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LOCALIZATION OF FREE VIBRATION MODES IN SYSTEMS OF NEARLY IDENTICAL VIBRATION ABSORBERS

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1. INTRODUCTION

It is known that when the degrees of freedom of a nominally periodic structure are weakly coupled and there are some small imperfections in the structure, then the free vibration modes will typically be spatially localized, resulting in confined regions of the structure where the vibration energy is concentrated. This type of localization was first considered in the field of solid state physics by Anderson [1], who showed that in a randomly disordered linear chain of particles, the wave function of the chain can exhibit spatially confined modes of motion.

One of the earliest studies of the phenomenon of localization in the field of structural dynamics was made by Hodges [2]. Pierre and Dowell [3] investigated localization phenomena for a chain of coupled oscillators, and Pierre *et al.* [4] theoretically and experimentally investigated localization of the free modes of vibration of disordered multi-span beams constrained at irregular intervals. Samples of other work on localization phenomenon in both linear and non-linear systems can be found in references [5–7].

In the present work we consider free vibrations of systems which consist of a primary inertia to which are attached several nearly identical subsystems. The motivation for studying this class of systems is that they represent situations in which one wishes to employ undamped, tuned vibration absorbers and distribute the total absorber inertia among several smaller absorber masses. We consider both translational and rotational systems which have no resilient connection to ground. While this is not typical for translational absorber systems, it is natural in the rotational setting. In the translational case, the absorbers are connected via springs, while in the case of torsional absorbers, the absorbers often use the centrifugal field for restoring and are tuned to a given order at all rotation speeds [8].

We consider the case in which the coupling is small compared to the level of imperfections, and use a perturbation technique to determine the mode shapes, as described by Pierre [3]. One distinction here is that the perfectly tuned system in the present case is completely degenerate, with several repeated frequencies and non-unique modes. This prevents one from employing the singular perturbation approach of Happawana *et al.* [9] to determine the modes for all ranges of imperfection and coupling.

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The note is organized as follows. Section 2 describes the two types of absorber systems and formulates the equations of motion for each in such a manner that the perturbation technique can be readily applied. Section 3 describes the analysis and presents sample results, and the paper closes with a brief discussion in Section 4.

2. FORMULATION

2.1. TRANSLATIONAL VIBRATION ABSORBERS

Consider a structure of mass M on which are mounted N vibration absorbers of masses m_i and spring stiffnesses k_i, $(i = 1, \cdots, N)$, as shown in Figure 1. We consider the case in which the natural frequency of the primary mass–spring system is much smaller than that of the absorbers, and use the limiting case in which the primary mass has no stiffness to ground, so that the overall system has a rigid-body mode. In the analysis, we will uncouple this mode via a simple change of co-ordinates. A similar system (only with stiffness to ground) has been considered by Weaver [10], who considered the response to random excitation of a system with a large number of substructures having a distribution of natural frequencies.

The energy-based equations of motion for free vibration of this system are

$$M\ddot{y} + \sum_{j=1}^{N} m_j(\ddot{z}_j + \ddot{y}) = 0, \tag{1}$$

$$m_i(\ddot{z}_i + \ddot{y}) + k_i z_i = 0. \tag{2}$$

We assume that the absorbers are of equal mass (i.e., $m_i = m$), and use the stiffnesses to introduce mistuning among the absorbers. The dynamics of the primary mass can be uncoupled from the absorbers' by substituting $\ddot{y}$ from equation (1) into equation (2), using some summations, and rearranging. This results in the following

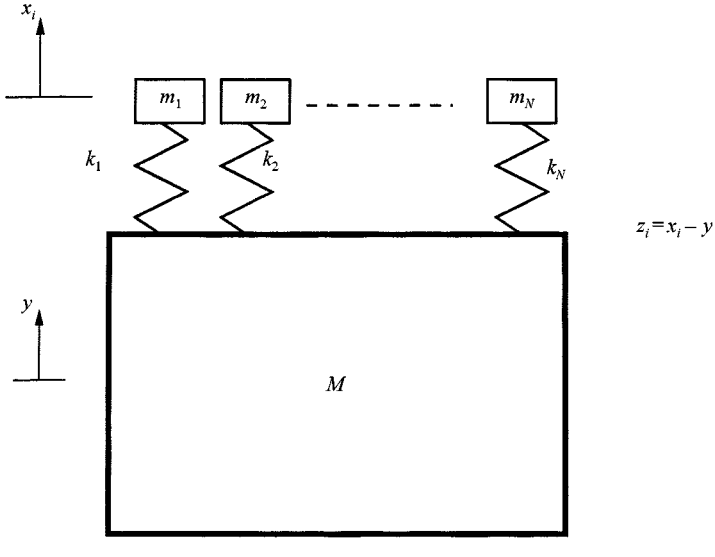

Figure 1. Translational vibration absorbers.

