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HOW MANY MODES ARE ACCEPTABLE AND HOW DO PEOPLE IMPROVE THE MODES?

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In this paper, an effective judgement and improvement regarding the truncation errors of the modal synthesis method are put forward. The truncation errors estimated are considered as "error forces" and their reasonable bound is defined to determine the number of acceptable modes. The initial eigenvalues and eigenvectors can be calculated from the modal synthesis method. The renewed eigenvalues are obtained from Rayleigh quotient and the renewed eigenvectors are obtained from an equation, which is essentially the expression for "error forces". Then the renewed eigenvectors are put into Rayleigh quotient again as an iterative process until the permitted "error forces" or definite errors of eigenvalues are reached. As a result, the proposed approach is called an "error force vector method". Obviously, the process is also an accuracy improvement for eigenpair updating.

1. INTRODUCTION

Complete structures are frequently very complex and their major components are often designed and produced by different organizations. Thus, it may be difficult to assemble a finite element model (FEM) of the entire structure in a timely manner. In addition, the finite element model of the entire structure might contain so many degrees of freedom that it would be infeasible to perform a dynamic analysis based on the finite element equations for the complete system. For these reasons, a powerful method has been developed which permits the structure to be subdivided into several components, or substructures, with much of the analysis being done on the subdivided components in order to develop an approximate mathematical model of the full structural system [1]. Normally, this method has come to be called the method of component modal synthesis or the modal synthesis method (MSM). It has made rapid progress and has been programmed in most commercial structural analysis software [2–4].

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It is understood that a so-called truncation error should be caused due to the imperfection of preselected Ritz bases. Thus, a very important question of common interest which comes up frequently is: how many modes are acceptable and how do people improve the acceptable modes so as to make them as accurate as possible? From the engineering point of view, analysts often tend to preselect the highest possible number of Ritz bases before synthesis and extract the lowest possible number of synthesized mode after synthesis to guarantee the accuracy of the dynamic properties of the entire structure. Unfortunately, it is in general not an economic way and moreover, the examination of accuracy should be completed normally by the modal test, which is not feasible in a structural design step.

An earlier analysis for the error in synthesis techniques was done by Natke [5] in 1982. He concluded that approximate synthesis formulations could be used with known modal submatrices and, if wanted, with supplemented arbitrary matrices. The linearized error analysis required the calculation of one scalar equation and the solution of one system of linear equation of reduced order for each degree of freedom. In order to check these values, one can take bilinear forms resulting from the quadratic eigenvalue problem with regard to the errors in a second approximation, while using the approximate eigensolutions of the modal synthesis or the solutions corrected by the linearized errors. If necessary, one could determine the errors in a second approximation by solving the quadratic eigenvalue problem. Later, many authors [6, 7] discussed the errors caused in modal synthesis of multiple substructures with interface damping, and the truncation error reduction by using the static parts of the truncated modes without the necessity of their being known. Special interest was also concentrated on some special engineering problems, for instance, the error of modal truncation in substructure testing [8, 9]. In reference [10], Curnier gave a formal proof of the exactness of all three variants (i.e., fixed, free and loaded interface variants) in the absence of modal truncation and studied the sensitivity of the truncation error to the different interface condition.

An "error force vector method" is developed in this paper to confirm the reliability of modes after truncation and to improve the accuracy of truncated eigenpair problem as much as possible.

2. THEORY AND METHOD

It is assumed that the mass matrices and the stiffness matrices of assembled substructures are known and they may also be modified by some methods of finite element model updating by using test modal data, if available, to improve their accuracy. Then, a reliable mass matrix $[M]$and stiffness matrix $[K]$ for the entire structure are assembled directly from elemental ones of the substructures. The approximate eigenvalues and eigenvectors given by the modal synthesis method are expressed as λ_i, $\{x_i\}$ and the assumed exact ones (unknown) are expressed as λ_i', $\{x_i'\}$ which should be obtained from the equation

$$([K] - \lambda_i'[M])\{x_i'\} = 0 \tag{1}$$

without truncation error at all.

Let

$$\lambda_i' = \lambda_i + \Delta\lambda_i, \qquad \{x_i'\} = \{x_i\} + \{\Delta x_i\}, \tag{2}$$

where $\Delta\lambda_i$ and $\{\Delta x_i\}$ caused by mode truncation are the error items of eigenvalue and eigenvector respectively. Of course, $\{x_i\}$ and $\{x_i'\}$ possess the same dimensions as the number of concerned d.o.f. for the structure. As shown in Figure 2, there are big differences between $\{x_i\}$ and $\{x_i'\}$, sometimes, for a high order mode.

