5354	←	←
	10^3		34·5661	33·9613	33·9692	32·9239	←	←
	10^4		100.9694	99·0068	99·0766	53·8906	←	←
	10^5		316.5356	177·2396	192·2575	54·0261	←	←
10	0	0	9.8696	9·7071	9·7270	9·7121	←	←
	10		10.3638	10·2057	10·2246	10·2078	←	←
	10^2		14.0502	13·9086	13·9224	13·8941	←	←
	10^3		33.1271	32·9615	32·9665	32·8494	←	←
	10^4		100.4859	100·0749	100·0749	98·1173	←	←
	10^5		316.3817	314·8251	314·8771	108·6430	←	←
	0	1	13.9577	13·8162	13·8297	13·8018	←	←
	10		14.3115	14·1709	14·1840	14·1548	←	←
	10^2		17.1703	17·0326	17·0434	17·0046	←	←
	10^3		34.5661	34·3963	34·4011	34·2741	←	←
	10^4		100.9694	100·5564	100·5564	98·5532	←	←
	10^5		316.5356	314·9778	315·0300	108·6431	←	←

CBT: Classical beam theory TBT: Timoshenko beam theory (shear coefficient $\kappa^2 = \frac{5}{6}$) $M = 1^{\dagger}$: TBT ($\sigma_{zz} = 0, \kappa^2 = 1$)

of classical beam theory (CBT) and Timoshenko beam theory (TBT) in which the effects of extension and rotatory inertia are included here is also made. The present CBT and TBT solutions of natural frequencies (and buckling stresses) agree perfectly with the previously published results in the figures of reference [5] for the same data. A significant overprediction of CBT solutions for deep beams due to the

TABLE 2

Convergence of the first buckling stresses and comparison with previously published results

L/H	K_1	K_2	CBT	TBT	$M = 1^{\dagger}$	$M = 2$	$M = 3$	$M = 4$
2	0	0	1·0000	0·5641	0·5971	0·5729	0·5723	←
	10		1·1027	0·6588	0·6908	0·6592	0·6587	←
	10^2		2·0266	1·5051	1·5284	1·3072	1·3067	←
	10^3		11·2660	8·6831	8·8238	3·1621	3·1514	←
	10^4		103·6598	12·3159	13·7732	3·7035	3·6849	←
	10^5		1027·5980	12·4328	13·9439	3·7548	3·7355	←
	0	1	2·0000	1·4809	1·5044	1·2914	1·2909	←
	10		2·1027	1·5734	1·5969	1·3516	1·3511	←
	10^2		3·0266	2·4080	2·4223	1·8081	1·8070	←
	10^3		12·2660	9·1917	9·4029	3·2142	3·2028	←
	10^4		104·6598	12·3173	13·7752	3·7041	3·6854	←
	10^5		1028·5980	12·4328	13·9439	3·7548	3·7355	←
5	0	0	1·0000	0·8830	0·8961	0·8861	0·8860	←
	10		1·1027	0·9829	0·9960	0·9842	←	←
	10^2		2·0266	1·8823	1.8946	1·8634	←	←
	10^3		11·2660	10·8687	10·8748	10·2623	←	←
	10^4		103·6598	99·6841	99·8213	29·8121	29·8108	←
	10^5		1027·5980	322·4889	379·4514	29·9646	29·9633	←
	0	1	2·0000	1·8564	1·8688	1·8382	←	←
	10		2·1027	1·9563	1·9686	1·9355	←	←
	10^2		3·0266	2·8556	2·8672	2·8069	←	←
	10^3		12·2660	11·8405	11·8460	11·1281	←	←
	10^4		104.6598	100·6308	100·7727	29·8145	29·8132	←
	10^5		1028.5980	322·4944	379·4609	29·9646	29·9634	←
10	0	0	1.0000	0·9675	0·9714	0·9683	←	←
	10		1.1027	1·0693	1·0732	1·0697	←	←
	10^2		2.0266	1·9859	1·9901	1·9818	←	←
	10^3		11.2660	11·1535	11·1571	11·0779	←	←
	10^4		103.6598	102·8135	102·8137	98·8306	98·8305	←
	10^5		1027.5980	1017·5110	1017·8470	121·1726	121·1723	←
	0	1	2·0000	1·9597	1·9636	1·9556	←	←
	10		2.1027	2·0616	2·0655	2·0569	←	←
	10^2		3.0266	2·9784	2·9820	2·9685	←	←
	10^3		12.2660	12·1458	12·1491	12·0596	←	←
	10^4		104.6598	103·8057	103·8057	99·7108	99·7107	←
	10^5		1028.5980	1018·4990	1018·8360	121·1726	121·1724	←