equations that describe the responses of the absorbers:

$$\ddot{z}_i + \omega_i^2 z_i + \frac{m_0}{NM}\sum_{j=1}^{N}\omega_j^2 z_j = 0, \tag{3}$$

where $m_o = Nm$ is the total mass of all absorbers and $\omega_i^2 = k_i/m$ are their individual, uncoupled natural frequencies.

In typical applications the absorber mass is significantly smaller than the primary mass. Therefore, we introduce as a small parameter the ratio of the total absorber mass to the primary mass, denoted as ε, i.e., $\varepsilon = m_o/M$. Since the absorbers are coupled to each other only through the primary mass, ε represents the magnitude of the coupling between the individual absorbers.

Equation (3) can now be expressed in matrix form as

$$\mathbf{I}\ddot{\tilde{z}} + \mathbf{A}\tilde{z} = \tilde{0}, \tag{4}$$

where

$$\mathbf{A} = \mathbf{A}_0 + \delta\mathbf{A}, \quad \mathbf{A}_0 = Diag\,[\omega_i^2], \tag{5}$$

$\delta\mathbf{A}$ is an $N \times N$ matrix with each element of the i^{th} column equal to $\varepsilon w_i^2/N$. Since the diagonal matrix $\mathbf{A}_o$ has readily obtained eigenvalues and eigenvectors, equation (4) is in a form suitable for the application of an eigenvalue perturbation method. Before doing so, we show that the dynamics of a system of torsional vibration absorbers can also be expressed in identical form.

2.2. CENTRIFUGAL PENDULUM VIBRATION ABSORBERS

We consider a system of N torsional vibration absorbers which are mounted on a rotor of inertia J_0, as shown in Figure 2. The ith absorber, which has a mass of m_i, is riding on a path specified by the function $R_i(S_i)$, which denotes the distance from a point on the path of the ith absorber to the center of the rotor. S_i is an arc length variable along the path, and is used to describe the position of m_i. R_{io} is the value of R_i at the vertex of the path, i.e., $R_{io} = R_i(S_i = 0)$. θ represents the angular orientation of the rotor, and we define s_i as the normalized arc length co-ordinate, i.e., $s_i = S_i/R_{io}$.

The free vibration equations of motion for this system, for arbitrary absorber amplitudes, are given by [8]

$$\ddot{s}_i + g_i(s_i)\ddot{\theta} - \frac{1}{2}\frac{\mathrm{d}x_i}{\mathrm{d}s_i}(s_i)\dot{\theta}^2 = 0, \tag{6}$$

$$J_o\ddot{\theta} + \sum_{i=1}^{N} m_i R_{io}^2\left[\frac{\mathrm{d}x_i}{\mathrm{d}s_i}(s_i)\dot{s}_i\dot{\theta} + x_i(s_i)\ddot{\theta} + g_i(s_i)\ddot{s}_i + \frac{\mathrm{d}g_i}{\mathrm{d}s_i}(s_i)\dot{s}_i^2\right] = 0, \tag{7}$$

where the functions $x_i(s_i)$ and $g_i(x_i)$ are defined as

$$\begin{aligned} g_i(s_i) &= \sqrt{x_i(s_i) - \frac{1}{4}\left(\frac{\mathrm{d}x_i}{\mathrm{d}s_i}(s_i)\right)^2} \\ x_i(s_i) &= r_i^2(s_i) = \left(\frac{R_i}{R_{io}}\right)^2 \\ &= 1 - \tilde{n}_i^2 s_i^2 + \text{HOT}, \end{aligned} \tag{8}$$