Substituting equation (2) into equation (1), regardless of the high negligible quantities, the following equation will be obtained:

$$([K] - \lambda_i[M])\,\{x_i\} + ([K] - \lambda_i[M])\{\Delta x_i\} - \Delta\lambda_i[M]\{x_i\} = \{0\}. \tag{3}$$

Let

$$\{f_i\} \equiv ([K] - \lambda_i[M])\{x_i\} \tag{4}$$

and define it, according to the dimensional analysis, as an "error force vector". Obviously, $\{f_i\} = 0$ if λ_i and $\{x_i\}$ are the ith exact eigenvalue and eigenvector respectively. Therefore, $\{f_i\}$ is a measurement of the mode truncation.

2.1. CRITERION OF ACCEPTABLE MODE

By computing the error force vectors $\{f_i\}$, $i = 1, 2, 3, \dots, n$, extracting the maximum element of $\{f_i\}$, defining it as $\max f_{i,j}$, $i = 1, 2, 3, \dots, n$, $j = 1$ or 2 or 3, …, or n (i.e., the maximum element is located at the jth element of the ith error force vector) and indicating the minimum one among $\max f_{i,j}$ as a constant m, the acceptable mode can then be determined if $\max f_{i,j}/m \leqslant \varepsilon$, where $\varepsilon \geqslant 1$ is a given limitation of error.

2.2. THE IMPROVEMENT OF ACCEPTABLE MODE

Considering equation (4), it is convenient to rewrite equation (3) as

$$\{f_i\} + ([K] - \lambda_i[M])\{\Delta x_i\} = \Delta\lambda[M]\{x_i\}. \tag{5}$$

Furthermore, the orthonormal mass

$$\{x_i'\}^{\mathrm{T}}[M]\{x_i'\} = 1, \tag{6}$$

i.e.,

$$(\{x_i\} + \{\Delta x_i\})^{\mathrm{T}}[M](\{x_i\} + \{\Delta x_i\}) = 1. \tag{7}$$

Then, the error eigenpair $\Delta\lambda_i$ and $\{\Delta x_i\}$ can be obtained from equations (5) and (7) in principle and the improved eigenpair is expressed by $\lambda_i' = \lambda_i + \Delta\lambda_i$ and $\{x_i'\} = \{x_i\} + \{\Delta x_i\}$. However, the process of solving simultaneous non-linear equations

(5) and (7) is tedious and time consuming. A very simple iteration approach is proposed here to obtain a sufficient good solution of λ_i and $\{\Delta x_i\}$ based on Rayleigh quotient.

Step 1: Computing $\{f_i^0\}$ (superscript 0 is related to the quantities obtained by the modal synthesis method directly, and the same below) via equation (4), we get

$$\{f_i^0\} = ([K] - \lambda_i [M])\{x_i\}, \tag{8}$$

where $\{x_i\}$ and λ_i are the initial eigenvector and eigenvalue, respectively, obtained by the modal synthesis approach. The process then will go to step 2, if $\{f_i^0\}$ cannot meet the criterion of acceptable modes; otherwise the process stops.

Step 2: Finding Rayleigh quotient λ_i^0, we get

$$\lambda_i^0 = \frac{\{x_i^0\}^{\mathrm{T}}[K]\{x_i^0\}}{\{x_i^0\}^{\mathrm{T}}[M]\{x_i^0\}}. \tag{9}$$

Moreover

$$\Delta\lambda_i^0 = \lambda_i^0 - \lambda_i. \tag{10}$$

Step 3: Obtaining $\{\Delta x_i^0\}$ from equation (5), we get

$$\{\Delta x_i^0\} = ([K] - \lambda_i^0[M])^{-1}(\Delta\lambda_i^0[M]\{x_i^0\} - \{f_i^0\}). \tag{11}$$

Step 4: The first improvement of eigenpair, $\{x_i^1\}$ and λ_i^1 can then be obtained from

$$\{x_i^1\} = \{x_i^0\} + \{\Delta x_i^0\} \tag{12}$$

and

$$\lambda_i^1 = \frac{\{x_i^1\}^{\mathrm{T}}[K]\{x_i^1\}}{\{x_i^1\}^{\mathrm{T}}[M]\{x_i^1\}}. \tag{13}$$