CBT: Classical beam theory TBT: Timoshenko beam theory (shear coefficient $\kappa^2 = \frac{5}{6}$) $M = 1^{\dagger}$: TBT ($\sigma_{zz} = 0, \kappa^2 = 1$)

neglect of the effects of transverse shear deformation, depth change and rotatory inertia can be seen in the results. The inherent deficiency of the Timoshenko beam theory is the presence of a correction factor κ^2, which cannot be found from within the assumptions of the theory itself. The complete effects of higher order deformations such as shear deformations with depth changes and rotatory inertia for the analysis of vibration and buckling problems of deep beam-columns on elastic foundations should be taken into account.

It is observed that the proper order of the present higher order approximate theories may be estimated according the level of L/H. Since the present results for $M = 1$–4 converge accurately enough within the present order of approximate theories, only the numerical results for $M = 5$ are discussed.

5.3. NATURAL FREQUENCIES WITHOUT AXIAL STRESS AND BUCKLING STRESSES

Both the first three natural frequencies ($r = 1$–3) of a beam-column without axial stresses and the first three buckling stresses are shown in Table 3 for all the values of L/H and for several combinations of elastic foundation parameters. The results converge enough for $M = 5$ with sufficient numerical accuracy.

The lowest two natural frequencies for $r = 1$ of a beam-column without axial stresses versus the first foundation parameter K_1 is plotted for $L/H = 2$ and 5 in Figure 2. The lower natural frequency Ω_1 corresponds to predominantly flexural modes with some shear deformations, whereas the upper natural frequency Ω_2 corresponds to predominantly extensional modes with depth changes. Natural frequencies vary gently with K_1 for deep beam-columns, but vary suddenly at a specific value of K_1 which may depend on L/H. For values of K_1 larger than this specific value, the vibration modes of Ω_1 and Ω_2 change places with each other. The second foundation parameter K_2 has an evident effect upon Ω_1 for comparatively small values of K_1.

5.4. NATURAL FREQUENCIES VERSUS AXIAL STRESS CURVES AND BUCKLING STRESSES

In Figure 3, the variation of the lowest two natural frequencies for $r = 1$ with respect to axial stresses is shown for $L/H = 2$ and 5. Ω_1 shows the natural frequencies which correspond to predominantly flexural modes and Ω_2, extensional modes with some shear deformation and depth change. When the lower natural frequency Ω_1 vanishes, the axial stresses reduce to the critical buckling stresses of beam-columns on elastic foundations. The lower frequency curve (Ω_1) will decrease rapidly prior to buckling and the frequency vanishes at the axial buckling stress.

The buckling stresses can be calculated usually through the stability equation (25) as eigenvalue problems. In the case of a simply supported beam-column on elastic foundations subjected to axial stress Λ, the natural frequency Ω_a can be expressed explicitly with reference to the natural frequency Ω_0 of a beam-column without axial stress. The relation between Ω_a and Ω_0 can be obtained from comparison of the equations of motion as follows:

$$\Omega_a^2 = \Omega_0^2 + r^2\pi^4\Lambda. \tag{26}$$

When the natural frequency Ω_a vanishes under the axial stress, elastic buckling occurs and the critical buckling stress Λ_{cr} relates to the natural frequency Ω_0 as

$$\Lambda_{cr} = -\frac{\Omega_0^2}{r^2\pi^4}. \tag{27}$$

TABLE 3

First three frequencies and buckling stresses ($r = 1$–3) *of simply supported beam-columns on two-parameter elastic foundation*