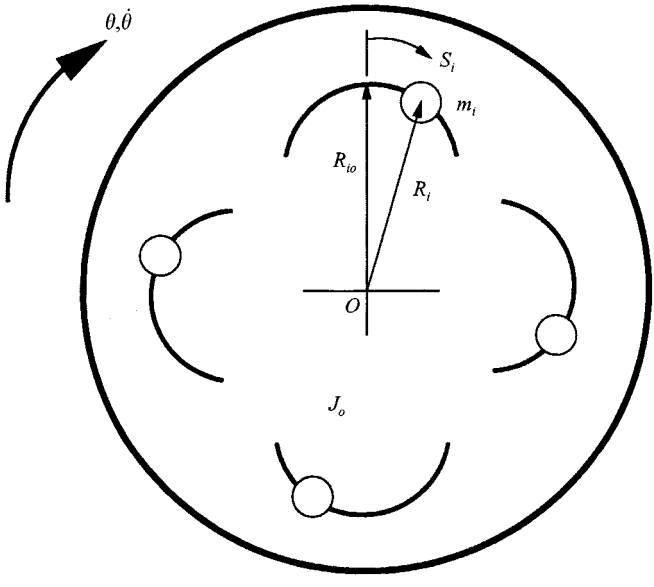

Figure 2. Torsional vibration absorbers.

where $\tilde{n}_i = n(1 + \sigma_i)$. Typically, $\sigma_i \ll 1$, which represents the variability of the tuning of each absorber from the nominal tuning of order n. (These absorbers are tuned to react to an applied torque that is of order n, that is, it generates n torque periods per revolution of the rotor [8]).

We again assume that the absorbers are of equal mass and the value of R_i at the vertex of the path is the same for all absorbers, i.e., $m_i = m$, and $R_{io} = R_0$. Linearizing equations (6) and (7) about $s_i = 0$, and $\dot{\theta} = \Omega$, where Ω is the nominal speed of the rotor, and making use of equation (8), the following linearized equations for the system dynamics are obtained:

$$J_o\ddot{\theta} + \sum_{i=1}^{N} mR_o^2(\ddot{\theta} + \ddot{s}_i) = 0, \tag{9}$$

$$\ddot{s}_i + \ddot{\theta} + \Omega^2\tilde{n}_i^2 s_i = 0. \tag{10}$$

Comparing equations (9) and (10) with equations (1) and (2), they are clearly seen to be equivalent. This implies that equation (4) also applies to the torsional vibration absorber problem. In this case, the small parameter ε represents the ratio of the total moment of inertia of the absorbers to the rotor moment of inertia, that is, $\varepsilon = NmR_0^2/J_0$, and ω_i is replaced here by $\Omega\tilde{n}_i$.

3. ANALYSIS AND DISCUSSION

3.1. GENERAL FEATURES OF THE SYSTEM

The system of equations (1, 2) has some interesting properties when the absorbers are identical. When $k_i = k \forall i$, the absorbers have identical uncoupled natural

frequencies. In this case, the overall $N+1$ degree-of-freedom system has two modes in which the absorber masses move in a synchronous manner; these correspond to the modes of an equivalent two-degree-of-freedom system wherein one is the rigid-body mode, and the other is oscillatory. The remaining $N-1$ modes have identical frequencies and mode shapes that correspond to the absorber masses moving in such a manner that they exert zero net force on the primary mass, rendering it stationary. The selection of the modes in this degenerate case is highly non-unique. Therefore, the level of imperfections cannot be used as a small parameter in a perturbation scheme, since one does not have a specified set of modes that can be used as the starting point in the perturbation scheme. This unperturbed system has absolute sensitivity to the mistuning in the sense that different arrangements of mistuning lead to completely different sets of modes.

On the other hand, the case of zero coupling and non-zero mistuning is unique, since the modes are represented by the ideally localized responses in which only one absorber is active and the others are stationary. These are naturally suited for use as the seed modes in a perturbation scheme [3].

Singular perturbation schemes have recently been used to capture localized modes for all relative ranges of mistuning and coupling [9, 11]. However, in the present case this is not feasible, due to the completely degenerate nature of the system in the case of zero mistuning.