Step 5: By substituting λ_i^1 into equation (4), $\{f_i^1\}$ is gained from

$$\{f_i^1\} = ([K] - \lambda_i^1[M])\{x_i^1\}. \tag{14}$$

Step 6: The iteration should stop if the results are satisfied with

$$|\lambda_i^1 - \lambda_i^0| \leqslant \varepsilon_1 \tag{15}$$

or

$$\left|\frac{\max f_{i,j}^1}{\max f_{i,j}^0}\right| \leqslant \varepsilon_2, \tag{16}$$

where ε_1 and ε_2 are defined as error limitation according to the requirement in practice; otherwise it will go to step 3 again and the superscript k is increased to $(k+1)$.

It should be explained that:

(1) The mass and stiffness matrices $[M]$ and $[K]$ of the entire structure used in the proposed approach are only for the purpose of derivation conciseness, and actually they can be kept in the form of a block component, and the calculation of equations (4)–(9) can be carried out by partitioned block multiplication.
(2) The assumed exact solution λ_i' and $\{x_i'\}$ in equation (1) are unknown and it is unnecessary to find them during the operation of the presented approach.

3. NUMERICAL RESULTS AND DISCUSSIONS

To demonstrate the principle and the algorithm described in this paper, several typical numerical examples have been cited. Here, only a 188-bar space truss (Figure 1) is selected to show the results due to limited space. The elastic modulus and mass density for the truss are equal to $2{\cdot}1 \times 10^{11}$ Pa and $7800\ \mathrm{kg/m^3}$ respectively. The entire structure is divided into two symmetric substructures. The initial results of eigenpair λ_i and $\{x_i\}$ are found by the constraint mode synthesis method. In the method, the number of fixed-interface normal mode is preselected to be 12 for each substructure. The resulting number of synthesis mode for the entire structure is 36. For the convenience of comparison, the first 10 natural frequencies obtained from the finite element method, the modal synthesis method and the improved method proposed in the paper are listed in Table 1. The results of the finite element method can be assumed as the exact ones because the mass and the stiffness matrices for the 188-bar truss structure are exact. It is shown that

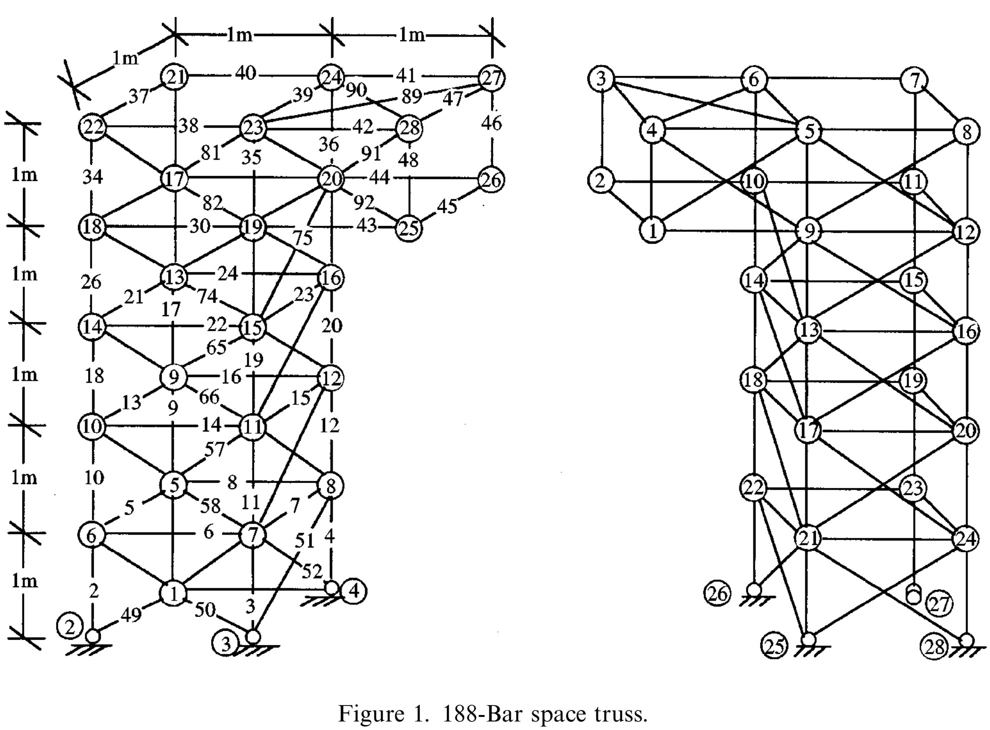

Figure 1. 188-Bar space truss.