			Ω			Λ		
L/H	K_1	K_2	$r=1$	$r=2$	$r=3$	$r=1$	$r=2$	$r=3$
2	0	0	7·4664	20·4448	33·8983	0·5723	1·0728	1·3107
	10		8·0102	20·6407	34·0260	0·6587	1·0934	1·3206
	10^2		11·2820	22·0350	34·9267	1·3067	1·2461	1·3915
	10^3		17·5208	25·6375	37·0821	3·1514	1·6869	1·5685
	10^4		18·9457	26·9350	37·7741	3·6849	1·8620	1·6276
	10^5		19·0754	27·0911	37·8547	3·7355	1·8836	1·6346
	0	1	11·2136	24·2169	36·9912	1·2909	1·5051	1·5608
	10		11·4721	24·2607	36·9993	1·3511	1·5106	1·5615
	10^2		13·2672	24·6050	37·0664	1·8070	1·5538	1·5672
	10^3		17·6630	25·9940	37·4162	3·2028	1·7342	1·5969
	10^4		18·9472	26·9414	37·7813	3·6854	1·8629	1·6282
	10^5		19·0754	27·0911	37·8548	3·7355	1·8836	1·6346
5	0	0	9·2903	32·3391	62·0126	0·8860	2·6841	4·3865
	10		9·7912	32·4756	62·0801	0·9842	2·7068	4·3960
	10^2		13·4726	33·6731	62.6813	1·8634	2·9101	4·4816
	10^3		31·6171	43·4716	68·1451	10·2623	4·8501	5·2970
	10^4		53·8874	82·1422	96·3702	29·8108	17·3170	10·5936
	10^5		54·0250	102·1233	124·0626	29·9633	26·7664	17·5566
	0	1	13·3811	37·2054	67·4499	1·8382	3·5526	5·1894
	10		13·7306	37·3197	67·5069	1·9354	3·5745	5·1982
	10^2		16·5353	38·3287	68·0148	2·8069	3·7704	5·2767
	10^3		32·9238	46·8774	72·6796	11·1281	5·6398	6·0254
	10^4		53·8895	82·9246	97·8569	29·8132	17·6485	10·9230
	10^5		54.0250	102·1281	124·1049	29·9634	26·7689	17·5685
10	0	0	9.7121	37·1610	78·4264	0·9683	3·5442	7·0159
	10		10.2078	37·2895	78·4849	1·0697	3·5687	7·0264
	10^2		13·8941	38·4259	79·0099	1·9818	3·7896	7·1207
	10^3		32.8494	48·3173	84·0657	11·0779	5·9917	8·0611
	10^4		98.1173	103·3225	122·7926	98·8305	27·3987	17·1990
	10^5		108.6429	214·8104	272·3120	121·1723	118·4271	84·5846
	0	1	13.8018	41·8706	83·3926	1·9556	4·4995	7·9326
	10		14.1548	41·9844	83·4474	2·0569	4·5239	7·9430
	10^2		17.0046	42·9947	83·9388	2·9685	4·7443	8·0368
	10^3		34.2741	52·0065	88·6905	12·0596	6·9415	8·9725
	10^4		98.5532	105·0088	125·8413	99·7107	28·3004	18·0636
	10^5		108.6430	214·8210	273·0282	121·1724	118·4388	85·0302

The critical buckling stresses of simply supported beam-columns subjected to axial stress can be predicted from the natural frequency of the beam-columns without axial stress.