3.2. PERTURBATION METHOD FORMULATION

The normal modes of vibration are represented by the eigenvectors of the matrix **A** in equation (4). These can be determined here using a standard perturbation method of the eigenvalue problem [12]. The following expressions are used for the eigenvalues (λ_i) and the eigenvectors ($\tilde{v}_i$), up to second order in approximation:

$$\lambda_i = \lambda_{oi} + \delta\lambda_i + \delta^2\lambda_i, \quad \tilde{v}_i = \tilde{v}_{oi} + \delta\tilde{v}_i + \delta^2\tilde{v}_i,$$

where λ_{oi} and $\tilde{v}_{oi}$ are the eigenvalues and eigenvectors of $\mathbf{A}_0$, respectively. The terms in the perturbation expansion are solved for by expanding and matching terms, and are given by

$$\delta\tilde{v}_i = \sum_{k=1}^{N} v_{ik}\tilde{v}_{ok}, \quad \delta^2\tilde{v}_i = \sum_{k=1}^{N} \eta_{ik}\tilde{v}_{ok}, \quad i = 1, \ldots, N,$$

$$\delta\lambda_i = \frac{\tilde{y}_{oi}^{\mathrm{T}}[\delta\mathbf{A}]\tilde{v}_{oi}}{\tilde{y}_{oi}^{\mathrm{T}}\tilde{v}_{oi}}, \quad \delta^2\lambda_i = \frac{\tilde{y}_{oi}^{\mathrm{T}}[\delta\mathbf{A}]\delta\tilde{v}_i - \delta\lambda_i\tilde{y}_{oi}^{\mathrm{T}}\delta\tilde{v}_i}{\tilde{y}_{oi}^{\mathrm{T}}\tilde{v}_{oi}}, \quad i = 1, \ldots, N,$$

where, for $j \neq i$, the coefficients are

$$v_{ij} = \frac{1}{\tilde{y}_{oj}^{\mathrm{T}}\tilde{v}_{oj}} \frac{\tilde{y}_{oj}^{\mathrm{T}}[\delta\mathbf{A}]\tilde{v}_{oi}}{(\lambda_{oi} - \lambda_{oj})},$$

$$\eta_{ij} = \frac{1}{\tilde{y}_{oj}^{\mathrm{T}}\tilde{v}_{oj}} \frac{\delta\lambda_i\tilde{y}_{oj}^{\mathrm{T}}\delta\tilde{v}_i - \tilde{y}_{oj}^{\mathrm{T}}[\delta\mathbf{A}]\delta\tilde{v}_i}{(\lambda_{oj} - \lambda_{oi})},$$

and for $j = i$,

$$v_{ii} = 0, \quad \eta_{ii} = -\frac{\delta\tilde{v}_i^{\mathrm{T}}\delta\tilde{v}_i}{2\tilde{v}_{oi}^{\mathrm{T}}\tilde{v}_{oi}}.$$

Here $\tilde{y}_{oi}$ represents the left eigenvectors of $\mathbf{A}_0$.

Note that these expressions are, as expected, singular if the frequencies are repeated. That is, this perturbation scheme accounts for the effects of coupling, but the results become invalid if the mistuning among the absorbers is small [3].

These perturbation results can be used to find the frequencies and modes of vibration. This is easy in the present case, since the eigenvectors of the matrix $\mathbf{A}_0$ can be taken as the canonical unit vectors (for the case of distinct eigenvalues). Two numerical examples are now presented.

3.3. EXAMPLES

The purpose of these examples is to demonstrate the localization phenomenon and the accuracy of the perturbation method.

3.3.1. *Six absorbers with moderate coupling*

In this example, we consider a system of six absorbers with the data given in Table 1, and with a coupling coefficient of 1/20. Two cases of mistuning are considered, as indicated in Table 1.

Figures 3 and 4 show the modes of free vibration obtained by the second order perturbation method and by the exact solution of the full eigenvalue problem. As expected, the perturbation results start to deviate from the exact solution in the case of small mistuning.

3.3.2. *Ten absorbers with small coupling*

Here we consider a system of 10 absorbers with the data given in Table 2. The coupling coefficient is 1/75, a value taken from an existing light aircraft engine. Figure 5 shows the free vibration modes for this system of absorbers. The modes are seen to be highly localized, and, as expected, the perturbation method works very well in this case.