TABLE 1

The comparison of natural frequency among different approaches (Hz)

	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Finite element method (without truncation)	20·866	26·424	30·968	42·759	46·622	92·956	107·25	108·47	111·35	122·99
Modal synthesis method	20·451	26·289	30·486	42·751	46·355	92·026	106·90	108·44	109·35	122·70
Proposed method	20·869	26·425	30·972	42·760	46·624	92·957	107·25	108·47	111·35	122·99

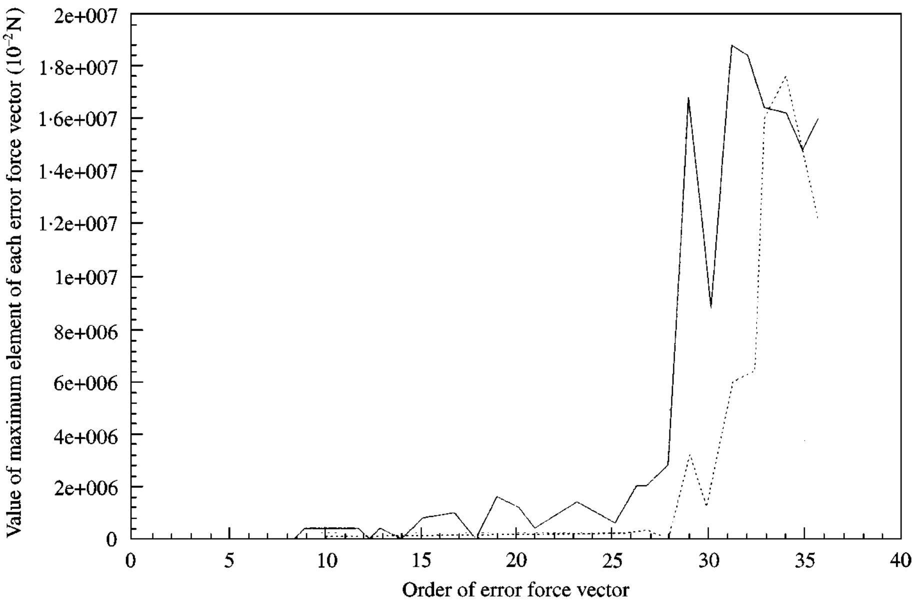

Figure 2. Maximum element of each error force vector. (——) synthesized results; (. . . .) improved results after 1st iteration.

the biggest error of natural frequency occurs at the 9th synthesized frequency. Moreover, it is clear that perfect results can be obtained for all first 10 natural frequencies after the improvement via the proposed method. Figure 2 shows the value of the biggest element of the error force vector. A definite conclusion can be drawn that as many as 28 modes can be acceptable among the 36 synthesized frequencies. Beyond the 28th mode, an obvious bigger error force is yielded. A comparison between the synthesized error force and the one of first iteration, in Figure 2, also shows that an excellent improvement can be gained from the proposed method. The first 36 natural frequencies obtained from the finite element method and the modal synthesis method are shown in Figure 3. In fact, all degrees of freedom of the truss were taken and no modal truncation occurred in FEM numerical analysis. Therefore, the eigenpairs solution from FEM with consistent mass matrices can be considered as the exact ones for the 188-bar space truss under