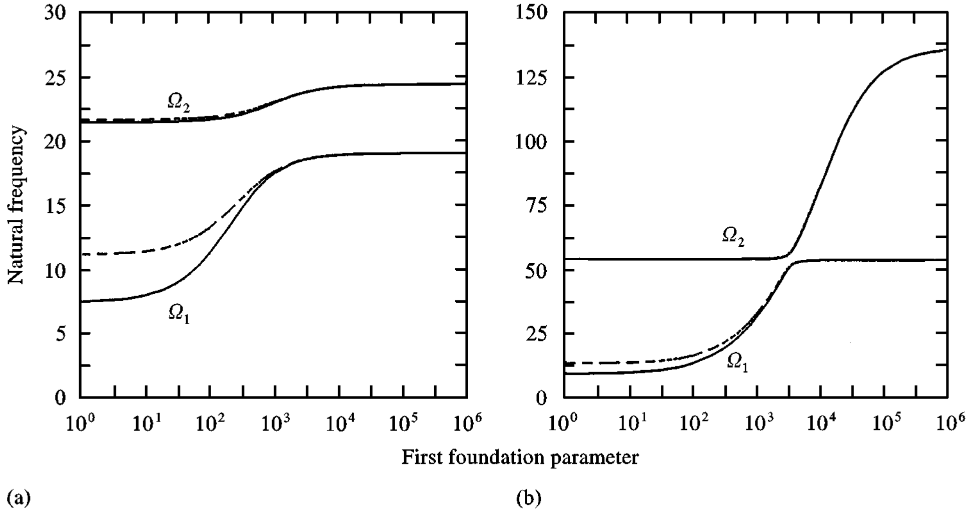

Figure 2. Natural frequency versus K_1 curves. (—— : $K_2 = 0$; - - - - -: $K_2 = 1$) (a) $L/H = 2$; (b) $L/H = 5$.

5.5. BUCKLING STRESSES VERSUS DISPLACEMENT MODE CURVES

For a simply supported beam on elastic foundations, Figure 4 shows the variation of the buckling stresses with respect to displacement modes. The lower (critical) buckling stresses Λ_1 which have flexural modes increase monotonically to asymptotic values at higher displacement modes and the upper buckling stresses Λ_2 which have extensional modes decrease rapidly from those for $r = 1$ to the same asymptotic values. This feature shows that both the flexural and extensional buckling instabilities can exist at higher displacement modes with wrinkling deformations. The buckling stresses for the first three displacement modes $r = 1$–3 are shown in Table 3.

The critical buckling stresses are compared with those of CBT and TBT in Table 4. A figure on the right shoulder of buckling stresses defines the buckling mode number r and the infinity sign shows a much higher-mode number. It can be noticed that the buckling stresses of CBT are overestimated for deep beam-columns, especially for large values of the first elastic constant of foundations K_1.

5.6. GENERALIZED DISPLACEMENTS OF VIBRATION AND BUCKLING MODES

The same displacement modes are obtained in the eigenvalue problems of vibration and buckling. In order to show the variation of vibration and buckling modes, the first three generalized displacements are plotted with respect to K_1 in Figure 5. $\overset{(0)}{U}_{\mathrm{I,II}}$ is the first axial generalized displacement and $\overset{(0)}{W}_{\mathrm{I,II}}$ and $\overset{(1)}{U}_{\mathrm{I,II}}$ are the first flexural generalized displacements. The Roman numeral I and II correspond to the lower (Ω_1, Λ_1) and upper (Ω_2, Λ_2) natural frequencies and buckling stresses