TABLE 1

Data for example 1

Absorber	Mistuning (a) (%)	Mistuning (b) (%)
1	2	1
2	− 2·5	− 1·2
3	3·5	1·7
4	− 4·5	− 2·2
5	− 1·0	− 0·5
6	0·5	0·2

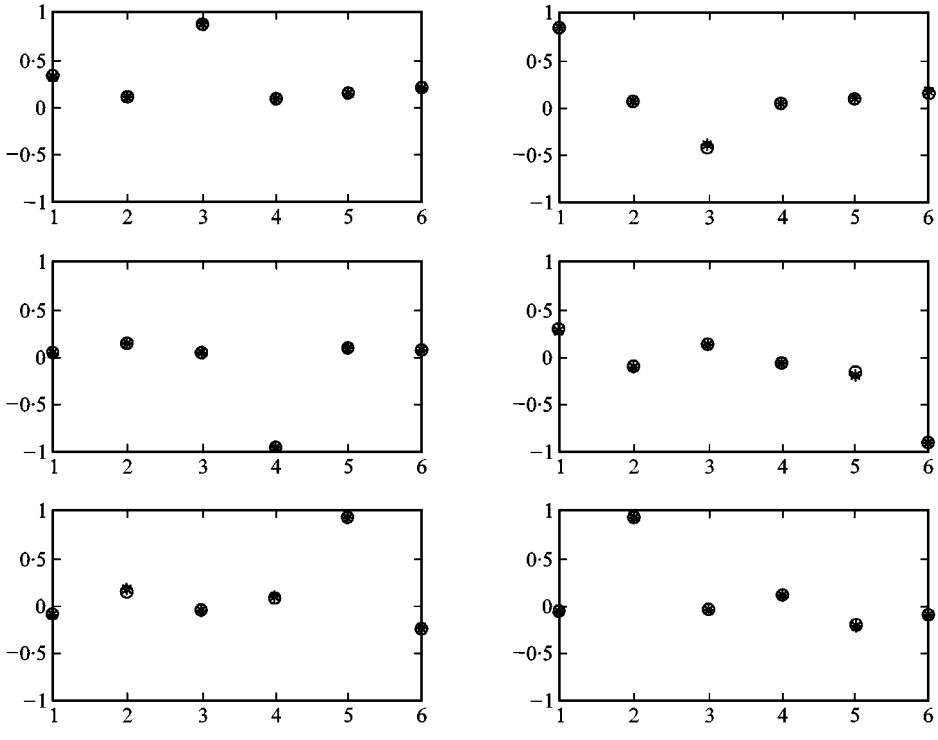

Figure 3. Modes of vibration for example 1(a): ∗, exact ◦, second order perturbation.