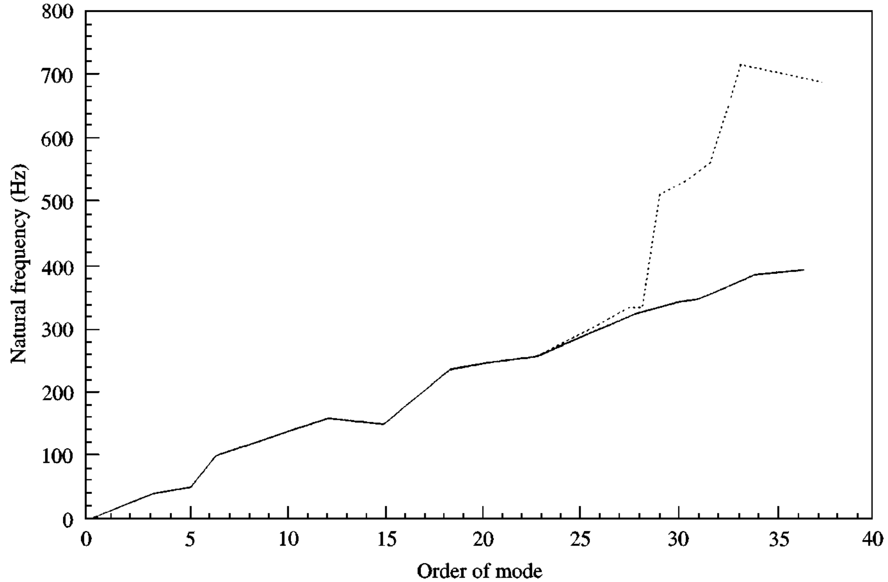

Figure 3. Results from FEM and MSM. (——) finite element results (without truncation); (. . . .) synthesized results.

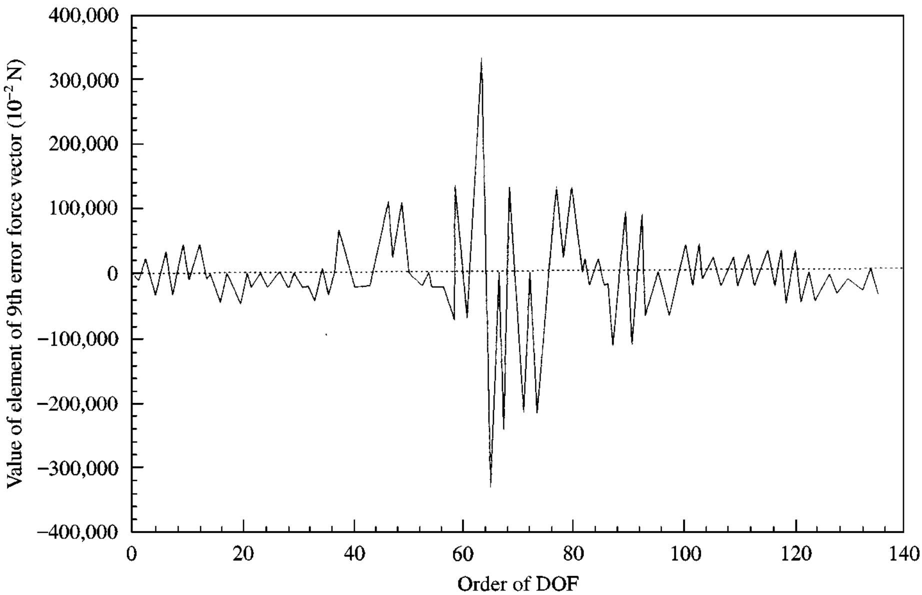

Figure 4. Value of the element of 9th error force. (——) synthesized results; (. . . .) improved results after 1st iteration.

the assumption of uniform element undergoing axial deformation. Actually, a real exact solution would not be obtained unless an assumption of elastic mechanic is adopted instead. However, it is unnecessary for the purpose of contrast. The effectiveness and the reliability of the method are confirmed again in Figure 3. Figure 4 shows all elements of the 9th error force vector and the ones after first iteration. A distinct improvement is presented again via the iteration method. It is worth pointing out that the modes beyond the 28th are totally unacceptable even when using the proposed method (Figure 2). The possible explanation is that the synthesized eigenvectors beyond the 28th have seriously drifted off their correct ones, so that the improvement of Rayleigh quotient will be irrational and ineffective. When it happens, the only way out is to increase the number of remaining normal modes in the modal synthesis process.

4. CONCLUSIONS AND REMARKS

Several points seem reasonably clear from the amount of information gathered from this investigation.

(1) The proposed method can be applicable to different kinds of modal synthesis methods because the deducting process has nothing to do with the property of the modal synthesis method.
(2) The method presented here may not only serve as a criterion of acceptable mode, but also be considered as a measure to update the synthesized modes.
(3) The proposed method can not only improve the accuracy of natural frequency but also the accuracy of eigenvector in step 4. Therefore, it is a good approach, in case the eigenvector improvement is required. Several numerical examples show that a good improvement can be gained just after the first iteration.
(4) The proposed method, only using Rayleigh quotient and a simple iteration process, possesses properties of simplicity and explicitness and is easy to operate.

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