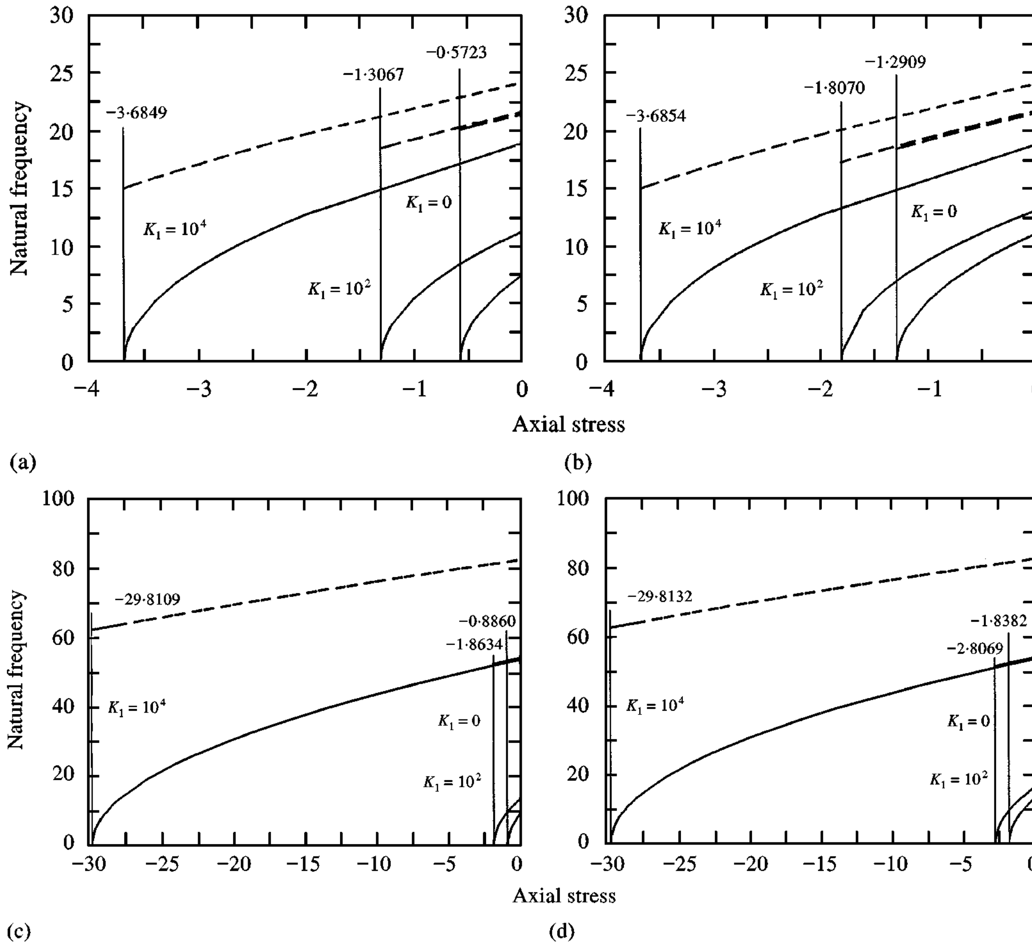

Figure 3. Natural frequency versus axial stress curves. (——: Ω_1; - - - - -: Ω_2) (a) $L/H = 2$, $K_2 = 0$; (b) $L/H = 2$, $K_2 = 1$; (c) $L/H = 5$, $K_2 = 0$; (d) $L/H = 5$, $K_2 = 1$.

respectively. The lower natural frequency Ω_1 and buckling stress Λ_1 arise at dominant flexural modes for small values of K_1. For values of K_1 larger than a specific value which may be affected by L/H, the displacement modes of the lower natural frequency Ω_1 and buckling stress Λ_1 transform into dominant axial modes. However, for the upper natural frequency Ω_2 and buckling stress Λ_2, this situation is completely reverse. The second foundation parameter K_2 does not affect the results so much if the thickness parameter L/H and/or the first foundation parameter K_1 are large.

6. CONCLUSIONS

Beyond the limits of applicability of the existing beam theories, various orders of the expanded approximate theories have been applied to analyze the vibration and