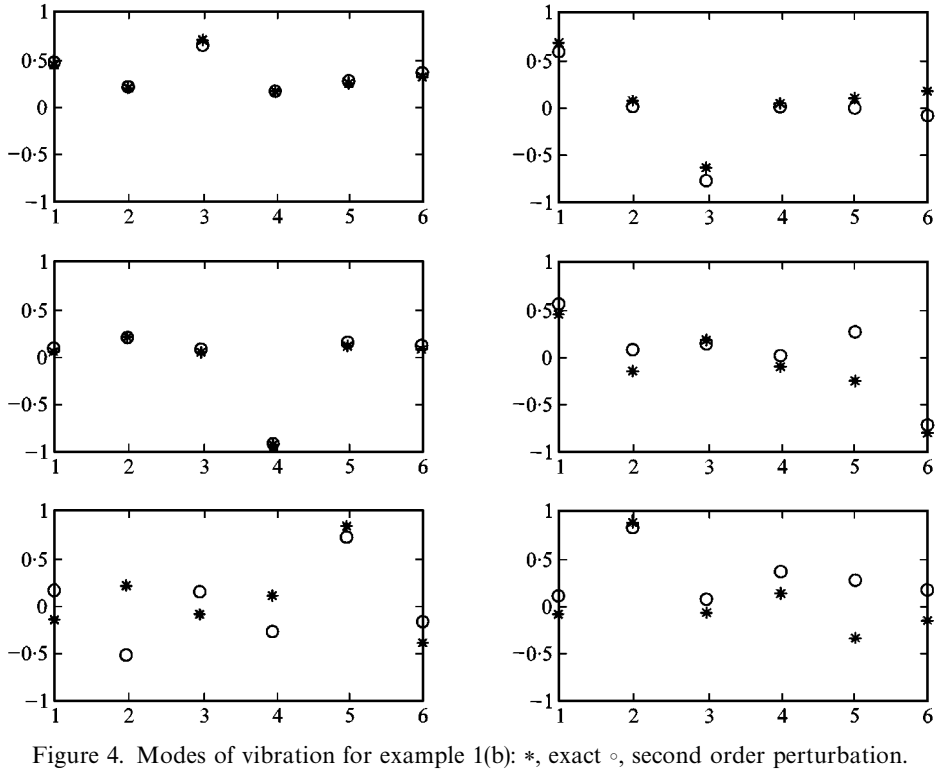

Figure 4. Modes of vibration for example 1(b): ∗, exact ◦, second order perturbation.

TABLE 2

Data for example 2

Absorber	1	2	3	4	5	6	7	8	9	10
% Mistuning	1·2	0·0	1·6	− 0·4	0·4	0·8	− 1·2	− 1·8	− 0·8	− 1·6

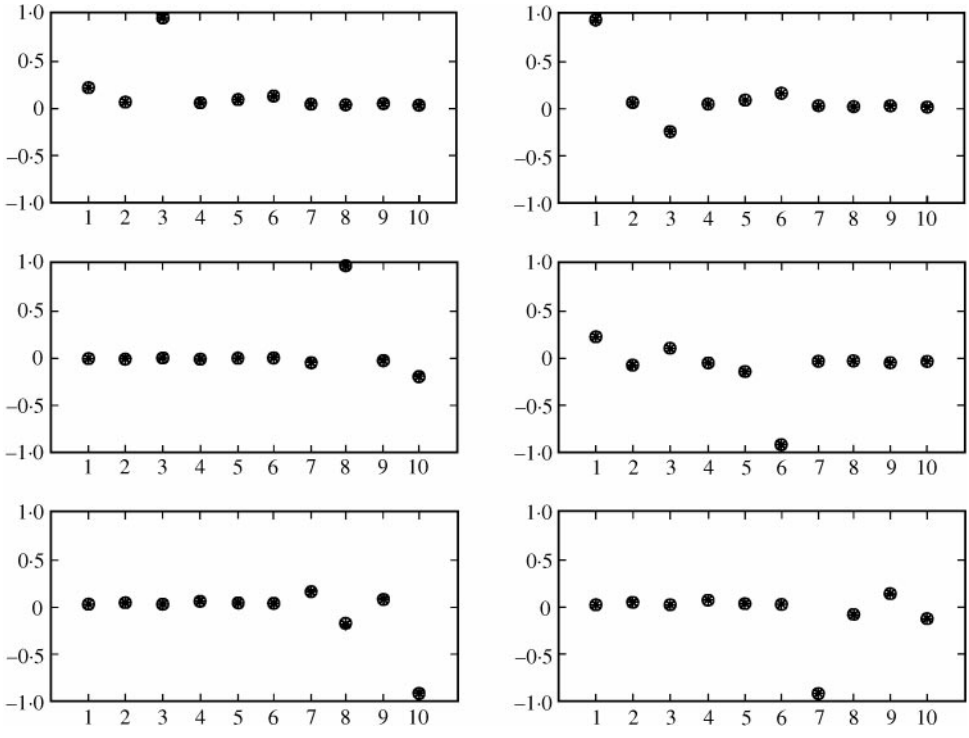

Figure 5. Some modes of vibration for example 2: ∗, exact ◦, second order perturbation.

4. CONCLUSIONS AND FUTURE WORK

The results obtained from the examples clearly show that localization indeed occurs for this class of systems, and that the perturbation method is a reliable tool for obtaining the modes of vibration for a range of small coupling relative to mistuning. These results also indicate that some interesting and unexpected behavior may be found in the forced response of systems of tuned vibration absorbers, in particular since these systems are excited at a frequency that is very close to the natural frequencies of the absorbers. A hint of this has been found in the localized, non-linear response of certain types of torsional absorbers with zero mistuning [7, 8], and this topic is currently under investigation.

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