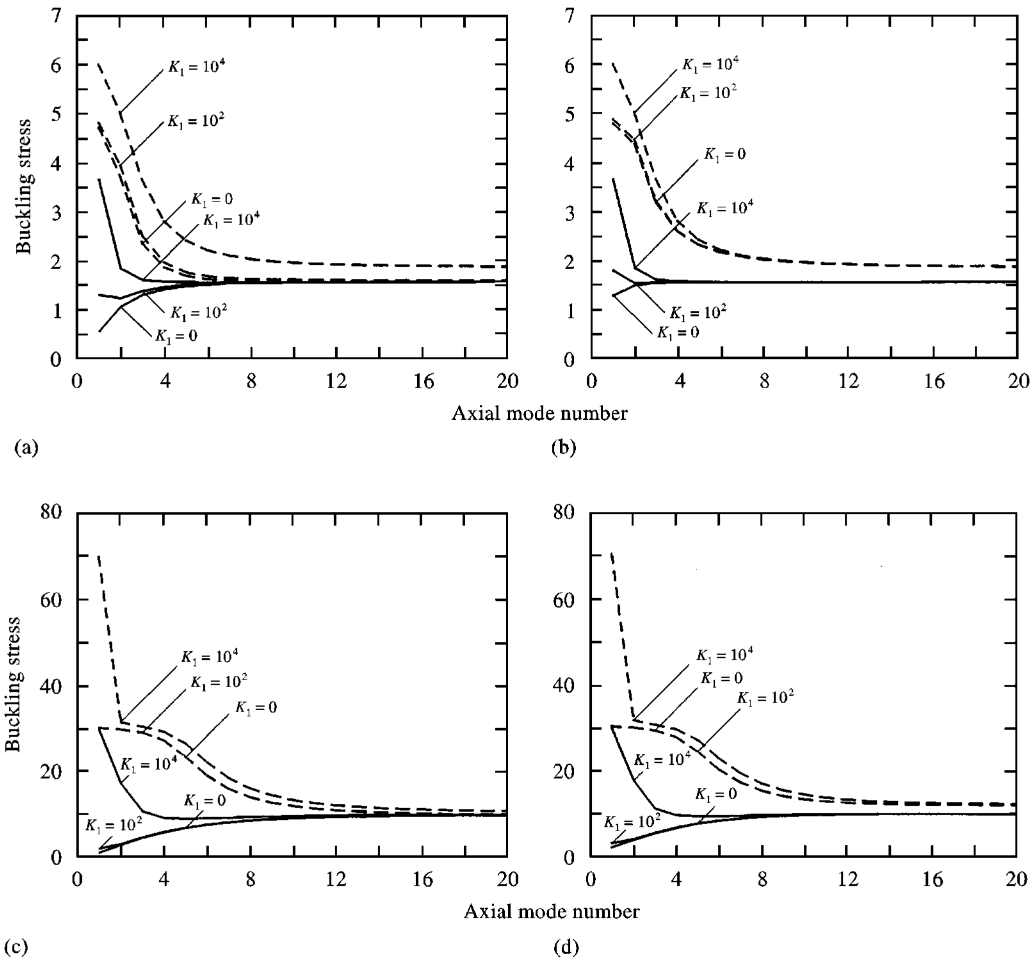

Figure 4. Buckling stress versus axial displacement mode number curves. (—— : Λ_1; - - - - - : Λ_2) (a) $L/H = 2$, $K_2 = 0$; (b) $L/H = 2$, $K_2 = 1$; (c) $L/H = 5$, $K_2 = 0$; (d) $L/H = 5$, $K_2 = 1$.

buckling problems of a simply supported deep beam-column subjected to axial stresses. In the present analysis, natural frequencies and buckling stresses of a deep beam-column resting on two-parameter elastic foundations have been obtained.

The following conclusions may be drawn from the present analysis:

(1) The natural frequencies and buckling stresses of deep beam-columns resting on two-parameter elastic foundations calculated by using the classical beam theory are usually overpredicted. It is very important to take into account the complete effects of higher order deformations such as shear deformations with depth changes and rotatory inertia for the analysis of vibration and buckling problems of deep beam-columns on elastic foundations.

(2) In order to verify the accuracy of the present results, the convergence properties of the numerical solutions according to the order of approximate

TABLE 4

Critical buckling stress with buckling mode number

			Λ_{cr}		
L/H	K_1	K_2	CBT	TBT	$M = 5$
2	0	0	1·0000^{1}	0·5641^{1}	0·5723^{1}
	10^1		1·1027^{1}	0·6588^{1}	0·6587^{1}
	10^2		2·0266^{1}	1·2765^{2}	1·2461^{2}
	10^3		6·5665^{2}	1·5588$^{\infty}$	1·5621^{4}
	10^4		20·4066^{3}	1·5588$^{\infty}$	1·5703^{8}
	10^5		64·5166^{6}	1·5588$^{\infty}$	1·5703^{9}
	0	1	2·0000^{1}	1·4809^{1}	1·2909^{1}
	10^1		2·1027^{1}	1·5743^{1}	1·3511^{1}
	10^2		3·0266^{1}	2·1714^{2}	1·5538^{2}
	10^3		7·5665^{2}	2·5588$^{\infty}$	1·5703^{8}
	10^4		21·4066^{3}	2·5588$^{\infty}$	1·5703^{9}
	10^5		65·5166^{6}	2·5588$^{\infty}$	1·5703^{9}
5	0	0	1·0000^{1}	0·8829^{1}	0·8860^{1}
	10^1		1·1027^{1}	0·9829^{1}	0·9842^{1}
	10^2		2·0266^{1}	1·8823^{1}	1·8634^{1}
	10^3		6·5665^{2}	5·0512^{2}	4·8501^{2}
	10^4		20·4066^{3}	9·7424$^{\infty}$	8·8436^{5}
	10^5		64·5166^{6}	9·7424$^{\infty}$	9·8137^{17}
	0	1	2·0000^{1}	1·8563^{1}	1·8382^{1}
	10^1		2·1027^{1}	1·9563^{1}	1·9354^{1}
	10^2		3·0266^{1}	2·8556^{1}	2·8069^{1}
	10^3		7·5665^{2}	5·9825^{2}	5·6398^{2}
	10^4		21·4066^{3}	10·4977^{9}	9·1342^{5}
	10^5		65·5166^{6}	10·7424$^{\infty}$	9·8143^{19}
10	0	0	1·0000^{1}	0·9675^{1}	0·9683^{1}
	10^1		1·1027^{1}	1·0693^{1}	1·0697^{1}
	10^2		2·0266^{1}	1·9859^{1}	1·9818^{1}
	10^3		6·5665^{2}	6·0302^{2}	5·9917^{2}
	10^4		20·4066^{3}	16·6148^{4}	16·2476^{4}
	10^5		64·5166^{6}	35·2902^{11}	32·6907^{8}
	0	1	2·0000^{1}	1·9597^{1}	1·9556^{1}
	10^1		2·1027^{1}	2·0616^{1}	2·0569^{1}
	10^2		3·0266^{1}	2·9784^{1}	2·9685^{1}
	10^3		7·5665^{2}	7·0036^{2}	6·9415^{2}
	10^4		21·4066^{3}	17·5477^{4}	17·0774^{4}
	10^5		65·5166^{6}	36·2012^{11}	33·1339^{8}

theories have been examined. The present results obtained for $M = 5$ are considered to be accurate enough for very deep beam-columns on elastic foundations. It may be noticed that the one-dimensional higher order beam theory in the present paper can predict the natural frequencies and buckling stresses of deep beam-columns on elastic foundations.

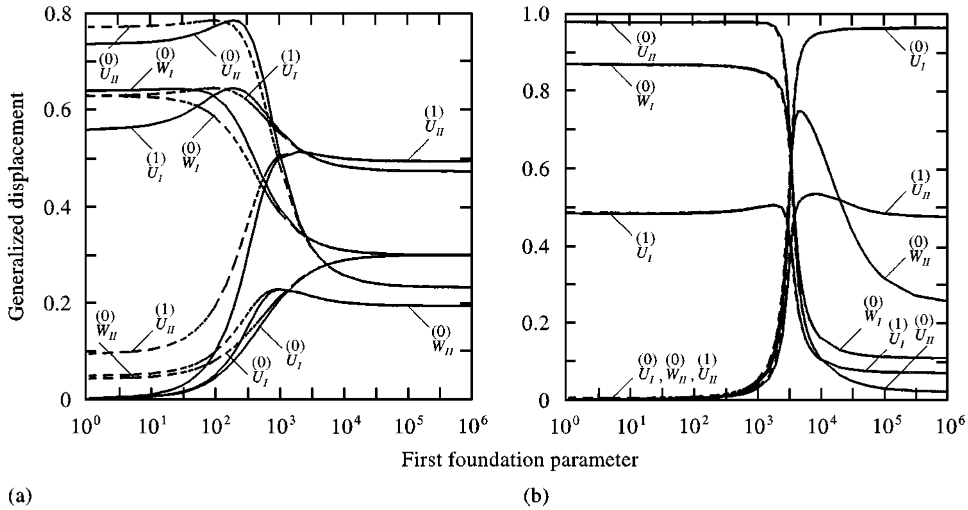

Figure 5. Generalized displacement versus K_1 curves. (——: $K_2 = 0$; - - - - -: $K_2 = 1$) (a) $L/H = 2$; (b) $L/H = 5$.

(3) In the case of a simply supported beam-column subjected to axial stress, the natural frequency can be expressed explicitly with reference to the natural frequency of a beam without axial stress. When the natural frequency reaches zero under axial compressions, elastic buckling occurs. The critical buckling stress can be predicted from the natural frequency of a beam-column without axial stress.

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